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Research Article

Approximate Riesz Algebra-Valued Derivations

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Let F be a Riesz algebra with an extended norm $\|\cdot\|_u$ such that $(F, \|\cdot\|_u)$ is complete. Also, let $\|\cdot\|_v$ be another extended norm in F weaker than $\|\cdot\|_u$ such that whenever (a) $x_n \to x$ and $x_n \cdot y \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$; (b) $y_n \to y$ and $x \cdot y_n \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$. Let ε and $\delta >$ be two nonnegative real numbers. Assume that a map $f : F \to F$ satisfies $\|f(x+y) - f(x) - f(y)\|_u \leq \varepsilon$ and $\|f(x \cdot y) - x \cdot f(y) - f(x) \cdot y\|_v \leq \delta$ for all $x, y \in F$. In this paper, we prove that there exists a unique derivation $d : F \to F$ such that $\|f(x) - d(x)\|_u \leq \varepsilon$, $(x \in F)$. Moreover, $x \cdot (f(y) - d(y)) = 0$ for all $x, y \in F$.

1. Introduction

Let E and E' be Banach spaces and let $\delta > 0$. A function $f : E \to E'$ is called δ-additive if $\|f(x+y) - f(x) - f(y)\| < \delta$ for all $x, y \in E$. The well-known problem of stability of functional equation $f(x+y) = f(x) + f(y)$ started with the following question of Ulam [1]. Does there exist for each $\varepsilon > 0$, a $\delta > 0$ such that, to each δ-additive function f of E into E' there corresponds an additive function l of E into E' satisfying the inequality $\|f(x) - l(x)\| \leq \varepsilon$ for each $x \in E$? In 1941, Hyers [2] answered this question in the affirmative way and showed that δ may be taken equal to ε. The answer of Hyers is presented in a great number of articles and books. For the theory of the stability of functional equations see Hyers et al [3].

Let F be an algebra. A mapping $d : F \to F$ is called a derivation if and only if it satisfies the following functional equations:

$$d(a+b) = d(a) + d(b), \tag{1.1}$$

$$d(ab) = ad(b) + d(a)b, \tag{1.2}$$

for all $a, b \in F$.

The stability of derivations was first studied by Jun and Park [4]. Further, approximate derivations were investigated by a number of mathematicians (see, e.g., [5–7]).

The aim of the present paper is to examine the stability problem of derivations for Riesz algebras with extended norms.

2. Preliminaries

A vector space F with a partial order $\leq$ satisfying the following two conditions:

(1) $x \leq y \Rightarrow \alpha x + z \leq \alpha y + z$ for all $z \in F$ and $0 \leq \alpha \in \mathbb{R}$,

(2) for all $x, y \in F$, the supremum $x \vee y$ and infimum $x \wedge y$ exist in F (hence, the modulus $|x| := x \vee (-x)$ exists for each $x \in F$),

is called a Riesz space or vector lattice. Typical examples of Riesz spaces are provided by the function spaces. $C(K)$ the spaces of real valued continuous functions on a topological space K, l_p real valued absolutely summable sequences, c the spaces of real valued convergent sequences, and c_0 the spaces of real valued sequences converging to zero are natural examples of Riesz spaces under the pointwise ordering. A Riesz space F is called Archimedean if $0 \leq u, v \in F$ and $nu \leq v$ for each $n \in \mathbb{N}$ imply $u = 0$. A subset S in a Riesz space F is said to be solid if it follows from $|u| \leq |v|$ in F and $v \in S$ that $u \in S$. A solid linear subspace of a Riesz space F is called an ideal. Every subset D of a Riesz space F is included in a smallest ideal F_D, called ideal generated *by* D. A principal ideal of a Riesz space F is any ideal generated by a singleton $\{u\}$. This ideal will be denoted by I_u. It is easy to see that

$$I_u = \{v \in F : \lambda \geq 0 \text{ such that } |v| \leq \lambda |u|\}. \tag{2.1}$$

Let F be a Riesz space and $0 \leq u \in F$. Firstly, we give the following definition.

Definition 2.1. (1) The sequence (x_n) in F is said to be u-uniformly convergent to the element $x \in F$ whenever, for every $\varepsilon > 0$, there exists n_0 such that $|x_{n_0+k} - x| \leq \varepsilon u$ holds for each k.

(2) The sequence (x_n) in F is said to be relatively uniformly convergent to x whenever x_n converges u-uniformly to $x \in F$ for some $0 \leq u \in F$.

When dealing with relative uniform convergence in an Archimedean Riesz space F, it is natural to associate with every positive element $u \in F$ an extended norm $\|\cdot\|_u$ in F by the following formula:

$$\|x\|_u = \inf\{\lambda \geq 0 : |x| \leq \lambda u\} \quad (x \in F). \tag{2.2}$$

Note that $\|x\|_u < \infty$ if and only if $x \in I_u$. Also $|x| \leq \delta u$ if and only if $\|x\|_u \leq \delta$.

A Banach lattice is a vector lattice F that is simultaneously a Banach space whose norm is monotone in the following sense.

For all $x, y \in F$, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Hence, $\|x\| = \||x|\|$ for all $x \in F$.

The sequence (x_n) in $(F, \|\cdot\|_u)$ is called an extended u-normed Cauchy sequence, if for every $\varepsilon > 0$ there exists k such that $\|x_{n+k} - x_{m+k}\|_u < \varepsilon$ for all m, n. If every extended u-normed Cauchy sequence is convergent in F, then F is called an extended u-normed Banach lattice.

A Riesz space F is called a Riesz algebra or a lattice ordered algebra if there exists an associative multiplication in F with the usual algebra properties such that $0 \leq u \cdot v$ for all $0 \leq u, v \in F$.

For more detailed information about Riesz spaces, the reader can consult the book *Riesz Spaces* by Luxemburg and Zaanen [8]. In the sequel, all the Riesz spaces are assumed to be Archimedean.

3. Main Result

Recently, Polat [9] generalized the Hyers' result [2] to Riesz spaces with extended norms and proved the following.

Theorem 3.1. *Let E be a linear space and F a Riesz space equipped with an extended norm $\|\cdot\|_u$ such that the space $(F, \|\cdot\|_u)$ is complete. If, for some $\delta > 0$, a map $f : E \to (F, \|\cdot\|_u)$ is δ-additive, then limit $l(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for each $x \in E$. $l(x)$ is the unique additive function satisfying the inequality $\|f(x) - l(x)\|_u \leq \delta$ for all $x \in E$.*

By using Theorem 3.1, we give the main result of the paper as follows.

Theorem 3.2. *Let F be a Riesz algebra with an extended norm $\|\cdot\|_u$ such that $(F, \|\cdot\|_u)$ is complete. Also, let $\|\cdot\|_v$ be another extended norm in F weaker than $\|\cdot\|_u$ such that whenever*

(a) *$x_n \to x$ and $x_n \cdot y \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$;*

(b) *$y_n \to y$ and $x \cdot y_n \to z$ in $\|\cdot\|_v$, then $z = x \cdot y$.*

Let ε and δ be two nonnegative real numbers. Assume that a map $f : F \to F$ satisfies

$$\left\|f(x+y) - f(x) - f(y)\right\|_u \leq \varepsilon, \tag{3.1}$$

$$\left\|f(x\cdot y) - x\cdot f(y) - f(x)\cdot y\right\|_v \leq \delta, \tag{3.2}$$

for all $x, y \in F$. Then, there exists a unique derivation $d : F \to F$ such that $\|f(x) - d(x)\|_u \leq \varepsilon$, $(x \in F)$. Moreover, $x \cdot (f(y) - d(y)) = 0$ for all $x, y \in F$.

Proof. By Condition (3.1), Theorem 3.1 shows that there exists a unique additive function $d : F \to F$ such that

$$\left\|f(x) - d(x)\right\|_u \leq \varepsilon, \tag{3.3}$$

for each $x \in F$. It is enough to show that d satisfies Condition (1.2). The inequality (3.3) implies that

$$\left\|f(nx) - d(nx)\right\|_u \leq \varepsilon \quad (x \in F, n \in \mathbb{N}). \tag{3.4}$$

By the additivity of d, we then have

$$\left\|\frac{1}{n} f(nx) - d(x)\right\|_u \leq \frac{1}{n}\varepsilon \quad (x \in F, n \in \mathbb{N}), \tag{3.5}$$

which means that

$$d(x) = \lim_{n\to\infty} \frac{1}{n} f(nx), \quad (x \in F), \tag{3.6}$$

with respect to $\|\cdot\|_u$ norm and so is with respect to $\|\cdot\|_v$ norm. Condition (3.2) implies that the function $r : F \times F \to F$ defined by $r(x,y) = f(x\cdot y) - x\cdot f(y) - f(x)\cdot y$ is bounded. Hence

$$\lim_{n\to\infty} \frac{1}{n} r(nx,y) = 0, \quad (x,y \in F), \tag{3.7}$$

with respect to $\|\cdot\|_v$ norm. Applying (3.6) and (3.7), we have

$$d(x\cdot y) = x\cdot f(y) + d(x)\cdot y, \quad (x,y \in F). \tag{3.8}$$

Indeed, we have the following with respect to $\|\cdot\|_v$ norm,

$$\begin{aligned} d(x\cdot y) &= \lim_{n\to\infty} \frac{1}{n} f(n(x\cdot y)) = \lim_{n\to\infty} \frac{1}{n} f((nx)\cdot y) \\ &= \lim_{n\to\infty} \frac{1}{n} (nx\cdot f(y) + f(nx)\cdot y + r(nx,y)) \\ &= \lim_{n\to\infty} \left(x\cdot f(y) + \frac{f(nx)}{n}\cdot y + \frac{r(nx,y)}{n} \right) \\ &= x\cdot f(y) + d(x)\cdot y, \quad (x,y \in F). \end{aligned} \tag{3.9}$$

Let $x,y \in F$ and $n \in \mathbb{N}$ be fixed. Then using (3.8) and additivity of d, we have

$$\begin{aligned} x\cdot f(ny) + nd(x)\cdot y = x\cdot f(ny) + d(x)\cdot ny &= d(x\cdot ny) \\ = d(nx\cdot y) = nx\cdot f(y) &+ d(nx)\cdot y \\ = nx\cdot f(y) &+ nd(x)\cdot y. \end{aligned} \tag{3.10}$$

Therefore,

$$x\cdot f(y) = x\cdot \frac{f(ny)}{n}, \quad (x,y \in F,\ n \in \mathbb{N}). \tag{3.11}$$

Sending n to infinity, by (3.6), we see that

$$x\cdot f(y) = x\cdot d(y), \quad (x,y \in F). \tag{3.12}$$

Combining this formula with (3.8), we have that d satisfies (1.2) which is the desired result. Moreover, the last formula yields $x\cdot (f(y) - d(y)) = 0$ for all $x,y \in F$. □

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