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Research Article

Positive Solutions for a Fractional Boundary Value Problem with Changing Sign Nonlinearity

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We discuss the existence of positive solutions to the following fractional m-point boundary value problem with changing sign nonlinearity $D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, 0<t<1, u(0)=0, D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i)$, where λ is a positive parameter, $1<\alpha\le 2$, $0<\beta<\alpha-1$, $0<\xi_1<\cdots<\xi_{m-2}<1$ with $\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-\beta-1}<1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, f and may be singular at $t=0$ and/or $t=1$ and also may change sign. The work improves and generalizes some previous results.

1. Introduction

In this paper, we consider the following fractional differential equation with m-point boundary conditions:

$$\begin{aligned} &D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, \quad 0<t<1, \\ &u(0)=0, \quad D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i), \end{aligned} \tag{1.1}$$

where $1<\alpha\le 2$, $\lambda>0$ is a parameter, $0<\beta<\alpha-1$, $0<\xi_1<\cdots<\xi_{m-2}<1$ with $\sum_{i=1}^{m-2}\eta_i\xi_i^{\alpha-\beta-1}<1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, and $f\in C((0,1)\times[0,+\infty)\to(-\infty,+\infty))$ may be singular at $t=0$ and/or $t=1$ and also may change sign. In this paper, by a positive solution to (1.1), we mean a function $u\in C[0,1]$ which is positive on $(0,1]$ and satisfies (1.1).

In recent years, great efforts have been made worldwide to study the existence of solutions for nonlinear fractional differential equations by using nonlinear analysis methods [1–24]. Fractional-order multipoint boundary value problems (BVP) have particularly attracted a great deal of attention (see, e.g., [13–19]). In [10], the authors discussed some properties of the Green function for the Direchlet-type BVP of nonlinear fractional differential equations

$$
\begin{gathered}
D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\
u(0) = 0, \quad u(1) = 0,
\end{gathered}
\tag{1.2}
$$

where $1 < \alpha < 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative and $f : C([0,1]\times[0,+\infty) \to [0,+\infty))$ is continuous. By using the Krasnosel'skii fixed-point theorem, the existence of positive solutions was obtained under some suitable conditions on f.

In [14], the authors investigated the existence and multiplicity of positive solutions by using some fixed-point theorems for the fractional differential equation given by

$$
\begin{gathered}
D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\
u(0) = 0, \quad D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\xi),
\end{gathered}
\tag{1.3}
$$

where $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 < \xi < 1$, $0 \leq a \leq 1$ with $a\xi^{\alpha-\beta-2} < 1-\beta$, $0 \leq \alpha - \beta - 1$, f is nonnegative.

It should be noted that in most of the works in literature the nonlinearity needs to be nonnegative in order to establish positive solutions. As far as we know, semipositone fractional nonlocal boundary value problems with $1 < \alpha \leq 2$ have been seldom studied due to the difficulties in finding and analyzing the corresponding Green function.

In [23], the authors investigated the following fractional differential equation with three-point boundary conditions:

$$
\begin{gathered}
D_{0+}^{\alpha} u(t) + f(t, u(t)) + e(t) = 0, \quad 0 < t < 1, \\
u(0) = 0, \quad D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\xi),
\end{gathered}
\tag{1.4}
$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 < \xi < 1$, $0 \leq a \leq 1$,$0 \leq \alpha - \beta - 1$, $e(t) \in L[0,1]$, and f satisfies the Caratheodory conditions. The authors obtained the properties of the Green function for (1.4) as follows:

$$
\frac{\beta t^{\alpha-1} s (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \leq G(t,s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)\left(1 - a\xi^{\alpha-\beta-1}\right)}. \tag{1.5}
$$

By using the Schauder fixed-point theorem, the authors obtained the existence of positive solution of (1.4) with the following assumptions:

(A_1) for each $L > 0$, there exists a function $\phi_L \succ 0$ such that $f(t, t^{\alpha-1}x) \geq \phi_L(t)$ for a.e. $t \in (0,1)$, for all $x \in (0, L]$;

(A_2) there exist $g(x), h(x)$, and $k(t) \succ 0$, such that

$$0 \le f(t,x) \le k(t)\{g(x)+h(x)\}, \quad \forall x \in (0,\infty), \text{ a.e. } t \in (0,1), \tag{1.6}$$

here $g : (0,\infty) \to [0,\infty)$ is continuous and nonincreasing, $h : [0,\infty) \to [0,\infty)$ is continuous, and h/g is nondecreasing;

(A_3) There exist two positive constants $R > r > 0$ such that

$$\begin{gathered} R > \Phi_{R1} + \gamma_* \ge r > 0, \\ \int_0^1 k(s)g\left(rs^{\alpha-1}\right)ds < +\infty, \\ R \ge \left(1+\frac{h(R)}{g(R)}\right)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)\left(1-a\xi^{\alpha-\beta-1}\right)}k(s)g\left(rs^{\alpha-1}\right)ds + \gamma_*. \end{gathered} \tag{1.7}$$

Here

$$\Phi_{R1} = \int_0^1 \frac{\beta s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\phi_R(s)ds. \tag{1.8}$$

The assumptions on nonlinearity are not suitable for frequently used conditions, such as superlinear or some sublinear. For instance, $f(t,x) = x^{\mu}$, $\mu > 0$, obviously, f does not satisfy (A_1).

Inspired by the previous work, the aim of this paper is to establish conditions for the existence of positive solutions of the more general BVP (1.1). Our work presented in this paper has the following new features. Firstly, we consider few cases of $1 < \alpha \le 2$ which has been studied before, and in dealing with the difficulties related to the Green function for this case, some new properties of the Green function have been discovered. Secondly, the BVP (1.1) possesses singularity; that is, f may be singular at $t = 0$ and/or $t = 1$. Thirdly, the nonlinearity f may change sign and may be unbounded from below. Finally, we impose weaker positivity conditions on the nonlocal boundary term; that is, some of the coefficients η_i may be negative.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are to be used to prove our main results. We also discover some new positive properties of the corresponding Green function. In Section 3, we discuss the existence of positive solutions of the semipositone BVP (1.1). In Section 4, we give an example to demonstrate the application of our theoretical results.

2. Basic Definitions and Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can also be found in the recent literature.

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \tag{2.1}$$

provided that the right-hand side is pointwisely defined on $(0, +\infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwisely defined on $(0, +\infty)$.

Lemma 2.3 (see [3]). *Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, $D_{0+}^{\alpha} u \in L(0,1)$;*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.3}$$

where $c_i \in R$, $i = 1, 2, \ldots, n$, $n - 1 < \alpha \le n$.

Set

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \end{cases} \tag{2.4}$$

$$p(s) = 1 - \sum_{s \le \xi_i} \eta_i \left(\frac{\xi_i - s}{1-s}\right)^{\alpha-\beta-1}, \tag{2.5}$$

$$G(t,s) = G_0(t,s) + q(s) t^{\alpha-1}, \tag{2.6}$$

where

$$q(s) = \frac{p(s) - p(0)}{\Gamma(\alpha) p(0)} (1-s)^{\alpha-\beta-1}, \quad p(0) = 1 - \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1}. \tag{2.7}$$

For convenience in presentation, we here list the assumption to be used throughout the paper.

(H_1) $p(0) > 0$, $q(s) \ge 0$ on $[0,1]$.

Remark 2.4. If $\eta_i = 0$ $(i = 1, \ldots, m-2)$, we have $p(0) = 1$ and $q(s) \equiv 0$. If $\eta_i \ge 0$ $(i = 1, \ldots, m-2)$ and $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$, we have $q(s) \ge 0$ on $[0,1]$.

Lemma 2.5 (see [14]). *Assume that* $g(t) \in L[0,1]$ *and* $\alpha > \beta > 0$. *Then*

$$D_{0+}^{\beta} \int_0^t (t-s)^{\alpha-1} g(s)ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s)ds. \tag{2.8}$$

Lemma 2.6. *Assume* (H_1) *holds, and* $y(t) \in L[0,1]$. *Then the unique solution of the problem*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\beta} u(\xi_i) \end{gathered} \tag{2.9}$$

is

$$u(t) = \int_0^1 G(t,s) y(s)ds, \tag{2.10}$$

where $G(t,s)$ *is the Green function of the boundary value problem* (2.9).

Proof. From Lemma 2.3, the solution of (2.9) is

$$u(t) = -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{2.11}$$

Consequently,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{2.12}$$

From $u(0) = 0$, we have $c_2 = 0$.

By Lemma 2.5, we have

$$D_{0+}^{\beta} u(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \tag{2.13}$$

Therefore,

$$\begin{gathered} D_{0+}^{\beta} u(1) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)}, \\ D_{0+}^{\beta} u(\xi_i) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} y(s)ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} \xi_i^{\alpha-\beta-1}. \end{gathered} \tag{2.14}$$

By $D_{0+}^{\beta}u(1) = \sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i)$, we have

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha-1}y(s)ds - \sum_{i=1}^{m-2}\eta_i \int_0^{\xi_i}(\xi_i - s)^{\alpha-\beta-1}y(s)ds}{\Gamma(\alpha)p(0)} = \frac{\int_0^1 (1-s)^{\alpha-\beta-1}p(s)y(s)ds}{\Gamma(\alpha)p(0)}. \tag{2.15}$$

Therefore, the solution of (2.9) is

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds = \int_0^1 G(t,s)y(s)ds. \tag{2.16}$$

□

Lemma 2.7. *The function $G_0(t,s)$ has the following properties:*

(1) $G_0(t,s) > 0$, *for* $t,s \in (0,1)$;

(2) $\Gamma(\alpha)G_0(t,s) \le t^{\alpha-1}$, *for* $t,s \in [0,1]$;

(3) $\beta t^{\alpha-1}h(s) \le \Gamma(\alpha)G_0(t,s) \le h(s)t^{\alpha-2}$, *for* $t,s \in (0,1)$,

where

$$h(s) = s(1-s)^{\alpha-\beta-1}. \tag{2.17}$$

Proof. (1) When $0 < t \le s < 1$, it is clear that

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1} > 0. \tag{2.18}$$

When $0 < s \le t < 1$, we have

$$t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \ge t^{\alpha-1}(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-s)^{\alpha-1} = t^{\alpha-1}(1-s)^{\alpha-\beta-1}\left[1-(1-s)^{\beta}\right] > 0. \tag{2.19}$$

(2) By (2.4), for any $t,s \in [0,1]$, we have

$$\Gamma(\alpha)G_0(t,s) \le t^{\alpha-1}(1-s)^{\alpha-\beta-1} \le t^{\alpha-1}. \tag{2.20}$$

In the following, we will prove (3).

(i) When $0 < s \le t < 1$, noticing that $0 < \beta < \alpha - 1 \le 1$, we have

$$\frac{\partial}{\partial \beta}\left\{ t^{\alpha-2}s(1-s)^{\alpha-\beta-1} - t^{\alpha-1}(1-s)^{\alpha-\beta-1} \right\} = t^{\alpha-2}(1-s)^{\alpha-\beta-1}(t-s)\ln(1-s) \le 0. \tag{2.21}$$

Therefore,

$$\begin{aligned} t^{\alpha-2}s(1-s)^{\alpha-\beta-1} - \left(t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \right) &\ge t^{\alpha-2}s - t^{\alpha-1} + (t-s)^{\alpha-1} \\ &= -t^{\alpha-2}(t-s) + (t-s)^{\alpha-1} \ge 0, \end{aligned} \tag{2.22}$$

which implies

$$\Gamma(\alpha)G_0(t,s) \le h(s)t^{\alpha-2}. \tag{2.23}$$

On the other hand, we have

$$\frac{d}{ds}\left\{ \beta s + (1-s)^{\beta} \right\} \le 0, \quad s \in [0,1). \tag{2.24}$$

Therefore, $\beta s + (1-s)^{\beta} \le 1$, which implies

$$\left[1 - (1-s)^{\beta} \right] \ge \beta s. \tag{2.25}$$

Then

$$\begin{aligned} \Gamma(\alpha)G_0(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ &\ge t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\beta}(t-ts)^{\alpha-\beta-1} \\ &= \left[1 - \left(1 - \frac{s}{t} \right)^{\beta} \right] t^{\alpha-1}(1-s)^{\alpha-\beta-1} \\ &\ge \left[1 - (1-s)^{\beta} \right] t^{\alpha-1}(1-s)^{\alpha-\beta-1} \\ &\ge \beta t^{\alpha-1}h(s). \end{aligned} \tag{2.26}$$

(ii) When $0 < t \le s < 1$, we have

$$\begin{aligned} \Gamma(\alpha)G_0(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} = t^{\alpha-2}t(1-s)^{\alpha-\beta-1} \\ &\le t^{\alpha-2}s(1-s)^{\alpha-\beta-1} = h(s)t^{\alpha-2}, \end{aligned} \tag{2.27}$$

On the other hand, clearly we have

$$\Gamma(\alpha)G_0(t,s) = t^{\alpha-1}(1-s)^{\alpha-\beta-1} \geq \beta t^{\alpha-1}h(s). \tag{2.28}$$

The inequalities (2.23)–(2.28) imply that (3) holds. □

By Lemma 2.7, we have the following results.

Lemma 2.8. *Assume* (H_1) *holds, then the Green function defined by* (2.6) *satisfies*

(1) $G(t,s) > 0$, *for all* $t,s \in (0,1)$;

(2) $G(t,s) \leq t^{\alpha-1}((1/(\Gamma(\alpha))) + q(s))$, *for all* $t,s \in [0,1]$;

(3) $\beta t^{\alpha-1}\Phi(s) \leq G(t,s) \leq t^{\alpha-2}\Phi(s)$, *for all* $t,s \in (0,1)$,

where

$$\Phi(s) = \left(\frac{h(s)}{\Gamma(\alpha)} + q(s)\right). \tag{2.29}$$

Lemma 2.9. *Assume* (H_1) *holds, then the function* $G^*(t,s) =: t^{2-\alpha}G(t,s)$ *satisfies*

(1) $G^*(t,s) > 0$, *for all* $t,s \in (0,1)$;

(2) $G^*(t,s) \leq t((1/(\Gamma(\alpha))) + q(s))$, *for all* $t,s \in [0,1]$;

(3) $\beta t\Phi(s) \leq G^*(t,s) \leq \Phi(s)$, *for all* $t,s \in [0,1]$.

For convenience, we list here four more assumptions to be used later:

(H_2) $f \in C((0,1) \times [0,+\infty), (-\infty,+\infty))$ satisfies

$$f(t,x) \geq -r(t), \quad f\left(t, t^{\alpha-2}x\right) \leq z(t)g(x), \quad t \in (0,1),\ x \in [0,+\infty), \tag{2.30}$$

where $r, z \in C((0,1), [0,+\infty))$, $g \in C([0,+\infty), [0,+\infty))$.

(H_3) $\int_0^1 r(s)ds < +\infty$, $0 < \int_0^1 z(s)ds < +\infty$.

(H_4) There exists $[a,b] \subset (0,1)$ such that

$$\liminf_{x\to+\infty} \min_{t\in[a,b]} \frac{f(t,x)}{x} = +\infty. \tag{2.31}$$

(H_5) There exists $[c,d] \subset (0,1)$ such that

$$\begin{gathered} \liminf_{x\to+\infty} \min_{t\in[c,d]} f(t,x) = +\infty, \\ \lim_{x\to+\infty} \frac{g(x)}{x} = 0. \end{gathered} \tag{2.32}$$

Remark 2.10. The second limit of (H_5) implies

$$\lim_{u\to+\infty}\frac{g^*(u)}{u}=0, \tag{2.33}$$

where

$$g^*(u)=\max_{x\in[0,u]}g(x). \tag{2.34}$$

Proof. By $\lim_{u\to+\infty}(g(u)/u)=0$, for any $\epsilon>0$, there exists $N_1>0$, such that for any $u>N_1$ we have

$$0\leq g(u)<\epsilon u. \tag{2.35}$$

Let $N=\max\{N_1,((g^*(N_1))/\epsilon)\}$, for any $u>N$ we have

$$0\leq g^*(u)<\epsilon u+g^*(N_1)<2\epsilon u. \tag{2.36}$$

Therefore, $\lim_{u\to+\infty}((g^*(u))/u)=0$. □

Lemma 2.11. *Assume (H_1) holds and $r(t)\in C(0,1)\cap L[0,1]$ is nonnegative, then the BVP*

$$\begin{gathered}D_{0+}^{\alpha}u(t)+r(t)=0,\quad 0<t<1,\\ u(0)=0,\quad D_{0+}^{\beta}u(1)=\sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i)\end{gathered} \tag{2.37}$$

has a unique solution $\omega(t)=\int_0^1 G(t,s)r(s)ds$ with $\omega(t)\leq kt^{\alpha-1}$, where

$$k=\int_0^1\left(\frac{1}{\Gamma(\alpha)}+q(s)\right)r(s)ds,\quad t\in[0,1]. \tag{2.38}$$

Proof. By Lemma 2.6, $\omega(t)=\int_0^1 G(t,s)r(s)ds$ is the unique solution of (2.37). By (2) of Lemma 2.8, we have

$$\omega(t)=\int_0^1 G(t,s)r(s)ds\leq t^{\alpha-1}\int_0^1\left(\frac{1}{\Gamma(\alpha)}+q(s)\right)r(s)ds. \tag{2.39}$$

□

Let $E=C[0,1]$ be endowed with the maximum norm $\|u\|=\max_{0\leq t\leq 1}|u(t)|$ and define a cone P by

$$P=\{u(t)\in E:\ \text{there exists } l_u>0\ \text{ such that }\ \beta t\|u\|\leq u(t)\leq l_u t\}, \tag{2.40}$$

and then set $B_r=\{u(t)\in E:\|u\|<r\}$, $P_r=P\cap B_r$, $\partial P_r=P\cap\partial B_r$.

Next we consider the following singular nonlinear BVP:

$$D_{0+}^{\alpha}u(t) + \lambda\left[f\left(t, [u(t) - \lambda\omega(t)]^{+}\right) + r(t)\right] = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad D_{0+}^{\beta}u(1) = \sum_{i=1}^{m-2}\eta_i D_{0+}^{\beta}u(\xi_i), \tag{2.41}$$

where $\lambda > 0$, $[v(t)]^{+} = \max\{v(t), 0\}$, $\omega(t)$ is defined in Lemma 2.11.

Let

$$Tu(t) = \lambda\int_0^1 G^*(t,s)\left[f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^{+}\right) + r(s)\right]ds. \tag{2.42}$$

Clearly, if $u(t) \in P$ is a fixed point of T, then $y(t) = t^{\alpha-2}u(t)$ is a positive solution of (2.41).

Lemma 2.12. *Suppose that (H_1)–(H_3) hold. Then $T : P \to P$ is a completely continuous operator.*

Proof. It is clear that T is well defined on P. For any $u \in P$, Lemma 2.9 implies

$$Tu(t) \geq \beta t\lambda\int_0^1 \Phi(s)\left[f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^{+}\right) + r(s)\right]ds. \tag{2.43}$$

On the other hand,

$$Tu(t) \leq \lambda\int_0^1 \Phi(s)\left[f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^{+}\right) + r(s)\right]ds. \tag{2.44}$$

Therefore, $Tu(t) \geq \beta t\,\|Tu\|$. Noticing that

$$Tu(t) \leq \lambda t\int_0^1 \left(\frac{1}{\Gamma(\alpha)} + q(s)\right)\left[f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^{+}\right) + r(s)\right]ds, \tag{2.45}$$

we have $T : P \to P$.

Using the Ascoli-Arzela theorem, we can then get that $T : P \to P$ is a completely continuous operator. □

Lemma 2.13 (see [25]). *Let E be a real Banach space and let $P \subset E$ be a cone. Assume that Ω_1 and Ω_2 are two-bounded open subsets of E with $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ a completely continuous operator such that either*

(1) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(2) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence of Positive Solutions

Theorem 3.1. *Suppose that* (H_1)–(H_4) *hold. Then there exists* $\lambda^* > 0$ *such that the BVP* (1.1) *has at least one positive solution for any* $\lambda \in (0, \lambda^*)$.

Proof. Choose $r_1 > k\beta^{-1}$. Let

$$\lambda^* = \min\left\{1, \frac{r_1}{(g^*(r_1)+1)\int_0^1 \Phi(s)(z(s)+r(s))ds}\right\}, \tag{3.1}$$

where

$$g^*(r) = \max_{x\in[0,r]} g(x). \tag{3.2}$$

In the following of the proof, we suppose $\lambda \in (0, \lambda^*)$.

For any $u \in \partial P_{r_1}$, noticing $u(t) \geq \beta t r_1$ and Lemma 2.11, we have

$$t^{\alpha-2}u(t) - \lambda\omega(t) \geq (\beta r_1 - \lambda k)t^{\alpha-1} \geq (\beta r_1 - k)t^{\alpha-1} \geq 0, \tag{3.3}$$

$$r_1 \geq u(t) - \lambda t^{2-\alpha}\omega(t) \geq (\beta r_1 - k)t \geq 0. \tag{3.4}$$

Therefore,

$$\begin{aligned} Tu(t) &= \lambda\int_0^1 G^*(t,s)\left(f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^+\right) + r(s)\right)ds \\ &\leq \lambda\int_0^1 \Phi(s)\left(z(s)g\left(\left[u(s) - \lambda s^{2-\alpha}\omega(s)\right]^+\right) + r(s)\right)ds \\ &\leq \lambda(g^*(r_1)+1)\int_0^1 \Phi(s)(z(s)+r(s))ds \\ &< \lambda^*(g^*(r_1)+1)\int_0^1 \Phi(s)(z(s)+r(s))ds \leq r_1. \end{aligned} \tag{3.5}$$

Thus,

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial P_{r_1}. \tag{3.6}$$

Now choose a real number

$$L > \frac{2}{\lambda\beta^2\int_a^b \Phi(s)s^{\alpha-1}ds}. \tag{3.7}$$

By (H_4), there exists a constant $N > 0$ such that

$$f(t,x) > Lx, \quad \text{for any } t \in [a,b],\ x \geq N. \tag{3.8}$$

Let

$$r_2 = r_1 + \frac{2k}{\beta} + \frac{2N}{\beta a^{\alpha-1}}. \tag{3.9}$$

Then for any $u \in \partial P_{r_2}$, we have

$$t^{\alpha-2}u(t) - \lambda\omega(t) \geq (\beta r_2 - k)t^{\alpha-1} \geq \frac{\beta r_2}{2}t^{\alpha-1}, \quad \forall t \in (0,1]. \tag{3.10}$$

Thus, for any $t \in [a,b]$, we have $t^{\alpha-2}u(t) - \lambda\omega(t) > N$. Hence, we get

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]}\lambda\int_0^1 G^*(t,s)\left[f\left(s,\left[s^{\alpha-2}u(s)-\lambda\omega(s)\right]^+\right)+r(s)\right]ds \\
&\geq \max_{t\in[0,1]}\lambda\int_a^b G^*(t,s)f\left(s,\left[s^{\alpha-2}u(s)-\lambda\omega(s)\right]\right)ds \\
&\geq \max_{t\in[0,1]}\lambda L\int_a^b G^*(t,s)\left(s^{\alpha-2}u(s)-\lambda\omega(s)\right)ds \\
&\geq \max_{t\in[0,1]}\lambda L\int_a^b G^*(t,s)\frac{\beta r_2}{2}s^{\alpha-1}ds \\
&\geq \max_{t\in[0,1]}\frac{\lambda L\beta^2 r_2}{2}t\int_a^b \Phi(s)s^{\alpha-1}ds \\
&= \frac{\lambda L\beta^2 r_2}{2}\int_a^b \Phi(s)s^{\alpha-1}ds \geq r_2.
\end{aligned} \tag{3.11}$$

Therefore,

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial P_{r_2}. \tag{3.12}$$

By Lemma 2.13, T has a fixed point $u \in P$ such that $r_1 \leq \|u\| \leq r_2$. Let $\overline{u}(t) = t^{\alpha-2}u(t) - \lambda\omega(t)$. Since $\|u\| \geq r_1$, by (3.3) we have $\overline{u}(t) \geq 0$ on $(0,1]$ and $\lim_{t\to 0^+}t^{\alpha-2}u(t) = 0$. Notice that $\omega(t)$ is the solution of (2.37) and $t^{\alpha-2}u(t)$ is the solution of (2.41). Thus, $\overline{u}(t)$ is a positive solution of the BVP (1.1). □

Theorem 3.2. *Suppose that* (H_1)–(H_3) *and* (H_5) *hold. Then there exists* $\lambda^* > 0$ *such that the BVP* (1.1) *has at least one positive solution for any* $\lambda \in (\lambda^*, +\infty)$.

Proof. By the first limit of (H_5), there exists $N > 0$ such that

$$f(t,x) \geq \frac{2k}{\beta^2 \int_c^d \Phi(s)ds}, \quad \text{for any } t \in [c,d],\ x \geq N. \tag{3.13}$$

Let

$$\lambda^* = \frac{N}{kc^{\alpha-1}}. \tag{3.14}$$

In the following part of the proof, we suppose $\lambda > \lambda^*$.

Let

$$R_1 = \frac{2\lambda k}{\beta}. \tag{3.15}$$

Then for any $u \in \partial P_{R_1}$, we have

$$t^{\alpha-2}u(t) - \lambda\omega(t) \geq (\beta R_1 - \lambda k)t^{\alpha-1} = \lambda k t^{\alpha-1} \geq \lambda^* k t^{\alpha-1}, \quad \forall t \in (0,1]. \tag{3.16}$$

Therefore, $t^{\alpha-2}u(t) - \lambda\omega(t) \geq N$, for any $t \in [c,d]$ and $u \in \partial P_{R_1}$. Then

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G^*(t,s)\left[f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^+\right) + r(s)\right]ds \\ &\geq \lambda \int_c^d G^*(t,s) f\left(s, \left[s^{\alpha-2}u(s) - \lambda\omega(s)\right]^+\right)ds \\ &\geq \frac{2\lambda k}{\beta^2 \int_c^d \Phi(s)ds} \int_c^d G^*(t,s)ds \\ &\geq \frac{2\lambda k t}{\beta \int_c^d \Phi(s)ds} \int_c^d \Phi(s)ds = R_1 t. \end{aligned} \tag{3.17}$$

This implies

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial P_{R_1}. \tag{3.18}$$

On the other hand, $g(x)$ is continuous on $[0,+\infty)$, and thus from the second limit of (H_5), we have

$$\lim_{x\to+\infty} \frac{g^*(x)}{x} = 0, \tag{3.19}$$

where $g^*(x)$ is defined by (3.2). For

$$\epsilon = \left(2\lambda\int_0^1 \Phi(s)z(s)ds\right)^{-1}, \tag{3.20}$$

there exists $X_0 > 0$ such that $g(u) \le \varepsilon x$ for any $x \ge X_0$ and $u \in [0, x]$.

Let

$$R_2 = X_0 + R_1 + 2\lambda\int_0^1 \Phi(s)r(s)ds. \tag{3.21}$$

For any $u \in \partial P_{R_2}$, by (3.16) we can get $R_2 \ge u(t) - \lambda t^{2-\alpha}\omega(t) \ge 0$, for all $t \in [0,1]$. Therefore,

$$\begin{aligned}
\|Tu\| &\le \lambda\int_0^1 \Phi(s)\left[z(s)g\left(\left[u(s) - \lambda s^{2-\alpha}\omega(s)\right]^+\right) + r(s)\right]ds \\
&\le \lambda\varepsilon R_2\int_0^1 \Phi(s)z(s)ds + \lambda\int_0^1 \Phi(s)r(s)ds \\
&\le \frac{R_2}{2} + \frac{R_2}{2} = R_2.
\end{aligned} \tag{3.22}$$

Thus,

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial P_{R_2}. \tag{3.23}$$

By Lemma 2.13, T has a fixed point $u \in P$ such that $R_1 \le \|u\| \le R_2$. Let $\overline{u}(t) = t^{\alpha-2}u(t) - \lambda\omega(t)$. Since $\|u\| \ge R_1$, by (3.16) we have $\overline{u}(t) \ge 0$ on $(0,1]$ and $\lim_{t\to 0^+} t^{\alpha-2}u(t) = 0$. Notice that $\omega(t)$ is a solution of (2.37) and $t^{\alpha-2}u(t)$ is a solution of (2.41). Thus, $\overline{u}(t)$ is a positive solution of the BVP (1.1). □

By the proof of Theorem 3.2, we have the following corollary.

Corollary 3.3. *The conclusion of Theorem 3.2 is valid if* (H_5) *is replaced by* (H_5^*). *There exist* $[c,d] \subset (0,1)$ *and* $N > 0$ *such that for any* $t \in [c,d]$ *and* $x \ge N$,

$$\begin{gathered}
f(t,x) \ge \frac{2k}{\beta^2\int_c^d \Phi(s)ds}, \\
\lim_{x\to+\infty}\frac{g(x)}{x} = 0.
\end{gathered} \tag{3.24}$$

4. Example

Example 4.1 (a 4-point BVP with coefficients of both signs). Consider the following problem:

$$D_{0+}^{7/4}u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1), \; u(0) = 0,$$
$$D_{0+}^{1/4}u(1) = D_{0+}^{1/4}u\left(\frac{1}{4}\right) - \frac{1}{2}D_{0+}^{1/4}u\left(\frac{4}{9}\right), \tag{4.1}$$

where

$$f(t, x) = x^2 + \ln t. \tag{4.2}$$

We have

$$G_0(t,s) = \frac{1}{\Gamma(7/4)}\begin{cases} t^{3/4}(1-s)^{1/2}, & 0 \le t \le s \le 1, \\ t^{3/4}(1-s)^{1/2} - (t-s)^{3/4}, & 0 \le s \le t \le 1, \end{cases} \tag{4.3}$$

$$p(s) = \begin{cases} 1 - \left(\dfrac{(1/4)-s}{1-s}\right)^{1/2} - \dfrac{1}{2}\left(\dfrac{(4/9)-s}{1-s}\right)^{1/2}, & 0 \le s \le \dfrac{1}{4}, \\ 1 - \dfrac{1}{2}\left(\dfrac{(4/9)-s}{1-s}\right)^{1/2}, & \dfrac{1}{4} < s \le \dfrac{4}{9}, \\ 1, & \dfrac{4}{9} < s \le 1. \end{cases} \tag{4.4}$$

By direct calculations, we have $p(0) = (1/6)$ and $q(s) \ge 0$, which implies that (H_1) holds.

Let $r(t) = -\ln t$, $z(t) = t^{-1/2}$, $g(x) = x^2$. It is easy to see that (H_2) and (H_3) hold. Moreover,

$$\liminf_{x \to +\infty} \min_{t \in [(1/4),(3/4)]} \frac{f(t,x)}{x} = +\infty. \tag{4.5}$$

Therefore, the assumptions of Theorem 3.1 are satisfied. Thus, Theorem 3.1 ensures that there exists $\lambda^* > 0$ such that the BVP (4.1) has at least one positive solution for any $\lambda \in (0, \lambda^*)$.

Remark 4.2. Noticing that λx^2 does not satisfy (A_1), therefore, the work in the present paper improves and generalizes the main results of [23].

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References

[1] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives (Theory and Applications)*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.

[2] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1999.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Nieto, *Theory and Applicational Diffential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.

[4] O. P. Agrawal, "Formulation of Euler-Lagrange equations for fractional variational problems," *Journal of Mathematical Analysis and Applications*, vol. 272, no. 1, pp. 368–379, 2002.

[5] V. Lakshmikantham and A. S. Vatsala, "General uniqueness and monotone iterative technique for fractional differential equations," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 828–834, 2008.

[6] N. Kosmatov, "Integral equations and initial value problems for nonlinear differential equations of fractional order," *Nonlinear Analysis*, vol. 70, no. 7, pp. 2521–2529, 2009.

[7] R. P. Agarwal, D. O'Regan, and S. Staněk, "Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 1, pp. 57–68, 2010.

[8] B. Ahmad and A. Alsaedi, "Existence of solutions for anti-periodic boundary value problems of nonlinear impulsive functional integro-differential equations of mixed type," *Nonlinear Analysis*, vol. 3, no. 4, pp. 501–509, 2009.

[9] S. Liang and J. Zhang, "Positive solutions for boundary value problems of nonlinear fractional differential equation," *Nonlinear Analysis*, vol. 71, no. 11, pp. 5545–5550, 2009.

[10] D. Jiang and C. Yuan, "The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application," *Nonlinear Analysis*, vol. 72, no. 2, pp. 710–719, 2010.

[11] C. S. Goodrich, "Existence of a positive solution to a class of fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1050–1055, 2010.

[12] A. Arara, M. Benchohra, N. Hamidi, and J. J. Nieto, "Fractional order differential equations on an unbounded domain," *Nonlinear Analysis*, vol. 72, no. 2, pp. 580–586, 2010.

[13] H. A. H. Salem, "On the fractional order m-point boundary value problem in reflexive Banach spaces and weak topologies," *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 565–572, 2009.

[14] C. F. Li, X. N. Luo, and Y. Zhou, "Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.

[15] M. Benchohra, S. Hamani, and S. K. Ntouyas, "Boundary value problems for differential equations with fractional order and nonlocal conditions," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 2391–2396, 2009.

[16] M. El-Shahed and J. J. Nieto, "Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3438–3443, 2010.

[17] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis*, vol. 72, no. 2, pp. 916–924, 2010.

[18] M. Rehman and R. A. Khan, "Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1038–1044, 2010.

[19] J. Allison and N. Kosmatov, "Multi-point boundary value problems of fractional order," *Communications in Applied Analysis*, vol. 12, no. 4, pp. 451–458, 2008.

[20] C. Yuan, D. Jiang, and X. Xu, "Singular positone and semipositone boundary value problems of nonlinear fractional differential equations," *Mathematical Problems in Engineering*, vol. 2009, Article ID 535209, 17 pages, 2009.

[21] B. Ahmad and J. J. Nieto, "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1838–1843, 2009.

[22] B. Ahmad and J. J. Nieto, "Existence results for higher order fractional differential inclusions with nonlocal boundary conditions," *Nonlinear Studies*, vol. 17, no. 2, pp. 131–138, 2010.

[23] X. Xu and X. Fei, "The positive properties of Green's function for three point boundary value problems of nonlinear fractional differential equations and its applications," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 4, pp. 1555–1565, 2012.

[24] K. Li and J. Jia, "Existence and uniqueness of mild solutions for abstract delay fractional differential equations," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1398–1404, 2011.

[25] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, NY, USA, 1985.