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Research Article

On a Fixed Point for Generalized Contractions in Generalized Metric Spaces

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Lakzian and Samet (2010) studied some fixed-point results in generalized metric spaces in the sense of Branciari. In this paper, we study the existence of fixed-point results of mappings satisfying generalized weak contractive conditions in the framework of a generalized metric space in sense of Branciari. Our results modify and generalize the results of Laksian and Samet, as well as, our results generalize several well-known comparable results in the literature.

1. Introduction and Preliminaries

Branciari in [1] initiated the notion of a generalized metric space as a generalization of a metric space in such a way that the triangle inequality is replaced by the "quadrilateral inequality," $d(x,y) \leq d(x,a) + d(a,b) + d(b,y)$ for all pairwise distinct points x, y, a, and b of X. Afterwards, many authors initiated and studied many existing fixed-point theorems in such spaces. For more details about fixed-point theory in generalized metric spaces, we refer the reader to [1–13].

The following definitions will be needed in the sequel.

Definition 1.1 (see [1]). Let X be a nonempty set and $d : X \times X \to [0, +\infty)$ such that for all $x, y \in X$ and for all distinct points $u, v \in X$ each of them different from x and y, one has (p1): $x = y \Leftrightarrow d(x,y) = 0$,

(p2): $d(x,y) = d(y,x)$,

(p3): $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Then, (X,d) is called a generalized metric space (or shortly g.m.s).

Any metric space is a generalized metric space, but the converse is not true [1].

Definition 1.2 (see [1]). Let (X,d) be a g.m.s, $\{x_n\}$ a sequence in X, and $x \in X$. We say that $\{x_n\}$ is g.m.s convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.

Definition 1.3 (see [1]). Let (X,d) be a g.m.s and $\{x_n\}$ a sequence in X. We say that $\{x_n\}$ is a g.m.s Cauchy sequence if and only if for each $\varepsilon > 0$ there exists a natural number N such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N$.

Definition 1.4 (see [1]). Let (X,d) be a g.m.s. Then, (X,d) is called a complete g.m.s if every g.m.s Cauchy sequence is g.m.s convergent in X.

Very recently, Lakzian and Samet [9] proved the following nice result.

Theorem 1.5. *Let (X,d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ is such that for all $x, y \in X$*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \phi(d(x,y)), \tag{1.1}$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0$, and $\phi : [0,\infty) \to [0,\infty)$ is continuous and $\phi(t) = 0$ if and only if $t = 0$. Then, there exists a unique point $u \in X$ such that $u = Tu$.

Note that Theorem 1.5 extends a result of Dutta and Choudhury [14] to the set of generalized metric spaces. Moreover, its proof is more technical compared with that of [9].

In this paper, we generalize in some cases Theorem 1.5 by replacing in (1.1) the term $d(x,y)$ by the quantity $\max\{d(x,y), d(x,Tx), d(y,Ty)\}$ and the continuity of ϕ by lower semicontinuity. Also, we derive some useful corollaries of this result.

2. Main Results

Let X be a nonempty set and $T : X \to X$ a given mapping. For all $x, y \in X$, set

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{2.1}$$

Also, let $\Psi = \{\psi \mid \psi : [0,\infty) \to [0,\infty)$ be continuous, nondecreasing, and $\psi(t) = 0$ if and only if $t = 0\}$, and $\Phi = \{\phi \mid \phi : [0,\infty) \to [0,\infty)$ is lower semi continuous, $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0\}$. Note that, if $\psi \in \Psi$, ψ is called an altering distance function [15].

The notion of a periodic point of a given mapping $T : X \to X$ is crucial for proving our main theorem. So we need the following definition.

Definition 2.1. Let X be a nonempty set. A given mapping $T : X \to X$ admits a periodic point if there exists $u \in X$ such that $u = T^p u$ for some $p \geq 1$. If $p = 1$, u is a fixed point.

Hence, each fixed point is also a periodic point of T.

Now, in the following, let us prove our main result.

Theorem 2.2. *Let (X, d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ is such that for all $x, y \in X$*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{2.2}$$

where $\psi \in \Psi$, $\phi \in \Phi$, and $M(x, y)$ is defined by (2.1)*. Then, there exists a unique point $u \in X$ such that $u = Tu$.*

Proof. First, it is obvious that $M(x, y) = 0$ if and only if $x = y$ is a fixed point of T. Let $x_0 \in X$ an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \forall n \geq 0. \tag{2.3}$$

Step 1. We claim that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

Substituting $x = x_n$ and $y = x_{n-1}$ in (2.2) and using properties of functions ψ and ϕ, we obtain

$$\begin{aligned} \psi(d(x_{n+1}, x_n)) &= \psi(d(Tx_n, Tx_{n-1})) \\ &\leq \psi(M(x_n, x_{n-1})) - \phi(M(x_n, x_{n-1})) \\ &\leq \psi(M(x_n, x_{n-1})) \end{aligned} \tag{2.5}$$

which implies that

$$d(x_{n+1}, x_n) \leq M(x_n, x_{n-1}) \quad \forall n \geq 1. \tag{2.6}$$

Note that

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\ &= \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned} \tag{2.7}$$

If for some $n \geq 1$, $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, then $M(x_n, x_{n-1}) = d(x_n, x_{n+1}) > 0$ and $\phi(d(x_{n+1}, x_n)) > 0$ by a property of ϕ, so (2.5) becomes

$$0 < \psi(d(x_{n+1}, x_n)) \leq \psi(d(x_{n+1}, x_n)) - \phi(d(x_{n+1}, x_{n+1})) < \psi(d(x_{n+1}, x_n)) \tag{2.8}$$

a contradiction. Thus, for all $n \geq 1$,

$$d(x_{n+1}, x_n) \leq d(x_{n-1}, x_n) = M(x_{n-1}, x_n). \tag{2.9}$$

From (2.9), the sequence $\{d(x_n, x_{n+1})\}$ is monotone nonincreasing and so bounded below. So there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} M(x_{n-1}, x_n) = r. \tag{2.10}$$

Letting $\limsup_{n\to\infty}$ in (2.5) and using the above limits with the continuity of ψ and the lower semicontinuity of ϕ, we get $\psi(r) \leq \psi(r) - \phi(r)$, which implies that $\phi(r) = 0$, so $r = 0$ by a property of ϕ. Thus, (2.4) is proved.

Step 2. We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.11}$$

By (2.2), we have

$$\begin{aligned}
\psi(d(x_{n+2}, x_n)) &= \psi(d(Tx_{n+1}, Tx_{n-1})) \\
&\leq \psi(M(x_{n+1}, x_{n-1})) - \phi(M(x_{n+1}, x_{n-1})) \\
&\leq \psi(M(x_{n+1}, x_{n-1}))
\end{aligned} \tag{2.12}$$

which implies that

$$d(x_{n+2}, x_n) \leq M(x_{n+1}, x_{n-1}) \quad \forall n \geq 1, \tag{2.13}$$

where

$$\begin{aligned}
M(x_{n+1}, x_{n-1}) &= \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, Tx_{n+1}), d(x_{n-1}, Tx_{n-1})\} \\
&= \max\{d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n)\} \\
&= \max\{d(x_{n+1}, x_{n-1}), d(x_{n-1}, x_n)\}.
\end{aligned} \tag{2.14}$$

Set $\alpha_n = d(x_{n+2}, x_n)$ and $\beta_n = d(x_n, x_{n+1})$. Thus, by (2.12), one can write

$$\psi(\alpha_n) \leq \psi\left(\max\left\{\alpha_{n-1}, \beta_{n-1}\right\}\right) - \phi\left(\max\left\{\alpha_{n-1}, \beta_{n-1}\right\}\right) \quad \forall n \geq 1 \tag{2.15}$$

which implies that

$$\alpha_n \leq \max\left\{\alpha_{n-1}, \beta_{n-1}\right\}. \tag{2.16}$$

On the other hand, having in mind that the sequence $\{d(x_n, x_{n+1})\} = \{\beta_n\}$ is monotone nonincreasing, so

$$\beta_n \le \beta_{n-1} \le \max\{\alpha_{n-1}, \beta_{n-1}\}. \tag{2.17}$$

From (2.16) and (2.17), we have

$$\max\{\alpha_n, \beta_n\} \le \max\{\alpha_{n-1}, \beta_{n-1}\} \quad \forall n \ge 1. \tag{2.18}$$

Therefore, the sequence $\{\max\{\alpha_n, \beta_n\}\}$ is monotone nonincreasing, so it converges to some $t \ge 0$. Assume that $t > 0$. Now, by (2.4), it is obvious that

$$\limsup_{n\to\infty} \alpha_n = \limsup_{n\to\infty} \max\{\alpha_n, \beta_n\} = \lim_{n\to\infty} \max\{\alpha_n, \beta_n\} = t. \tag{2.19}$$

Taking the $\limsup_{n\to\infty}$ in (2.15) and using (2.19) and the properties of ψ and ϕ, we obtain

$$\begin{aligned}
\psi(t) &= \psi\left(\limsup_{n\to\infty} \alpha_n\right) \\
&= \limsup_{n\to\infty} \psi(\alpha_n) \\
&\le \limsup_{n\to\infty} \psi(\max\{\alpha_{n-1}, \beta_{n-1}\}) - \liminf_{n\to\infty} \phi(\max\{\alpha_{n-1}, \beta_{n-1}\}) \\
&\le \psi\left(\lim_{n\to\infty} \max\{\alpha_{n-1}, \beta_{n-1}\}\right) - \phi\left(\lim_{n\to\infty} \max\{\alpha_{n-1}, \beta_{n-1}\}\right) \\
&= \psi(t) - \phi(t)
\end{aligned} \tag{2.20}$$

which implies that $\phi(t) = 0$, so $t = 0$, a contradiction. Thus, from (2.19),

$$\limsup_{n\to\infty} \alpha_n = 0, \tag{2.21}$$

and hence $\lim_{n\to\infty} \alpha_n = 0$, so (2.11) is proved.

Step 3. We claim that T has a periodic point.

We argue by contradiction. Assume that T has no periodic point. Then, $\{x_n\}$ is a sequence of distinct points, that is, $x_n \ne x_m$ for all $m \ne n$. We will show that, in this case, $\{x_n\}$ is g.m.s Cauchy. Suppose to the contrary. Then, there is a $\varepsilon > 0$ such that for an integer k there exist integers $m(k) > n(k) > k$ such that

$$d(x_{n(k)}, x_{m(k)}) > \varepsilon. \tag{2.22}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.22) and such that

$$d\left(x_{n(k)}, x_{m(k)-1}\right) \leq \varepsilon. \tag{2.23}$$

Now, using (2.22), (2.23), and the rectangular inequality (because $\{x_n\}$ is a sequence of distinct points), we find that

$$\begin{aligned}\varepsilon < d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{m(k)-2}\right) + d\left(x_{m(k)-2}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)}\right)\\ &\leq d\left(x_{m(k)}, x_{m(k)-2}\right) + d\left(x_{m(k)-2}, x_{m(k)-1}\right) + \varepsilon.\end{aligned} \tag{2.24}$$

Then, by (2.4) and (2.11), it follows that

$$\lim_{k \to +\infty} d\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon. \tag{2.25}$$

Now, by rectangular inequality, we have

$$\begin{aligned}d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right)\\ d\left(x_{m(k)-1}, x_{n(k)-1}\right) &\leq d\left(x_{m(k)-1}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)-1}\right).\end{aligned} \tag{2.26}$$

Letting $k \to \infty$ in the above inequalities, using (2.4) and (2.25), we obtain

$$\lim_{k \to \infty} d\left(x_{m(k)-1}, x_{n(k)-1}\right) = \varepsilon. \tag{2.27}$$

Therefore, by (2.4) and (2.27), we get that

$$\begin{aligned}M\left(x_{m(k)-1}, x_{n(k)-1}\right) = \max\left\{d\left(x_{m(k)-1}, x_{n(k)-1}\right), d\left(x_{m(k)-1}, x_{m(k)}\right), d\left(x_{n(k)-1}, x_{n(k)}\right)\right\} \longrightarrow \varepsilon\\ \text{as } k \longrightarrow \infty.\end{aligned} \tag{2.28}$$

Applying (2.2) with $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$, we have

$$\psi\left(d\left(x_{m(k)}, x_{n(k)}\right)\right) = \psi\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right) \leq \psi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) - \phi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right). \tag{2.29}$$

Letting $k \to \infty$ in the above inequality and using (2.25) and (2.28), we obtain

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) \tag{2.30}$$

which yields that $\phi(\varepsilon) = 0$, so $\varepsilon = 0$, which is a contradiction.

Hence, $\{x_n\}$ is g.m.s Cauchy. Since (X, d) is a complete g.m.s, there exists $u \in X$ such that $x_n \to u$. Applying (2.2) with $x = x_n$ and $y = u$, we obtain

$$\psi(d(x_{n+1}, Tu)) = \psi(d(Tx_n, Tu)) \leq \psi(M(x_n, u)) - \phi(M(x_n, u)) \leq \psi(M(x_n, u)) \tag{2.31}$$

which implies that

$$d(x_{n+1}, Tu) \leq M(x_n, u), \tag{2.32}$$

where

$$M(x_n, u) = \max\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\}. \tag{2.33}$$

Since $\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, so we obtain that

$$\lim_{n\to\infty} M(x_n, u) = d(u, Tu). \tag{2.34}$$

It follows that

$$\limsup_{n\to\infty} d(x_{n+1}, Tu) \leq d(u, Tu). \tag{2.35}$$

Next, we shall find a contradiction of the fact that T has no periodic point in each of the two following cases.

(i) If, for all $n \geq 2$, $x_n \neq u$ and $x_n \neq Tu$, then by rectangular inequality

$$d(u, Tu) \leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu), \tag{2.36}$$

and, using (2.4), we get that

$$d(u, Tu) \leq \limsup_{n\to\infty} d(x_{n+1}, Tu). \tag{2.37}$$

From (2.35) and (2.37),

$$\limsup_{n\to\infty} d(x_{n+1}, Tu) = d(u, Tu). \tag{2.38}$$

Taking the $\limsup_{n\to\infty}$ in (2.31) and using (2.34), (2.38), and the properties of ψ and ϕ, we obtain

$$\psi(d(u, Tu)) \leq \psi(d(u, Tu)) - \phi(d(u, Tu)) \tag{2.39}$$

which implies that $d(u, Tu) = 0$, so $u = Tu$, that is, u is a fixed point of T, so u is a periodic point of T. It contradicts the fact that T has no periodic point.

(ii) Let for some $q \geq 2$, $x_q = u$ or $x_q = Tu$. Since T has no periodic point, then obviously $u \neq x_0$. Indeed, if $x_q = u = x_0$, so $T^q x_0 = x_0$, that is, x_0 is a periodic point of T, while if $x_q = Tu$ and $x_0 = u$, so $Tx_0 = Tu = x_q = T^q x_0 = T^{q-1}(Tx_0)$, that is, Tx_0 is a periodic point of T.

For all $n \geq 0$, we have

$$d(T^n u, u) = d(T^n x_q, u) = d(x_{n+q}, u) \quad \text{or}$$
$$d(T^n u, u) = d\left(T^{n-1} Tu, u\right) = d\left(T^{n-1} x_q, u\right) = d(x_{n+q-1}, u). \tag{2.40}$$

In the two precedent identities, the integer $q \geq 2$ is fixed, and so $\{x_{n+q}\}$ and $\{x_{n+q-1}\}$ are subsequences from $\{x_n\}$, and since $\{x_n\}$ g.m.s. converges to u in (X, d) which is assumed to be Hausdorff, so the two subsequences g.m.s. converge to same unique limit u, that is,

$$\lim_{n \to \infty} d(x_{n+q}, u) = \lim_{n \to \infty} d(x_{n+q-1}, u) = 0. \tag{2.41}$$

Thus,

$$\lim_{n \to \infty} d(T^n u, u) = 0. \tag{2.42}$$

Again, since (X, d) is Hausdorff, then by (2.42),

$$\lim_{n \to \infty} d\left(T^{n+2} u, u\right) = 0. \tag{2.43}$$

On the other hand, since T has no periodic point, it follows that

$$T^s u \neq T^r u \quad \text{for any } s, r \in \mathbb{N},\ s \neq r. \tag{2.44}$$

Using (2.44) and the rectangular inequality, we may write

$$\left| d\left(T^{n+1} u, Tu\right) - d(u, Tu) \right| \leq d\left(T^{n+1} u, T^{n+2} u\right) + d\left(T^{n+2} u, u\right). \tag{2.45}$$

Letting $n \to \infty$ in the above limit and proceeding as (2.4) (since the point x_0 is arbitrary), using (2.43), we obtain

$$\lim_{n \to \infty} d\left(T^{n+1} u, Tu\right) = d(u, Tu). \tag{2.46}$$

Now, by (2.2),

$$\psi\left(d\left(T^{n+1} u, Tu\right)\right) \leq \psi(M(T^n u, u)) - \phi(M(T^n u, u)), \tag{2.47}$$

where

$$M(T^n u, u) = \max\left\{d(T^n u, u), d\left(T^n u, T^{n+1} u\right), d(u, Tu)\right\} \longrightarrow d(u, Tu) \quad \text{as } n \longrightarrow \infty. \tag{2.48}$$

Letting $n \to \infty$ in (2.47) and using (2.46) and the above limit, we get that

$$\psi(d(u, Tu)) \leq \psi(d(u, Tu)) - \phi(d(u, Tu)) \tag{2.49}$$

which holds only if $d(u, Tu) = 0$, that is, $Tu = u$, which implies that u is a periodic point of T. This contradicts the fact that T has no periodic point.

Consequently, T admits a periodic point, that is, there exists $u \in X$ such that $u = T^p u$ for some $p \geq 1$.

Step 4. Existence of a fixed point of T.

If $p = 1$, then $u = Tu$, that is, u is a fixed point of T. Suppose now that $p > 1$. We will prove that $a = T^{p-1}u$ is a fixed point of T. Suppose that it is not the case, that is, $T^{p-1}u \neq T^p u$. Then, $d(T^{p-1}u, T^p u) > 0$ and $\phi(d(T^{p-1}u, T^p u)) > 0$, which implies that $\phi(M(T^{p-1}u, T^p u)) > 0$. Now, using inequality (2.2), we obtain

$$\begin{aligned}
\psi(d(u, Tu)) &= \psi\left(d\left(T^p u, T^{p+1} u\right)\right) \\
&= \psi\left(d\left(T\left(T^{p-1}u\right), T(T^p u)\right)\right) \\
&\leq \psi\left(M\left(T^{p-1}u, T^p u\right)\right) - \phi\left(M\left(T^{p-1}u, T^p u\right)\right) \\
&< \psi\left(M\left(T^{p-1}u, T^p u\right)\right)
\end{aligned} \tag{2.50}$$

which by the monotone nondecreasing property of ψ implies

$$d(u, Tu) < M\left(T^{p-1}u, T^p u\right), \tag{2.51}$$

where

$$\begin{aligned}
M\left(T^{p-1}u, T^p u\right) &= \max\left\{d\left(T^{p-1}u, T^p u\right), d\left(T^{p-1}u, T^p u\right), d\left(T^p u, T^{p+1} u\right)\right\} \\
&= \max\left\{d\left(T^{p-1}u, T^p u\right), d(u, Tu)\right\} = d\left(T^{p-1}u, T^p u\right)
\end{aligned} \tag{2.52}$$

because otherwise we get a contradiction with (2.51). Thus, (2.51) becomes

$$d(u, Tu) < d\left(T^{p-1}u, T^p u\right). \tag{2.53}$$

Again, using (2.2), we have

$$\begin{aligned}\psi\left(d\left(T^{p-1}u,T^{p}u\right)\right)&=\psi\left(d\left(T\left(T^{p-2}u\right),T\left(T^{p-1}u\right)\right)\right)\\&\leq\psi\left(M\left(T^{p-2}u,T^{p-1}u\right)\right)-\phi\left(M\left(T^{p-2}u,T^{p-1}u\right)\right)\\&<\psi\left(M\left(T^{p-2}u,T^{p-1}u\right)\right).\end{aligned}\tag{2.54}$$

Again, this implies that

$$d\left(T^{p-1}u,T^{p}u\right)<M\left(T^{p-2}u,T^{p-1}u\right),\tag{2.55}$$

Where

$$\begin{aligned}M\left(T^{p-2}u,T^{p-1}u\right)&=\max\left\{d\left(T^{p-2}u,T^{p-1}u\right),d\left(T^{p-2}u,T^{p-1}u\right),d\left(T^{p-1}u,T^{p}u\right)\right\}\\&=\max\left\{d\left(T^{p-2}u,T^{p-1}u\right),d\left(T^{p-1}u,T^{p}u\right)\right\}=d\left(T^{p-2}u,T^{p-1}u\right)\end{aligned}\tag{2.56}$$

because of (2.55). Thus, from (2.55),

$$d\left(T^{p-1}u,T^{p}u\right)<d\left(T^{p-2}u,T^{p-1}u\right).\tag{2.57}$$

Continuing this process as (2.53) and (2.57), we find that

$$d(u,Tu)<d\left(T^{p-1}u,T^{p}u\right)<d\left(T^{p-2}u,T^{p-1}u\right)<\cdots<d(u,Tu)\tag{2.58}$$

which is a contradiction. We deduce that $a=T^{p-1}u$ is a fixed point of T.

Step 5. Uniqueness of the fixed point of T.

Suppose that there are two distinct points $b,c\in X$ such that $Tb=b$ and $Tc=c$. Then, $M(b,c)=\max\{d(b,c),d(b,Tb),d(c,Tc)\}=d(b,c)$ and $\phi(d(b,c))>0$. By (2.2), we obtain

$$\begin{aligned}\psi(d(b,c))=\psi(d(Tb,Tc))&\leq\psi(M(b,c))-\phi(M(b,c))\\&=\psi(d(b,c))-\phi(d(b,c))<\psi(d(b,c))\end{aligned}\tag{2.59}$$

a contradiction. Thus, T has a unique fixed point. This completes the proof of Theorem 2.2. □

Now, we state some corollaries of Theorem 2.2, which are given in the following.

Corollary 2.3. *Let (X,d) be a Hausdorff and complete generalized metric space. Suppose that $T: X\to X$ is such that, for all $x,y\in X$, there exists $k\in[0,1)$ and*

$$d(Tx,Ty)\leq k\max\{d(x,y),d(x,Tx),d(y,Ty)\},\tag{2.60}$$

then T has a unique fixed point.

Proof. It suffices to take $\psi(t) = t$ and $\phi(t) = (1-k)t$ in Theorem 2.2. □

Corollary 2.4. *Let (X, d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ is such that, for all $x, y \in X$, there exists $\alpha \in [0, 1/2)$ and*

$$(d(Tx, Ty)) \le \alpha[d(x, Tx) + d(y, Ty)], \tag{2.61}$$

then T has a unique fixed point.

Proof. Let $k = 2\alpha$, so $k \in [0, 1)$. Also, if (2.61) holds, so

$$\begin{aligned}(d(Tx, Ty)) &\le \alpha[d(x, Tx) + d(y, Ty)] = k\frac{d(x, Tx) + d(y, Ty)}{2} \\ &\le k \max\{d(x, y), d(x, Tx), d(y, Ty)\}.\end{aligned} \tag{2.62}$$

Then, it suffices to apply Corollary 2.3. □

Another easy consequence of Corollary 2.3 (a Reich contraction type) is the following.

Corollary 2.5. *Let (X, d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ is such that, for all $x, y \in X$, there exists $k \in [0, 1/3)$ and*

$$(d(Tx, Ty)) \le k[d(x, y) + d(x, Tx) + d(y, Ty)], \tag{2.63}$$

then T has a unique fixed point.

Corollary 2.6. *Let T satisfy the conditions of Theorem 2.2, except that condition (2.2) is replaced by the following: there exist positive Lebesgue integrable functions u and v on $\mathbb{R}_+$ such that $\int_0^\varepsilon u(t)dt > 0$ and $\int_0^\varepsilon v(t)dt > 0$ for each $\varepsilon > 0$ and that*

$$\int_0^{\psi(d(Tx,Ty))} u(t)dt \le \int_0^{\psi(M(x,y))} u(t)dt - \int_0^{\phi(M(x,y))} v(t)dt. \tag{2.64}$$

Then, T has a unique fixed point.

Proof. Consider the functions

$$\varphi_0(x) = \int_0^x u(t)dt, \qquad \varphi_1(x) = \int_0^x v(t)dt. \tag{2.65}$$

Then, (2.64) becomes

$$(\varphi_0 \circ \psi)(d(Tx, Ty)) \le (\varphi_0 \circ \psi)(M(x, y)) - (\varphi_1 \circ \phi)(M(x, y)), \tag{2.66}$$

And, putting $\psi_0 = \varphi_0 \circ \psi$ and $\phi_0 = \varphi_1 \circ \phi$ and applying Theorem 2.2, we obtain the proof of Corollary 2.6 (it is easy to verify that $\psi_0 \in \Psi$ and $\phi_0 \in \Phi$). □

Corollary 2.7. *Let (X, d) be a Hausdorff and complete generalized metric space. Let $T : X \to X$. Assume there exist positive Lebesgue integrable functions u and v on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} u(t)dt > 0$ and $\int_0^{\varepsilon} v(t)dt > 0$ for each $\varepsilon > 0$ and for all $x, y \in X$, and*

$$\int_0^{d(Tx,Ty)} u(t)dt \leq \int_0^{M(x,y)} u(t)dt - \int_0^{M(x,y)} v(t)dt, \tag{2.67}$$

then T has a unique fixed point.

Proof. It follows by taking $\psi(t) = \phi(t) = t$ in Corollary 2.6. □

Corollary 2.8. *Let (X, d) be a Hausdorff and complete generalized metric space. Let $T : X \to X$. Assume there exist $k \in [0, 1)$ and a positive Lebesgue integrable function u on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} u(t)dt > 0$ for each $\varepsilon > 0$ and for all $x, y \in X$, and*

$$\int_0^{d(Tx,Ty)} u(t)dt \leq k \int_0^{\max\{d(x,y), d(x,Tx), d(y,Ty)\}} u(t)dt, \tag{2.68}$$

then T has a unique fixed point.

Proof. It suffices to take $v(t) = (1 - k)u(t)$ in Corollary 2.7. □

Finally, let us finish this paper by noticing the following remark.

Remark 2.9. (i) Theorem 2.2 extends Theorem 3.1 of Lakzian and Samet [9].

(ii) Corollary 2.3 extends the results of Branciari [1], Azam and Arshad [2], and Sarma et al. [13].

(iii) Corollary 2.8 extends Theorem 2 of Samet [11].

(iv) Several publications attempting to generalize fixed-point theorems in metric spaces to g.m.s are plagued by the use of some false properties given in [1] (see, e.g., [2–5]). This was observed by Das and Dey [7] who proved a fixed-point theorem without using the false properties. Subsequently, but independently, this was also observed by Samet [12] and Sarma et al. [13] who proved fixed-point theorems assuming that the generalized metric space is Hausdorff. Here, we give a rigorous proof of Theorem 2.2 by taking the same assumption.

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