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Research Article

Ulam Stability of a Quartic Functional Equation

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The oldest quartic functional equation was introduced by J. M. Rassias in (1999), and then was employed by other authors. The functional equation $f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$ is called a *quartic functional equation*, all of its solution is said to be a *quartic function*. In the current paper, the Hyers-Ulam stability and the superstability for quartic functional equations are established by using the fixed-point alternative theorem.

1. Introduction

We say a functional equation $\mathcal{F}$ is *stable* if any function f satisfying the equation $\mathcal{F}$ approximately is near to true solution of $\mathcal{F}$. Moreover, a functional equation $\mathcal{F}$ is *superstable* if any function f satisfying the equation $\mathcal{F}$ approximately is a true solution of $\mathcal{F}$ (see [1] for another notion of the superstability which may be called *superstability modulo the bounded functions*).

The stability problem for functional equations originated from a question by Ulam [2] in 1940, concerning the stability of group homomorphisms: let $(G_1,\cdot)$ be a group, and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist $\delta>0$ such that, if a mapping $h: G_1\to G_2$ satisfies the inequality $d(h(s\cdot t),h(s)*h(t))<\delta$ for all $s,t\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d(h(s),H(s))<\epsilon$ for all $s\in G_1$? In other words, under what condition a functional equation is stable? In the following year, Hyers [3] gave a partial affirmative answer to the question of Ulam for Banach spaces. In 1978, the generalized Hyers' theorem was independently rediscovered by Th. M. Rassias [4] by obtaining a unique linear mapping under certain continuity assumption.

The functional equations

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$
$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

are called *quadratic* and *cubic* functional equations, respectively. During the last decades, several stability problems for functional equations especially the quadratic and cubic and their generalized have been extensively investigated by many mathematicians (for instances, [5–9]).

In [10], Lee et al. considered the following quartic functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to check that for every $a \in \mathbb{R}$, the function $f(x) = ax^4$ is a solution of the above functional equation. They solved (1.2) and in fact showed that a function $f : \mathcal{X} \to \mathcal{Y}$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are real vector spaces is quadratic if and only if there exists a symmetric biquadratic function $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that $f(x) = F(x,x)$ for all $x \in \mathcal{X}$. They also proved the stability of (1.2). Zhou Xu et al. in [11] used the fixed-point alternative (Theorem 2.1 of the current paper) to establish Hyers-Ulam-Rassias stability of the general mixed additive-cubic functional equation, where functions map a linear space into a complete quasifuzzy p-normed space. The generalized Hyers-Ulam stability of a general mixed AQCQ-functional in multi-Banach spaces is also proved by using the mentioned theorem in [12].

Recently, Bodaghi et al. in [13, 14] investigated the stability and the superstability of quadratic and cubic functional equations by a fixed-point method and applied this method to prove the stability of (quadratic, cubic) multipliers on Banach algebras.

In this paper we prove the generalized Hyers-Ulam stability and the superstability for quartic functional equation (1.2) by using the alternative fixed point (Theorem 2.1) under certain conditions.

2. Main Results

Throughout this paper, assume that $\mathcal{X}$ is a normed vector space and $\mathcal{Y}$ is a Banach space. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$, we consider

$$Df(x,y) := f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y), \tag{2.1}$$

for all $x, y \in \mathcal{X}$.

To achieve our aim, we need the following known fixed-point theorem which has been proved in [15].

Theorem 2.1. *Suppose that (Δ, d) is a complete generalized metric space, and let $\mathcal{J} : \Delta \to \Delta$ be a strictly contractive mapping with Lipschitz constant $L < 1$, Then for each element $g \in \Delta$, either $d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*

(i) *$d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) < \infty$, for all $n \geq n_0$,*

(ii) *the sequence $\{\mathcal{J}^n g\}$ is convergent to a fixed-point g^* of $\mathcal{J}$,*

(iii) *g^* is the unique fixed point of $\mathcal{J}$ in the set*

$$\Omega = \{g \in \Delta : d(\mathcal{J}^{n_0} g, g) < \infty\}; \tag{2.2}$$

(iv) $d(g, g^*) \le (1/(1-L)) d(g, \mathcal{J} g)$, *for all* $g \in \Omega$.

Theorem 2.2. *Assume that $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a function satisfying*

$$\|Df(x,y)\| \le \phi(x,y), \tag{2.3}$$

for all $x, y \in \mathcal{X}$. Let a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfy $f(0) = 0$. If there exists $K \in (0,1)$ such that

$$\phi(x,y) \le 2^4 K \phi\left(\frac{x}{2}, \frac{y}{2}\right), \tag{2.4}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{32(1-K)} \phi(x, 0), \tag{2.5}$$

for all $x \in \mathcal{X}$.

Proof. By recurrence method, we can conclude from (2.4) that $\phi(2^n x, 2^n y)/2^{4n} \le K^n \phi(x,y)$ for all $x, y \in \mathcal{X}$. Passing to the limit, we get

$$\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.6}$$

for all $x, y \in \mathcal{X}$. Here, we intend to build the conditions of Theorem 2.1 and so consider the set $\Delta := \{h : \mathcal{X} \to \mathcal{Y} \mid h(0) = 0\}$ and the mapping d defined on $\Delta \times \Delta$ by

$$d(g,h) := \inf\{C \in (0,\infty) : \|g(x) - h(x)\| \le C\phi(x,0)\ \forall x \in \mathcal{X}\} \tag{2.7}$$

if there exists such constant C, and $d(g,h) = \infty$ otherwise. It is easy to see that $d(h,h) = 0$ and $d(g,h) = d(h,g)$, for all $g, h \in \Delta$. For each $g, h, p \in \Delta$, we have

$$\begin{aligned} &\inf\{C \in (0,\infty) : \|g(x) - h(x)\| \le C\phi(x,0)\ \forall x \in \mathcal{X}\} \\ &\quad \le \inf\{C \in (0,\infty) : \|g(x) - p(x)\| \le C\phi(x,0)\ \forall x \in \mathcal{X}\} \\ &\qquad + \inf\{C \in (0,\infty) : \|p(x) - h(x)\| \le C\phi(x,0)\ \forall x \in \mathcal{X}\}. \end{aligned} \tag{2.8}$$

Hence, $d(g,h) \le d(g,p) + d(p,h)$. Now if $d(g,h) = 0$, then for every fixed $x_0 \in \mathcal{X}$, we have $\|g(x_0) - h(x_0)\| \le C\phi(x_0, 0)$, for all $C > 0$. This implies $g = h$. Let $\{h_n\}$ be a d-Cauchy sequence in Δ, then $d(h_m, h_n) \to 0$, and thus $\|h_m(x) - h_n(x)\| \to 0$, for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, then there exists $h \in \Delta$ such that $h_n \xrightarrow{d} h$ in Δ. Therefore, d is a generalized metric on Δ, and the metric space (Δ, d) is complete. Now, we define the mapping $\mathcal{J} : \Delta \to \Delta$ by

$$\mathcal{J}g(x) = \frac{1}{2^4}g(2x), \quad (x \in \mathcal{X}). \tag{2.9}$$

Fix a $C \in (0, \infty)$ and take $g, h \in \Delta$ such that $d(g, h) < C$. The definitions of d and $\mathcal{J}$ show that

$$\left\| \frac{1}{2^4}g(2x) - \frac{1}{2^4}h(2x) \right\| \leq \frac{1}{2^4}C\phi(2x, 0), \tag{2.10}$$

for all $x \in \mathcal{X}$. By using (2.4), we have

$$\left\| \frac{1}{2^4}g(2x) - \frac{1}{2^4}h(2x) \right\| \leq CK\phi(x, 0), \tag{2.11}$$

for all $x \in \mathcal{X}$. It follows from the above inequality that $d(\mathcal{J}g, \mathcal{J}h) \leq Kd(g, h)$, for all $g, h \in \Delta$. Hence, $\mathcal{J}$ is a strictly contractive mapping on Δ with a Lipschitz constant K. Putting $y = 0$ in (2.3) and dividing both sides of the resulting inequality by 32, we have

$$\left\| f(x) - \frac{1}{16}f(2x) \right\| \leq \frac{1}{32}\phi(x, 0), \tag{2.12}$$

for all $x \in X$. Thus, $d(f, \mathcal{J}f) \leq 1/32 < \infty$. Note that by Theorem 2.1, $d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) < \infty$, for all $n \geq 0$. Thus, we get $n_0 = 0$ in this theorem, so (iii) and (iv) of Theorem 2.1 are true on the whole Δ. However, the sequence $\{\mathcal{J}^n f\}$ converges to a unique fixed-point $Q : \mathcal{X} \to \mathcal{Y}$ in the set $\{g \in \Delta; d(f, g) < \infty\}$, that is,

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^{4n}}, \tag{2.13}$$

for all $x \in X$. By the part (iv) of Theorem 2.1, we have

$$d(f, Q) \leq \frac{d(f, \mathcal{J}f)}{1 - K} \leq \frac{1}{32(1 - K)}. \tag{2.14}$$

From (2.14), we observe that the inequality (2.5) holds for all $x \in \mathcal{X}$. Substituting x, y by $2^n x, 2^n y$ in (2.3), respectively, and applying (2.6) and (2.13), we have

$$\|DQ(x, y)\| = \lim_{n \to \infty} \frac{1}{2^{4n}} \|Df(2^n x, 2^n y)\| \leq \lim_{n \to \infty} \frac{1}{2^{4n}} \phi(2^n x, 2^n y) = 0, \tag{2.15}$$

for all $x \in \mathcal{X}$. Therefore, Q is a quartic mapping which is unique by part (iii) of Theorem 2.1. □

Corollary 2.3. *Let p, θ be nonnegative real numbers such that $p < 4$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping (with $f(0) = 0$ when $p = 0$) satisfying*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p), \tag{2.16}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{32 - 2^{p+1}}\|x\|^p, \tag{2.17}$$

for all $x \in \mathcal{X}$.

Proof. The result follows from Theorem 2.2 by using $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$. □

Now, we establish the superstability of quartic mapping on Banach spaces under some conditions.

Corollary 2.4. *Let p, q, θ be nonnegative real numbers such that $p + q \in (0,4)$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x,y)\| \leq \theta\|x\|^p\|y\|^q, \tag{2.18}$$

for all $x, y \in \mathcal{X}$, then f is a quartic mapping on $\mathcal{X}$.

Proof. Letting $\phi(x,y) = \theta\|x\|^p\|y\|^q$ in Theorem 2.2, we have

$$\lim_{n\to\infty}\frac{\phi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.19}$$

which shows (2.6) holds for ϕ. Putting $x = y = 0$ in (2.18), we get $f(0) = 0$. Furthermore, if we put $y = 0$ in (2.18), then we have $f(2x) = 2^4 f(x)$, for all $x \in \mathcal{X}$. It is easy to see that by induction, we have $f(2^n x) = 2^{4n} f(x)$, and so $f(x) = f(2^n x)/2^{4n}$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Now, it follows from Theorem 2.2 that f is a quartic mapping. □

Let θ and p be positive real numbers. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies

$$\|Df(x,y)\| \leq \theta\|y\|^p, \tag{2.20}$$

for all $x, y \in \mathcal{X}$, then by considering $\phi(x,y) = \theta\|y\|^p$ in Theorem 2.2, the mapping f is again a quartic mapping on $\mathcal{X}$.

The following result is proved in [16, Theorem 1].

Theorem 2.5. *Let $\mathcal{X}$ be a linear space, and let $\mathcal{Y}$ be a Banach space. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exists a function $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that*

$$\tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 2^{-4k} \varphi\left(2^k x, 2^k y\right) < \infty, \qquad \|Df(x, y)\| \leq \delta + \varphi(x, y) \tag{2.21}$$

for all $x, y \in \mathcal{X}$, where $\delta \geq 0$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\| f(x) - Q(x) + \frac{1}{5} f(0) \right\| \leq \frac{1}{30}\delta + \frac{1}{32}\tilde{\varphi}(x, 0) \tag{2.22}$$

for all $x \in \mathcal{X}$.

One should note that in the above theorem, $f(0)$ is not necessarily zero, but in the following result, we assume that $f(0) = 0$ and also consider the case $\delta = 0$. By these hypotheses and by applying Theorem 2.1, we obtain the specific result which is a way to prove the superstability of a quartic functional equation.

Theorem 2.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$, and let $\psi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function satisfying*

$$\lim_{n \to \infty} 2^{4n} \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{2.23}$$

$$\|Df(x, y)\| \leq \psi(x, y), \tag{2.24}$$

for all $x, y \in \mathcal{X}$. If there exists $L \in (0, 1)$ such that

$$\psi(x, 0) \leq 2^{-4} L \psi(2x, 0), \tag{2.25}$$

for all $x \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{32(1 - L)} \psi(x, 0), \tag{2.26}$$

for all $x \in \mathcal{X}$.

Proof. We take the set $\Omega := \{g : \mathcal{X} \to \mathcal{Y} \mid g(0) = 0\}$ and consider the generalized metric on Ω,

$$d(g_1, g_2) := \inf\{C \in (0, \infty) : \|g_1(x) - g_2(x)\| \leq C\psi(x, 0)\ \forall x \in \mathcal{X}\}, \tag{2.27}$$

if there exists such a constant C, and $d(g_1, g_2) = \infty$ otherwise. It follows from the proof of Theorem 2.2 that the metric space (Ω, d) is complete (see the proof of Theorem 2.2).

We will show that the mapping $\mathcal{J} : \Omega \to \Omega$ defined by $\mathcal{J}g(x) = 2^4 g(x/2)(x \in \mathcal{X})$ is strictly contractive. Fix a $C \in (0, \infty)$ and take $g_1, g_2 \in \Omega$ such that $d(g_1, g_2) < C$, then we have

$$\left\|2^4 g_1\left(\frac{x}{2}\right) - 2^4 g_2\left(\frac{x}{2}\right)\right\| \leq 2^4 C\varphi\left(\frac{x}{2}, 0\right), \tag{2.28}$$

for all $x \in \mathcal{X}$. By using (2.25), we obtain

$$\left\|2^4 g_1\left(\frac{x}{2}\right) - 2^4 g_2\left(\frac{x}{2}\right)\right\| \leq CL\varphi(x, 0), \tag{2.29}$$

for all $x \in \mathcal{X}$. It follows from the last inequality that $d(\mathcal{J}g_1, \mathcal{J}g_2) \leq Ld(g_1, g_2)$, for all $g_1, g_2 \in \Omega$. Hence, $\mathcal{J}$ is a strictly contractive mapping on Ω with a Lipschitz constant L. By putting $y = 0$, replacing x by $x/2$ in (2.24) and using (2.25), and then dividing both sides of the resulting inequality by 2, we have

$$\left\|2^4 f\left(\frac{x}{2}\right) - f(x)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right) \leq 2^{-5} L\varphi(x, 0), \tag{2.30}$$

for all $x \in \mathcal{X}$. Hence, $d(f, \mathcal{J}f) \leq 2^{-5}L < \infty$. By applying the fixed-point alternative Theorem 2.1, there exists a unique mapping $Q : \mathcal{X} \to \mathcal{Y}$ in the set $\Omega_1 = \{g \in \Omega; d(f, g) < \infty\}$ such that

$$Q(x) = \lim_{n\to\infty} 2^{4n} f\left(\frac{x}{2^n}\right), \tag{2.31}$$

for all $x \in \mathcal{X}$. Again Theorem 2.1 shows that

$$d(f, Q) \leq \frac{d(f, \mathcal{J}f)}{1 - L} \leq \frac{2^{-5}L}{1 - L}. \tag{2.32}$$

Hence, inequality (2.32) implies (2.26). Replacing x, y by $2^n x, 2^n y$ in (2.24), respectively, and using (2.23) and (2.31), we conclude that

$$\begin{aligned} \|DQ(x, y)\| &= \lim_{n\to\infty} 2^{4n}\left\|Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 2^{4n}\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \end{aligned} \tag{2.33}$$

for all $x \in \mathcal{X}$. Therefore, Q is a quartic mapping. □

Corollary 2.7. *Let p and λ be nonnegative real numbers such that $p > 4$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying*

$$\|Df(x, y)\| \leq \lambda(\|x\|^p + \|y\|^p), \tag{2.34}$$

for all $x, y \in \mathcal{X}$, then there exists a unique quartic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{\lambda}{2(2^p - 2^4)}\|x\|^p \tag{2.35}$$

for all $x \in \mathcal{X}$.

Proof. It is enough to let $\varphi(x, y) = \lambda(\|x\|^p + \|y\|^p)$ in Theorem 2.6. □

Corollary 2.8. *Let p, q, λ be nonnegative real numbers such that $p + q \in (4, \infty)$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x, y)\| \leq \lambda\|x\|^p\|y\|^q \tag{2.36}$$

for all $x, y \in \mathcal{X}$. Then f is a quartic mapping on $\mathcal{X}$.

Proof. Putting $\varphi(x, y) = \theta\|x\|^p\|y\|^q$ in Theorem 2.6, we have

$$\lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y)}{2^{4n}} = 0, \tag{2.37}$$

and thus, (2.6) holds. If we put $x = y = 0$ in (2.36), then we get $f(0) = 0$. Again, letting $y = 0$ in (2.36), we conclude that $f(x) = 2^4 f(x/2)$, and thus, $f(x) = 2^{4n} f(x/2^n)$, for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Now, we can obtain the desired result by Theorem 2.6.

From Corollaries 2.4 and 2.8 we deduce the following result. □

Corollary 2.9. *Let p, q, and λ be nonnegative real numbers such that $p + q > 0$ and $p + q \neq 4$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies* (2.36)*, for all $x, y \in \mathcal{X}$ then f is a quartic mapping on $\mathcal{X}$.*

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