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Research Article

Well-Posedness of the First Order of Accuracy Difference Scheme for Elliptic-Parabolic Equations in Hölder Spaces

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A first order of accuracy difference scheme for the approximate solution of abstract nonlocal boundary value problem $-d^2u(t)/dt^2 + \text{sign}(t)Au(t) = g(t)$, $(0 \le t \le 1)$, $du(t)/dt + \text{sign}(t)Au(t) = f(t)$, $(-1 \le t \le 0)$, $u(0+) = u(0-)$, $u'(0+) = u'(0-)$, and $u(1) = u(-1) + \mu$ for differential equations in a Hilbert space H with a self-adjoint positive definite operator A is considered. The well-posedness of this difference scheme in Hölder spaces without a weight is established. Moreover, as applications, coercivity estimates in Hölder norms for the solutions of nonlocal boundary value problems for elliptic-parabolic equations are obtained.

1. Introduction

Nonlocal boundary value problems for partial differential equations have been applied by various researchers in order to model numerous processes in different fields of applied sciences when they are unable to determine the boundary values of the unknown function (see, e.g., [1–15] and the references therein).

Well-posedness of difference schemes of elliptic-parabolic equations with nonlocal boundary conditions in Hölder spaces with a weight was studied in [16–19].

In paper [20], the well-posedness of abstract nonlocal boundary value problem

$$-\frac{d^2u(t)}{dt^2} + \text{sign}(t)Au(t) = g(t), \quad (0 \le t \le 1),$$
$$\frac{du(t)}{dt} + \text{sign}(t)Au(t) = f(t), \quad (-1 \le t \le 0),$$

$$u(0+) = u(0-), \qquad u'(0+) = u'(0-),$$
$$u(1) = u(-1) + \mu \tag{1.1}$$

in Hölder spaces without a weight was established. The coercivity inequalities for solutions of the boundary value problem for elliptic-parabolic equations were obtained.

In the present paper, the first order of accuracy difference scheme

$$\begin{aligned}
&-\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k,\\
&g_k = g(t_k), \quad t_k = k\tau, \quad 1 \le k \le N-1,\\
&\tau^{-1}(u_k - u_{k-1}) - Au_{k-1} = f_k, \quad f_k = f(t_{k-1}),\\
&t_{k-1} = (k-1)\tau, \quad -N+1 \le k \le -1,\\
&u_N = u_{-N} + \mu, \qquad u_1 - u_0 = u_0 - u_{-1}
\end{aligned} \tag{1.2}$$

for the approximate solution of problem (1.1) is considered. The well-posedness of difference scheme (1.2) in Hölder spaces without a weight is established. As an application, coercivity inequalities for solutions of difference scheme for elliptic-parabolic equations are obtained.

Throughout the paper, H denotes a Hilbert space and A is a self-adjoint positive definite operator with $A \ge \delta I$ for some $\delta > \delta_0 > 0$. Then, it is wellknown that $B = (1/2)(\tau A + \sqrt{A(4+\tau^2 A)})$ is a self-adjoint positive definite operator and $B \ge \delta^{1/2} I$. Furthermore, $R = (I+\tau B)^{-1}$ and $P = P(\tau A) = (I+\tau A)^{-1}$ which are defined on the whole space H, are bounded operators, where I is the identity operator.

2. Well-Posedness of (1.2)

First of all, let us start with some auxiliary lemmas that are used throughout the paper.

Lemma 2.1. *The following estimates are satisfied [19, 21, 22]:*

$$\begin{aligned}
&\left\|P^k\right\|_{H\to H} \le M(\delta)(1+\delta\tau)^{-k}, \qquad k\tau\left\|AP^k\right\|_{H\to H} \le M(\delta),\\
&\left\|R^k\right\|_{H\to H} \le M(\delta)(1+\delta\tau)^{-k}, \qquad k\tau\left\|BR^k\right\|_{H\to H} \le M(\delta),\\
&\left\|P^k - e^{-k\tau A}\right\|_{H\to H} \le \frac{M(\delta)}{k}, \qquad \left\|R^k - e^{-k\tau A^{1/2}}\right\|_{H\to H} \le \frac{M(\delta)}{k},\\
&\left\|\left(I - R^{2N}\right)^{-1}\right\|_{H\to H} \le M(\delta), \quad k \ge 1,\ \delta > 0,
\end{aligned} \tag{2.1}$$

for some $M(\delta) > 0$, which is independent of τ is a positive small number.

Let $F_\tau(H) = F([a,b]_\tau, H)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_{N_a}^{N_b}$ defined on $[a,b]_\tau = \{t_k = kh, N_a \le k \le N_b, N_a\tau = a, N_b\tau = b\}$ with values in the Hilbert space H. Next, $C([a,b]_\tau, H)$, $C^\alpha([-1,1]_\tau, H)$, $C^{\alpha/2}([-1,0]_\tau, H)$, and $C^\alpha([0,1]_\tau, H)$ $(0 < \alpha < 1)$ denote Banach spaces on $F_\tau(H)$ with norms:

$$\begin{aligned}
\|\varphi^\tau\|_{C([a,b]_\tau,H)} &= \max_{N_a \le k \le N_b} \|\varphi_k\|_H, \\
\|\varphi^\tau\|_{C^\alpha([-1,1]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,1]_\tau,H)} + \sup_{-N \le k < k+r \le 0} \|\varphi_{k+r} - \varphi_k\|_H r^{-\alpha/2} \\
&\quad + \sup_{1 \le k < k+r \le N-1} \|\varphi_{k+r} - \varphi_k\|_H r^{-\alpha}, \\
\|\varphi^\tau\|_{C^{\alpha/2}([-1,0]_\tau,H)} &= \|\varphi^\tau\|_{C([-1,0]_\tau,H)} + \sup_{-N \le k < k+r \le 0} \|\varphi_{k+r} - \varphi_k\|_H r^{-\alpha/2}, \\
\|\varphi^\tau\|_{C^\alpha([0,1]_\tau,H)} &= \|\varphi^\tau\|_{C([0,1]_\tau,H)} + \sup_{1 \le k < k+r \le N-1} \|\varphi_{k+r} - \varphi_k\|_H r^{-\alpha}.
\end{aligned} \tag{2.2}$$

With the help of the self-adjoint positive definite operator B in a Hilbert space H, the Banach space $E_\alpha = E_\alpha(B,H)$ $(0 < \alpha < 1)$ consists of those $v \in H$ for which the norm (see [22, 23]):

$$\|v\|_{E_\alpha} = \sup_{z>0} z^\alpha \left\| B(z+B)^{-1} v \right\|_H + \|v\|_H, \tag{2.3}$$

is finite. By the definition of $E_\alpha(B,H)$,

$$D(B) \subset E_\alpha(B,H) \subset E_\beta(B,H) \subset H, \tag{2.4}$$

for all $\beta < \alpha$.

Lemma 2.2. *For $0 < \alpha < 1$, the norms of the spaces $E_\alpha(B,H)$ and $E_{\alpha/2}(A,H)$ are equivalent (see [24]).*

Theorem 2.3. *Suppose $\mu \in D(A)$, $A\mu \in E_\alpha(B,H)$, $f_0 + g_0 \in E_{\alpha/2}(A,H)$, $f_{-N} + g_N \in E_\alpha(B,H)$, $g(t) \in C^\alpha([0,1]_\tau, H)$, and $f(t) \in C^{\alpha/2}([-1,0]_\tau, H)$, $0 < \alpha < 1$. Boundary value problem (1.2) is wellposed in Hölder space $C^\alpha([-1,1]_\tau, H)$ and the following coercivity inequality holds:*

$$\begin{aligned}
&\left\| \left\{ \tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) \right\}_1^{N-1} \right\|_{C^\alpha([0,1]_\tau,H)} + \left\| \{Au_k\}_{-N}^{N-1} \right\|_{C^\alpha([-1,1]_\tau,H)} \\
&\quad + \left\| \left\{ \tau^{-1}(u_k - u_{k-1}) \right\}_{-N+1}^{0} \right\|_{C^{\alpha/2}([-1,0]_\tau,H)} \\
&\le M \Bigg[\|A\mu\|_{E_\alpha(B,H)} + \frac{1}{\alpha(1-\alpha)} \left[\|f^\tau\|_{C^{\alpha/2}([-1,0]_\tau,H)} + \|g^\tau\|_{C^\alpha([0,1]_\tau,H)} \right] \\
&\quad + \left\| (I+\tau B)(f_0 + g_0) \right\|_{E_{\alpha/2}(A,H)} + \left\| (I+\tau B)(f_{-N} + g_N) \right\|_{E_\alpha(B,H)} \Bigg],
\end{aligned} \tag{2.5}$$

where M is independent of not only f^τ, g^τ, and μ but also of τ and α.

Proof. First of all, let us get the formulae for solution of problem (1.2). By [21, 25],

$$\begin{aligned} u_k &= \left(I - R^{2N}\right)^{-1}\Bigg\{\left[R^k - R^{2N-k}\right]\xi + \left[R^{N-k} - R^{N+k}\right]\varphi \\ &\quad - \left[R^{N-k} - R^{N+k}\right](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s} - R^{N+s}\right]g_s\tau\Bigg\} \\ &\quad + (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{|k-s|} - R^{k+s}\right]g_s\tau, \quad 1 \le k \le N \end{aligned} \tag{2.6}$$

is the solution of boundary value difference problem:

$$\begin{gathered} -\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = g_k \\ g_k = g(t_k), \quad t_k = k\tau, \quad 1 \le k \le N-1, \qquad u_0 = \xi, \qquad u_N = \varphi, \end{gathered} \tag{2.7}$$

$$u_k = P^{-k}\xi - \tau\sum_{s=k+1}^{0} P^{s-k} f_s, \quad -N \le k \le -1 \tag{2.8}$$

is the solution of inverse Cauchy problem:

$$\begin{gathered} \tau^{-1}(u_k - u_{k-1}) - Au_{k-1} = f_k, \quad f_k = f(t_{k-1}), \\ t_{k-1} = (k-1)\tau, \quad -(N-1) \le k \le 0, \qquad u_0 = \xi. \end{gathered} \tag{2.9}$$

Combining the conditions $\varphi = u_{-N} + \mu$, $\xi = u_0$ and formulas (2.6), (2.8), we get formulas

$$\begin{aligned} u_k &= \left(I - R^{2N}\right)^{-1}\Bigg\{\left[R^k - R^{2N-k}\right]u_0 + \left[R^{N-k} - R^{N+k}\right]\left[P^N u_0 - \tau\sum_{s=-N+1}^{0} P^{s+N} f_s + \mu\right] \\ &\quad - \left[R^{N-k} - R^{N+k}\right](I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{N-s} - R^{N+s}\right]g_s\tau\Bigg\} \\ &\quad + (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{s=1}^{N-1}\left[R^{|k-s|} - R^{k+s}\right]g_s\tau, \quad 1 \le k \le N, \end{aligned} \tag{2.10}$$

$$u_k = P^{-k}u_0 - \tau\sum_{s=k+1}^{0} P^{s-k} f_s, \quad -N \le k \le -1. \tag{2.11}$$

Operator equation

$$
\begin{aligned}
2u_0 - Pu_0 + \tau P f_0 = \left(I - R^{2N}\right)^{-1} \Bigg\{ & \left[R - R^{2N-1}\right] u_0 + \left[R^{N-1} - R^{N+1}\right] \\
& \times \left[P^N u_0 - \tau \sum_{s=-N+1}^{0} P^{s+N} f_s + \mu\right] \\
& - \left[R^{N-1} - R^{N+1}\right] (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s}\right] g_s \tau \Bigg\} \\
& + (I + \tau B)(2I + \tau B)^{-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{s-1} - R^{1+s}\right] g_s \tau
\end{aligned}
\tag{2.12}
$$

follows from formulas (2.10), (2.11), and the condition $u_1 - u_0 = u_0 - u_{-1}$. As the operator

$$
\begin{aligned}
& I + (I + \tau A)(I + 2\tau A)^{-1} R^{2N-1} + B^{-1} A (I + 2\tau A)^{-1} \left(I - R^{2N-1}\right) \\
& - (2I + \tau B)(I + 2\tau A)^{-1} R^N P^{N-1}
\end{aligned}
\tag{2.13}
$$

has an inverse

$$
\begin{aligned}
T_\tau = \Big(& I + (I + \tau A)(I + 2\tau A)^{-1} R^{2N-1} + B^{-1} A (I + 2\tau A)^{-1} \left(I - R^{2N-1}\right) \\
& - (2I + \tau B)(I + 2\tau A)^{-1} R^N P^{N-1} \Big)^{-1},
\end{aligned}
\tag{2.14}
$$

it follows that

$$
\begin{aligned}
u_0 = T_\tau (I + 2\tau A)^{-1} (I + \tau A) \Bigg\{ \Bigg\{ & (2 + \tau B) R^N \left[-\tau \sum_{s=-N+1}^{0} P^{s+N} f_s + \mu\right] \\
& - R^{N-1} B^{-1} \sum_{s=1}^{N-1} \left[R^{N-s} - R^{N+s}\right] g_s \tau \Bigg\} \\
& + \left(I - R^{2N}\right) B^{-1} \sum_{s=1}^{N-1} R^{s-1} g_s \tau - \left(I - R^{2N}\right)(I + \tau B) B^{-1} P f_0 \Bigg\}
\end{aligned}
\tag{2.15}
$$

for the solution of operator equation (2.12). Hence, we have formulas (2.10), (2.11), and (2.15) for the solution of difference problem (1.2).

Using formulae (2.10) and (2.15), we can get

$$\begin{aligned}
Au_0 = {} & T_\tau (I+2\tau A)^{-1}(I+\tau A) \\
& \times \Bigg\{ \Bigg\{ (2+\tau B)R^N \Bigg[-\tau \sum_{s=-N+1}^{0} AP^{s+N}(f_s - f_{-N+1}) + A\mu \Bigg] \\
& \qquad - R^{N-1}AB^{-2} \Bigg\{ \sum_{s=1}^{N-1} BR^{N-s}(g_s - g_{N-1})\tau + \sum_{s=1}^{N-1} BR^{N+s}(g_1 - g_s)\tau \Bigg\} \Bigg\} \\
& \quad + \left(I - R^{2N}\right) AB^{-2} \sum_{s=1}^{N-1} BR^{s-1}(g_s - g_1)\tau \Bigg\} \\
& + T_\tau (I+2\tau A)^{-1}(I+\tau A) \\
& \times \Big\{ \Big\{ (2+\tau B)R^N \left(P^N - I\right) f_{-N+1} \\
& \qquad - R^{N-1}AB^{-2} \Big\{ \left(I - R^{N-1}\right) g_{N-1} - \left(R^{N-2} - R^{2N-1}\right) g_1 \Big\} \Big\} \\
& \quad + \left(I - R^{2N}\right) AB^{-2} \left(I - R^{N-1}\right) g_1 - \left(I - R^{2N}\right)(I+\tau B)B^{-1}APf_0 \Big\},
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
Au_N = {} & P^N \Bigg\{ T_\tau (I+2\tau A)^{-1}(I+\tau A) \\
& \times \Bigg\{ \Bigg\{ (2+\tau B)R^N \Bigg[-\tau \sum_{s=-N+1}^{0} AP^{s+N}(f_s - f_{-N+1}) + A\mu \Bigg] \\
& \qquad - R^{N-1}AB^{-2} \Bigg\{ \sum_{s=1}^{N-1} BR^{N-s}(g_s - g_{N-1})\tau + \sum_{s=1}^{N-1} BR^{N+s}(g_1 - g_s)\tau \Bigg\} \Bigg\} \\
& \quad + \left(I - R^{2N}\right) AB^{-2} \sum_{s=1}^{N-1} BR^{s-1}(g_s - g_1)\tau \Bigg\} \Bigg\} \\
& - \tau \sum_{s=-N+1}^{0} AP^{s+N}(f_s - f_{-N+1}) + A\mu + \left(P^N - I\right) f_{-N+1} \\
& + P^N \Big\{ T_\tau (I+2\tau A)^{-1}(I+\tau A) \\
& \qquad \times \Big\{ \Big\{ (2+\tau B)R^N \left(P^N - I\right) f_{-N+1} \\
& \qquad\qquad - R^{N-1}AB^{-2} \Big\{ \left(I - R^{N-1}\right) g_{N-1} - \left(R^{N-2} - R^{2N-1}\right) g_1 \Big\} \Big\} \\
& \qquad + \left(I - R^{2N}\right) AB^{-2} \left(I - R^{N-1}\right) g_1 - \left(I - R^{2N}\right)(I+\tau B)B^{-1}APf_0 \Big\} \Big\}.
\end{aligned} \tag{2.17}$$

Finally, we will get coercivity estimate (2.5). It is based on estimates

$$\left\|\left\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\right\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)}+\left\|\{Au_k\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)}$$
$$\le M\left[\frac{1}{\alpha(1-\alpha)}\|g^\tau\|_{C^\alpha([0,1]_\tau,H)}+\|Au_0-g_0\|_{E_\alpha(B,H)}+\|Au_N-g_N\|_{E_\alpha(B,H)}\right] \tag{2.18}$$

for the solution of boundary value difference problem (2.7),

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^{0}\right\|_{C^{\alpha/2}([-1,0]_\tau,H)}+\left\|\{Au_k\}_{-N}^{0}\right\|_{C^{\alpha/2}([-1,0]_\tau,H)}$$
$$\le M\left[\frac{1}{(\alpha/2)(1-\alpha/2)}\|f^\tau\|_{C^{\alpha/2}([-1,0]_\tau,H)}+\|Au_0+f_0\|_{E_\alpha(A,H)}\right], \tag{2.19}$$

for the solution of inverse Cauchy difference problem (2.9), and

$$\begin{aligned}
\|Au_0+f_0\|_{E_{\alpha/2}(A,H)} &\le \frac{M}{\alpha(1-\alpha)}\left[\|g\|_{C^\alpha([0,1],H)}+\|f\|_{C^{\alpha/2}([-1,0],H)}\right]\\
&\quad+M\left[\|A\mu\|_{E_\alpha(B,H)}+\|f_0+g_0\|_{E_{\alpha/2}(A,H)}\right],\\
\|Au_0-g_0\|_{E_\alpha(B,H)} &\le \frac{M}{\alpha(1-\alpha)}\left[\|f\|_{C^{\alpha/2}([-1,0],H)}+\|g\|_{C^\alpha([0,1],H)}\right]\\
&\quad+M\left[\|A\mu\|_{E_\alpha(B,H)}+\|f_0+g_0\|_{E_{\alpha/2}(A,H)}\right],\\
\|Au_N-g_N\|_{E_\alpha(B,H)} &\le \frac{M}{\alpha(1-\alpha)}\left[\|f\|_{C^{\alpha/2}([-1,0],H)}+\|g\|_{C^\alpha([0,1],H)}\right]\\
&\quad+M\left[\|A\mu\|_{E_\alpha(B,H)}+\|f_0+g_0\|_{E_{\alpha/2}(A,H)}+\|f_{-N}+g_N\|_{E_\alpha(B,H)}\right]
\end{aligned} \tag{2.20}$$

for the solution of problem (1.2). Estimates (2.18) and (2.19) were established in [21, 25], respectively.

Estimates (2.20) are derived from the formulas (2.16) and (2.17) for the solution of problem (1.2), estimates (2.1) and following estimates

$$\begin{gathered}
\left\| R^k(\tau B)\right\|_{H\to H} \le M, \quad 1\le k\le N,\\
\left\| \left(I - R^{2N}(\tau B)\right)^{-1}\right\|_{H\to H} \le M, \quad k\ge 1,\\
\left\| R^{k+r}(\tau B) - R^k(\tau B)\right\|_{H\to H} \le M\frac{(r)^\alpha}{(k+r)^\alpha}, \quad 1\le k<k+r\le N,\ 0\le\alpha\le 1,\\
\left\| (I-R(\tau B))^2(\tau B)^{-2}\right\|_{H\to H} \le M,\\
\left\| (I+R(\tau B))^{-1}\right\|_{H\to H} \le M,\\
\| T_\tau\|_{H\to H} \le M, \qquad \| BRPT_\tau\|_{H\to H} \le M,
\end{gathered} \tag{2.21}$$

which were established in [26]. This finalizes the proof of Theorem 2.3. □

3. An Application

In this section, an application of the abstract Theorem 2.3 is considered. First, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1\le k\le n)$ with boundary S, $\overline{\Omega} = \Omega\cup S$. In $[-1,1]\times\Omega$, the mixed boundary value problem for multidimensional mixed equation:

$$\begin{gathered}
-u_{tt} - \sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} = g(t,x), \quad 0<t<1,\ x\in\Omega,\\
u_t + \sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} = f(t,x), \quad -1<t<0,\ x\in\Omega,\\
f(0,x)+g(0,x)=0, \quad f(-1,x)+g(1,x)=0,\ x\in\overline{\Omega},\\
u(t,x)=0, \quad x\in S,\ -1\le t\le 1; \qquad u(1,x)=u(-1,x)+\mu(x),\ x\in\overline{\Omega},\\
u(0+,x)=u(0-,x), \qquad u_t(0+,x)=u_t(0-,x),\ x\in\overline{\Omega}
\end{gathered} \tag{3.1}$$

is considered. Here, $a_r(x)$ $(x\in\Omega)$, $\mu(x)$ $(\mu(x)=0,\ x\in S)$, $g(t,x)$ $(t\in(0,1),\ x\in\overline{\Omega})$, and $f(t,x)$ $(t\in(-1,0),\ x\in\overline{\Omega})$ are given smooth functions and $a_r(x)\ge a>0$.

The discretization of problem (3.1) is carried out in two steps. In the first step, the grid sets

$$\begin{gathered}\tilde{\Omega}_h = \{x = x_m = (h_1 m_1, \dots, h_n m_n),\ m = (m_1, \dots, m_n),\\ 0 \leq m_r \leq N_r,\ h_r N_r = 1,\ r = 1, \dots, n\},\\ \Omega_h = \tilde{\Omega}_h \cap \Omega, \qquad S_h = \tilde{\Omega}_h \cap S\end{gathered} \tag{3.2}$$

are defined. To the differential operator A generated by problem (3.1), the difference operator A_h^x is assigned by formula:

$$A_h^x u^h = -\sum_{r=1}^{n} \left(a_r(x) u_{\bar{x}_r}^h \right)_{x_r, m_r} \tag{3.3}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. With the help of A_h^x, we arrive at the nonlocal boundary-value problem

$$\begin{gathered}-\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = g^h(t,x), \quad 0 < t < 1,\ x \in \Omega_h,\\ \frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), \quad -1 < t < 0,\ x \in \Omega_h,\\ u^h(-1,x) = u^h(1,x) + \mu^h(x), \quad x \in \tilde{\Omega}_h,\\ u^h(0+,x) = u^h(0-,x), \qquad \frac{du^h(0+,x)}{dt} = \frac{du^h(0-,x)}{dt}, \quad x \in \tilde{\Omega}_h,\end{gathered} \tag{3.4}$$

for an infinite system of ordinary differential equations.

In the second step, problem (3.4) is replaced by difference scheme (1.2) (see [21]):

$$\begin{gathered}-\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = g_k^h(x),\\ g_k^h(x) = g^h(t_k, x), \quad t_k = k\tau, \quad 1 \leq k \leq N-1, \quad N\tau = 1, \quad x \in \Omega_h,\\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} - A_h^x u_{k-1}^h(x) = f_k^h(x),\\ f_k^h(x) = f^h(t_k, x), \quad t_{k-1} = (k-1)\tau, \quad -N+1 \leq k \leq -1,\ x \in \Omega_h,\\ u_{-N}^h(x) = u_N^h(x) + \mu^h(x), \quad x \in \tilde{\Omega}_h,\\ u_1^h(x) - u_0^h(x) = u_0^h(x) - u_{-1}^h(x), \quad x \in \tilde{\Omega}_h.\end{gathered} \tag{3.5}$$

To formulate the result, we introduce the Hilbert spaces $L_{2h} = L_2(\widetilde{\Omega}_h)$, $W_{2h}^1 = W_2^1(\widetilde{\Omega}_h)$, and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, \dots, h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms:

$$\begin{aligned}
\left\|\varphi^h\right\|_{L_{2h}} &= \left(\sum_{x\in\widetilde{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_n\right)^{1/2}, \\
\left\|\varphi^h\right\|_{W_{2h}^1} &= \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\left(\varphi^h\right)_{x_r}\right|^2 h_1 \cdots h_n\right)^{\frac{1}{2}}, \\
\left\|\varphi^h\right\|_{W_{2h}^2} &= \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\left(\varphi^h\right)_{x_r}\right|^2 h_1 \cdots h_n\right)^{1/2} \\
&\quad + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^{n} \left|\left(\varphi^h\right)_{x_r \bar{x}_r, m_r}\right|^2 h_1 \cdots h_n\right)^{1/2}.
\end{aligned} \tag{3.6}$$

Theorem 3.1. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then, the solutions of difference scheme* (3.5) *satisfy the following coercivity stability estimate:*

$$\begin{aligned}
&\left\| \left\{ \tau^{-2}\left(u_{k+1}^h - 2u_k^h + u_{k-1}^h\right) \right\}_1^{N-1} \right\|_{C^{\alpha}([0,1]_\tau, L_{2h})} \\
&\quad + \left\| \left\{ \tau^{-1}\left(u_k^h - u_{k-1}^h\right) \right\}_{-N+1}^{0} \right\|_{C^{\alpha/2}([-1,0]_\tau, L_{2h})} + \left\| \left\{ u_k^h \right\}_{-N}^{N-1} \right\|_{C^{\alpha}([-1,1]_\tau, W_{2h}^2)} \\
&\quad \le M\left[\left\|\mu^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)} \left[\left\| \left\{ f_k^h \right\}_{-N+1}^{-1} \right\|_{C^{\alpha/2}([-1,0]_\tau, L_{2h})} + \left\| \left\{ g_k^h \right\}_1^{N-1} \right\|_{C^{\alpha}([0,1]_\tau, L_{2h})} \right] \right],
\end{aligned} \tag{3.7}$$

where M is not dependent on τ, h, $\mu^h(x)$, $g_k^h(x)$, $1 \le k \le N-1$, and f_k^h, $-N+1 \le k \le 0$.

The proof of Theorem 3.1 is based on Theorem 2.3, the symmetry properties of the difference operator A_h^x defined by formula (3.3), and along with the following theorem on the coercivity inequality for the solution of elliptic difference equation in L_{2h}.

Theorem 3.2. *For the solution of elliptic difference problem:*

$$\begin{aligned}
A_h^x u^h(x) &= \omega^h(x), \quad x \in \Omega_h, \\
u^h(x) &= 0, \quad x \in S_h,
\end{aligned} \tag{3.8}$$

the following coercivity inequality holds [27]:

$$\sum_{r=1}^{n} \left\| \left(u^h \right)_{x_r \overline{x}_r, m_r} \right\|_{L_{2h}} \leq M \left\| \omega^h \right\|_{L_{2h}}. \tag{3.9}$$

Here, M depends neither on h nor $\omega^h(x)$.

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