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Research Article

Solution of Nonlinear Elliptic Boundary Value Problems and Its Iterative Construction

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We study a kind of nonlinear elliptic boundary value problems with generalized p-Laplacian operator. The unique solution is proved to be existing and the relationship between this solution and the zero point of a suitably defined nonlinear maximal monotone operator is investigated. Moreover, an iterative scheme is constructed to be strongly convergent to the unique solution. The work done in this paper is meaningful since it combines the knowledge of ranges for nonlinear operators, zero point of nonlinear operators, iterative schemes, and boundary value problems together. Some new techniques of constructing appropriate operators and decomposing the equations are employed, which extend and complement some of the previous work.

1. Introduction

The study on nonlinear boundary value problems with p-Laplacian operator, Δ_p, is a hot topic since it has a close relationship with practical problems. Some significant work has been done by us, see [1–8], and so forth.

Specifically, in 2004, we studied the following nonlinear elliptic boundary value problem involving the generalized p-Laplacian operator:

$$\begin{gathered}
-\operatorname{div}\left[\left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right]+|u|^{p-2}u+g(x,u(x))=f(x), \quad \text{a.e. in } \Omega \\
-\left\langle \vartheta, \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right\rangle = 0, \quad \text{a.e. on } \Gamma.
\end{gathered} \tag{1.1}$$

In Wei and Hou [4], we proved that under some conditions (1.1) has solutions in $L^2(\Omega)$, where $2 \le p < +\infty$. In [5, 6], we extended our work to the following problem:

$$
\begin{gathered}
-\operatorname{div}\left[\left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right]+|u|^{p-2}u+g(x,u(x))=f(x), \quad \text{a.e. in } \Omega \\
-\left\langle \vartheta, \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma.
\end{gathered} \tag{1.2}
$$

In Wei and Zhou [5], we established that (1.2) has solutions in $L^p(\Omega)$, where $2 \le p < +\infty$, and in Wei [6] we proved that (1.2) has solutions in $L^s(\Omega)$, where $\max(N,2) \le p \le s < +\infty$. As the summary and extension of [5, 6], we studied the following nonlinear boundary value problem:

$$
\begin{gathered}
-\operatorname{div}\left[\left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x), \quad \text{a.e. in } \Omega \\
-\left\langle \vartheta, \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma.
\end{gathered} \tag{1.3}
$$

It was shown by Wei and Agarwal [7] that (1.3) had a solution in $L^s(\Omega)$, where $2N/(N+1) < p \le s < +\infty$, $1 \le q < +\infty$ if $p \ge N$ and $1 \le q \le Np/(N-p)$ if $p < N$, for $N \ge 1$.

Clearly, if $C(x) \equiv 0$, then (1.1), (1.2), and (1.3) reduce the cases of involving p-Laplacian operators.

It is worth to mention that all of the work done in [4–7] is based on a perturbation result of the ranges of m-accretive mappings by Calvert and Gupta [9].

In this paper, we will study the following nonlinear elliptic boundary value problem:

$$
\begin{gathered}
-\operatorname{div}\left[\left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right]+\varepsilon|u|^{q-2}u+g(x,u(x),\nabla u(x))=f(x), \quad \text{a.e. in } \Omega \\
-\left\langle \vartheta, \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u\right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma.
\end{gathered} \tag{1.4}
$$

Necessary details of (1.4) will be provided in Section 3.

We may notice that the principal part of the concerned equation is almost the same as those in (1.1), (1.2), and (1.3) while the nonlinear item $g(x,u(x))$ is replaced by the item $g(x,u(x),\nabla u(x))$, which is rather general. It seems that the difference is minor; however, the previous method cannot be employed. In this paper, we will use some perturbation results of the ranges for maximal monotone operators by Pascali and Sburlan [10] to prove that (1.4) has a unique solution in $W^{1,p}(\Omega)$ and later we will prove that the unique solution is the zero point of a suitably defined maximal monotone operator. Finally, we will employ an iterative scheme to approximate strongly to the unique solution. Some new ideas of combining the knowledge of the ranges of nonlinear operators, zero point of nonlinear operators, iterative schemes, and the solution of nonlinear boundary value problems are demonstrated in this paper.

2. Preliminaries

Now, we list some of the knowledge we need in the sequel.

Let X be a real Banach space with a strictly convex dual space X^*. We use $(\cdot,\cdot)$ to denote the generalized duality pairing between X and X^*. We use "$\to$" to denote strong convergence. Let "$X \hookrightarrow Y$" denote the space X embedded continuously in space Y. For any subset G of X, we denote by $\operatorname{int} G$ its interior.

Function Φ is called a proper convex function on X [11] if Φ is defined from X to $(-\infty, +\infty]$, not identically $+\infty$ such that $\Phi((1-\lambda)x + \lambda y) \le (1-\lambda)\Phi(x) + \lambda\Phi(y)$, whenever $x, y \in X$ and $0 \le \lambda \le 1$.

Function $\Phi : X \to (-\infty, +\infty]$ is said to be lower semicontinuous on X [11] if $\liminf_{y \to x}\Phi(y) \ge \Phi(x)$, for any $x \in X$.

Given a proper convex function Φ on X and a point $x \in X$, we denote by $\partial\Phi(x)$ the set of all $x^* \in X^*$ such that $\Phi(x) \le \Phi(y) + (x - y,\ x^*)$, for every $y \in X$. Such elements x^* are called subgradients of Φ at x, and $\partial\Phi(x)$ is called the subdifferential of Φ at x [11].

A single-valued mapping $T : D(T) = X \to X^*$ is said to be hemicontinuous [11] if $w - \lim_{t \to 0} T(x + ty) = Tx$, for any $x, y \in X$.

A multivalued mapping $A : X \to 2^{X^*}$ is said to be monotone [10] if its graph $G(A)$ is a monotone subset of $X \times X^*$ in the sense that

$$(u_1 - u_2,\ w_1 - w_2) \ge 0, \tag{2.1}$$

for any $[u_i, w_i] \in G(A)$, $i = 1, 2$. The mapping A is said to be strictly monotone if the equality in (2.1) implies that $u_1 = u_2$. The monotone operator A is said to be maximal monotone if $G(A)$ is maximal among all monotone subsets of $X \times X^*$ in the sense of inclusion. The mapping A is said to be coercive [10] if $\lim_{n \to +\infty}(x_n, x_n^*)/\|x_n\| = +\infty$ for all $[x_n, x_n^*] \in G(A)$ such that $\lim_{n \to +\infty}\|x_n\| = +\infty$. A point $x \in D(A)$ is said to be a zero point of A if $0 \in Ax$, and we denote by $A^{-1}(0) = \{x \in X : 0 \in Ax\}$ the set of zero points of A.

Lemma 2.1 (Adams [12]). *Let Ω be a bounded conical domain in R^N. If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $mp < N$ and $q = Np/(N - mp)$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; if $mp = N$ and $p > 1$, then for $1 \le q < +\infty$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.*

Lemma 2.2 (Pascali and Sburlan [10]). *If $B : X \to 2^{X^*}$ is an everywhere defined, monotone, and hemicontinuous operator, then B is maximal monotone.*

Lemma 2.3 (Pascali and Sburlan [10]). *If $\Phi : X \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function, then $\partial\Phi$ is maximal monotone from X to X^*.*

Lemma 2.4 (Pascali and Sburlan [10]). *If B_1 and B_2 are two maximal monotone operators in X such that $(\operatorname{int} D(B_1)) \bigcap D(B_2) \ne \emptyset$, then $B_1 + B_2$ is maximal monotone.*

Lemma 2.5 (Pascali and Sburlan [10]). *If $A : X \to 2^{X^*}$ is maximal monotone and coercive, then $R(A) = X^*$.*

Definition 2.6 (Kamimura and Takahashi [13]). Let X be a real smooth Banach space. Then the *Lyapunov functional* $\varphi : X \times X \to R^+$ is defined as follows:

$$\varphi(x, y) = \|x\|^2 - 2(x, Jy) + \|y\|^2, \quad \forall x, y \in X, \tag{2.2}$$

where $J : X \to 2^{X^*}$ is the duality mapping defined by $Jx = \{f \in X^* : (x, f) = \|x\|\|f\|, \|f\| = \|x\|\}$, for $x \in X$.

Lemma 2.7 (Kamimura and Takahashi [13]). *Let X be a real reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed and convex subset of X, and let $x \in X$. Then there exists a unique element $x_0 \in C$ such that*

$$\varphi(x_0, x) = \min\{\varphi(z, x) : z \in C\}. \tag{2.3}$$

Define a mapping Π_C from X onto C by $\Pi_C x = x_0$ for all $x \in X$. Then Π_C is called the *generalized projection mapping* from X onto C. It is easy to see that Π_C coincides with the metric projection P_C in a Hilbert space.

3. Main Results

3.1. Notations and Assumptions of (1.4)

In the following of this paper, unless otherwise stated, we will assume that $2N/(N+1) < p < +\infty$, $1 \le q < +\infty$ if $p \ge N$, and $1 \le q \le Np/(N-p)$ if $p < N$, for $N \ge 1$. Let $1/p + 1/p' = 1$. We use $\|\cdot\|_p$, $\|\cdot\|_{p'}$, and $\|\cdot\|_{1,p,\Omega}$ to denote the norm of spaces $L^p(\Omega)$, $L^{p'}(\Omega)$, and $W^{1,p}(\Omega)$, respectively.

In nonlinear boundary value problem (1.4), Ω is a bounded conical domain of an Euclidean space R^N with its boundary $\Gamma \in C^1$ (see Wei and He [1]). We will assume that Green's formula is available. $f(x) \in L^{p'}(\Omega)$ is a given function. $0 \le C(x) \in L^p(\Omega)$, ε is a nonnegative constant and ϑ denotes the exterior normal derivative of Γ.

Let $\varphi : \Gamma \times R \to R$ be a given function such that, for each $x \in \Gamma$, $\varphi_x = \varphi(x, \cdot) : R \to R$ is a proper, convex, and lower semicontinuous function with $\varphi_x(0) = 0$. Let β_x be the subdifferential of φ_x, that is, $\beta_x \equiv \partial\varphi_x$. Suppose that $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Suppose that $g : \Omega \times R^{N+1} \to R$ is a given function satisfying the following conditions, which can be found in Zeidler [14].

(a) Carathéodory's conditions:

$$\begin{gathered} x \longrightarrow g(x, r) \text{ is measurable on } \Omega \quad \forall r \in R^{N+1}, \\ r \longrightarrow g(x, r) \text{ is continuous on } R^{N+1} \quad \text{for almost all } x \in \Omega. \end{gathered} \tag{3.1}$$

(b) Growth condition:

$$g(x, r_1, \ldots, r_{N+1}) \le h_1(x) + b \sum_{i=1}^{N+1} |r_i|^{p-1}, \tag{3.2}$$

where $(r_1, r_2, \ldots, r_{N+1}) \in R^{N+1}$, $h_1(x) \in L^p(\Omega)$ and b is a fixed positive constant.

(c) Monotone condition: g is monotone with respect to r_1, that is,

$$\left(g(x, r_1, \ldots, r_{N+1}) - g(x, t_1, \ldots, t_{N+1})\right)(r_1 - t_1) \geq 0, \tag{3.3}$$

for all $x \in \Omega$ and $(r_1, \ldots, r_{N+1}), (t_1, \ldots, t_{N+1}) \in R^{N+1}$.

3.2. Existence and Uniqueness of the Solution of (1.4)

Lemma 3.1. *Define the mapping* $B_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *by*

$$(v, B_{p,q}u) = \int_\Omega \left\langle \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u, \nabla v \right\rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) v(x) dx \tag{3.4}$$

for any $u, v \in W^{1,p}(\Omega)$. *Then,* $B_{p,q}$ *is everywhere defined, strictly monotone, hemicontinuous, and coercive.*

Moreover, Lemma 2.2 implies that $B_{p,q}$ is maximal monotone.

Proof. From Lemma 3.2 by Wei and Agarwal [7], we know that $B_{p,q}$ is everywhere defined, monotone, hemicontinuous, and coercive. Then we only need to show that $B_{p,q}$ is strictly monotone.

In fact, for any $u, v \in W^{1,p}(\Omega)$,

$$\begin{aligned}
&(u - v, B_{p,q}u - B_{p,q}v) \\
&\quad = \int_\Omega \left\langle \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u - \left(C(x) + |\nabla v|^2\right)^{(p-2)/2} \nabla v, \nabla u - \nabla v \right\rangle dx \\
&\qquad + \varepsilon \int_\Omega \left(|u|^{q-2}u - |v|^{q-2}v\right)(u - v) dx \\
&\quad \geq \int_\Omega \left[\left(C(x) + |\nabla u|^2\right)^{(p-2)/2} |\nabla u|^2 - \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} |\nabla u||\nabla v| \right. \\
&\qquad\qquad \left. - \left(C(x) + |\nabla v|^2\right)^{(p-2)/2} |\nabla v||\nabla u| + \left(C(x) + |\nabla v|^2\right)^{(p-2)/2} |\nabla v|^2\right] dx \\
&\qquad + \varepsilon \int_\Omega \left(|u|^q - |u|^{q-1}|v| - |v|^{q-1}|u| + |v|^q\right) dx \\
&\quad = \int_\Omega \left[\left(C(x) + |\nabla u|^2\right)^{(p-2)/2} |\nabla u| - \left(C(x) + |\nabla v|^2\right)^{(p-2)/2} |\nabla v|\right] (|\nabla u| - |\nabla v|) dx \\
&\qquad + \varepsilon \int_\Omega \left(|u|^{q-1} - |v|^{q-1}\right)(|u| - |v|) dx.
\end{aligned} \tag{3.5}$$

Define $h(t) = (k + t^2)^{(p-2)/2} t$, for all $t > 0$, where k is a nonnegative constant.

Case 1. If $p \geq 2$, then $h'(t) = (p - 2)(k + t^2)^{p/2-2} t^2 + (k + t^2)^{p/2-1} > 0$, for all $t > 0$. That is, $h : R \to R$ is strictly increasing. Thus we can see from (3.5) that if $(u - v, B_{p,q}u - B_{p,q}v) = 0$, then $u(x) = v(x)$ and $\partial u/\partial x_i = \partial v/\partial x_i$ $(i = 1, 2, \ldots, N)$, a.e. in Ω, which implies that $u(x) = v(x)$ in $W^{1,p}(\Omega)$. Therefore, $B_{p,q}$ is strictly monotone.

Case 2. If $2N/(N+1) < p < 2$, then for all $t > 0$,

$$\begin{aligned} h'(t) &= (p-2)\left(k+t^2\right)^{p/2-2} t^2 + \left(k+t^2\right)^{p/2-1} \\ &= (p-2)\left(k+t^2\right)^{p/2-1} - (p-2)\left(k+t^2\right)^{p/2-2} k + \left(k+t^2\right)^{p/2-1} \\ &= (p-1)\left(k+t^2\right)^{p/2-1} - (p-2)\left(k+t^2\right)^{p/2-2} k \\ &= (p-1)\left(k+t^2\right)^{p/2-1} + (2-p)\left(k+t^2\right)^{p/2-2} \quad k > 0, \end{aligned} \tag{3.6}$$

which implies that $h : R \to R$ is also strictly increasing. In the same way as Case 1, we know that $B_{p,q}$ is strictly monotone.

This completes the proof. □

Lemma 3.2 (see Wei and Agarwal [7]). *The mapping $\Phi_p : W^{1,p}(\Omega) \to R$ defined by $\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x))d\Gamma(x)$, for any $u \in W^{1,p}(\Omega)$, is proper, convex, and lower semicontinuous on $W^{1,p}(\Omega)$.*

Therefore, $\partial\Phi_p$ is maximal monotone in view of Lemma 2.3.

Lemma 3.3. *Define the mapping $F : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by*

$$(v, Fu(x)) = \int_\Omega g(x, u(x), \nabla u(x)) v(x) dx, \tag{3.7}$$

for all $u, v \in W^{1,p}(\Omega)$; then F is everywhere defined, monotone, and hemicontinuous on $W^{1,p}(\Omega)$.

Moreover, Lemma 2.2 implies that F is maximal monotone.

Proof. We split our proof into four steps.

Step 1. For $u \in W^{1,p}(\Omega)$, $x \to g(x, u(x), \nabla u(x))$ is measurable on Ω.

From the facts that $u(x), \partial u/\partial x_i \in L^p(\Omega)$, $i = 1, 2, \ldots, N$, we know that $x \to (u(x), \partial u/\partial x_1, \ldots, \partial u/\partial x_N)$ is measurable on Ω. Combining with the fact that g satisfies Carathéodory's conditions, we know that $x \to g(x, u(x), \nabla u(x))$ is measurable on Ω.

Step 2. F is everywhere defined.

From Lemma 2.1, we know that $W^{1,p}(\Omega) \hookrightarrow C_B(\Omega)$, when $p > N$, and $W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega)$, when $p = N$. Thus, for all $w \in W^{1,p}(\Omega)$,

$$\|w\|_{p'} \le k\|w\|_{1,p,\Omega}, \tag{3.8}$$

where $k > 0$ is a constant.

When $p < N$, we know from Lemma 2.1 that $W^{1,p}(\Omega) \hookrightarrow L^{Np/(N-p)}(\Omega)$. Since $2N/(N+1) < p < +\infty$, then $Np/(N-p) > p'$ and $L^{Np/(N-p)}(\Omega) \hookrightarrow L^{p'}(\Omega)$, which implies that (3.8) is still true.

Now, for $u, v \in W^{1,p}(\Omega)$, we have from (3.8) that

$$\begin{aligned} |(v, Fu)| &\leq \int_\Omega |h_1(x)||v|dx + b\int_\Omega |u|^{p-1}|v|dx + b\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p-1}|v|dx \\ &\leq \|h_1(x)\|_p\|v\|_{p'} + b\|u\|_p^{p/p'}\|v\|_p + b\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_p^{p/p'}\|v\|_p \\ &\leq \|h_1(x)\|_p\|v\|_{1,p,\Omega} + b(N+1)\|u\|_{1,p,\Omega}^{p/p'}\|v\|_{1,p,\Omega}, \end{aligned} \tag{3.9}$$

which implies that F is everywhere defined.

Step 3. F is monotone.

Since $g(x, r_1, \ldots, r_{N+1})$ is monotone with respect to r_1, then F is monotone.

Step 4. F is hemicontinuous.

In fact, it suffices to show that, for any $u, v, w \in W^{1,p}(\Omega)$ and $t \in [0,1]$, $(w, F(u+tv) - Fu) \to 0$, as $t \to 0$.

Noticing the facts that g is measurable on Ω and g satisfies Carathéodory's conditions, by using Lebesque's dominated convergence theorem, we have

$$\begin{aligned} 0 &\leq \lim_{t\to 0}|(w, F(u+tv) - Fu)| \\ &\leq \int_\Omega \lim_{t\to 0}\left|g(x, u(x)+tv(x), \nabla u + t\nabla v) - g(x, u, \nabla u)\right||w|dx = 0, \end{aligned} \tag{3.10}$$

and hence F is hemicontinuous.

This completes the proof. □

In view of Lemma 2.4, we can easily obtain the following result.

Lemma 3.4. $B_{p,q} + F + \partial\Phi_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *is maximal monotone.*

Lemma 3.5. *Define the mapping* $S : L^p(\Gamma) \to L^{p'}(\Gamma)$ *by* $Su = \beta_x(u(x))$, *for any* $u \in L^p(\Gamma)$. *Define the mapping* $K : W^{1,p}(\Omega) \to L^p(\Gamma)$ *by* $Kv = v|_\Gamma$ *for any* $v \in W^{1,p}(\Omega)$. *Then* $K^*SK = \partial\Phi_p$, *where* Φ_p *is the same as that in Lemma 3.2.*

Proof. We will prove the result under the additional condition $|\beta_x(u)| \leq a|u|^{p/p'} + b(x)$, where $b(x) \in L^{p'}(\Gamma)$ and $a \in R$. Refer to the result of Brezis [15] for the general case.

It is obvious that S is continuous. For all $u(x), v(x) \in L^p(\Gamma)$, since β_x is monotone, then $(u - v, Su - Sv) = \int_\Gamma (\beta_x(u(x)) - \beta_x(v(x)))(u(x) - v(x))d\Gamma(x) \geq 0$, which implies that S is monotone. Thus, $S : L^p(\Gamma) \to L^{p'}(\Gamma)$ is maximal monotone in view of Lemma 2.2.

Define $\Psi : L^p(\Gamma) \to R$ by $\Psi(u) = \int_\Gamma \varphi_x(u(x))d\Gamma(x)$, for all $u(x) \in L^p(\Gamma)$; then it is easy to see that Ψ is a proper, convex, and lower semicontinuous function on $L^p(\Gamma)$, which implies that $\partial\Psi : L^p(\Gamma) \to L^{p'}(\Gamma)$ is maximal monotone. Since $\Psi(u) - \Psi(v) = \int_\Gamma [\varphi_x(u(x)) - \varphi_x(v(x))]d\Gamma(x) \geq \int_\Gamma \beta_x(v(x))(u(x) - v(x))d\Gamma(x) = (Sv, u - v)$, for all $u(x), v(x) \in L^p(\Gamma)$, then $Bv \in \partial\Psi(v)$. So $S = \partial\Psi$.

Now clearly, $K^*SK : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is maximal monotone since both K, S are continuous. Finally, for any $u, v \in W^{1,p}(\Omega)$, we have

$$\begin{aligned} \Phi_p(v) - \Phi_p(u) &= \Psi(Kv) - \Psi(Ku) \\ &= \int_\Gamma \left[\varphi_x(v|_\Gamma(x)) - \varphi_x(u|_\Gamma(x))\right] d\Gamma(x) \\ &\geq \int_\Gamma \beta_x(u|_\Gamma(x))(v|_\Gamma(x) - u|_\Gamma(x))d\Gamma(x) \\ &= (SKu, Kv - Ku) = (K^*SKu, v - u). \end{aligned} \tag{3.11}$$

Hence we get $K^*SK \subset \partial\Phi_p$ and so $K^*SK = \partial\Phi_p$.

This completes the proof. □

Lemma 3.6. *One has* $R(B_{p,q} + F + \partial\Phi_p) = (W^{1,p}(\Omega))^*$.

Proof. From Lemma 3.5, we know that for all $u \in W^{1,p}(\Omega)$,

$$(u, \partial\Phi_p(u)) = \int_\Gamma \beta_x(u|_\Gamma(x))\, u|_\Gamma(x) d\Gamma(x). \tag{3.12}$$

Since $0 \in \beta_x(0)$, then for all $u \in W^{1,p}(\Omega)$, $(u, \partial\Phi_p(u)) \geq 0$.

Since F is monotone, then $(u, Fu - F0) \geq 0$. Moreover, in view of (3.8), we have

$$\begin{aligned} \frac{|(u, F0)|}{\|u\|_{1,p,\Omega}} &\leq \frac{\int_\Omega |h_1(x)||u(x)|dx}{\|u\|_{1,p,\Omega}} \\ &\leq \|h_1(x)\|_p \frac{\|u\|_{p'}}{\|u\|_{1,p,\Omega}} \leq k\|h_1(x)\|_p < +\infty. \end{aligned} \tag{3.13}$$

Then by using Lemma 3.1, we have

$$\frac{(u, B_{p,q}u + Fu + \partial\Phi_p(u))}{\|u\|_{1,p,\Omega}} \geq \frac{(u, B_{p,q}u)}{\|u\|_{1,p,\Omega}} + \frac{(u, F0)}{\|u\|_{1,p,\Omega}} \longrightarrow +\infty, \tag{3.14}$$

as $\|u\|_{1,p,\Omega} \to +\infty$, which implies that $B_{p,q} + F + \partial\Phi_p$ is coercive. Then Lemmas 3.4 and 2.5 ensure that the result is true.

This completes the proof. □

Theorem 3.7. *For $f \in L^{p'}(\Omega)$, nonlinear boundary value problem* (1.4) *has a unique solution $u \in W^{1,p}(\Omega)$.*

Proof. From Lemma 3.6, we know that for $f \in L^{p'}(\Omega)$, there exists $u \in W^{1,p}(\Omega)$ such that

$$f = B_{p,q}u + \partial\Phi_p(u) + Fu. \tag{3.15}$$

Now, we will show that this u is unique.

Otherwise, there exists $v \in W^{1,p}(\Omega)$ satisfying (3.15). Then notice the facts that $B_{p,q}$, $\partial\Phi_p$ and F are all monotone, we have

$$\begin{aligned} 0 &\le (u - v, B_{p,q}u - B_{p,q}v) \\ &= (u - v, f - Fu - \partial\Phi_p(u) - f + Fv + \partial\Phi_p(v)) \\ &= (u - v, Fv - Fu) + (u - v, \partial\Phi_p(v) - \partial\Phi_p(u)) \le 0. \end{aligned} \tag{3.16}$$

Since $B_{p,q}$ is strictly monotone, then $u = v$.

Next, we will show that this u is the solution of (1.4).

Since $\Phi_p(u+\varphi) = \Phi_p(u)$ for any $u \in W^{1,p}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, where $2N/(N+1) < p < +\infty$ and $N \ge 1$, we have $(\varphi, \partial\Phi(u)) = 0$. Since $f - Fu = B_{p,q}u + \partial\Phi_p(u)$, then for all $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} &\int_\Omega (f - g(x, u(x), \nabla u(x)))\varphi dx \\ &\quad= \int_\Omega \left\langle \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u, \nabla\varphi \right\rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x)\varphi(x) dx \\ &\quad= -\int_\Omega \operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u\right] \varphi dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x)\varphi dx, \end{aligned} \tag{3.17}$$

which implies that the result

$$-\operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u\right] + \varepsilon|u|^{q-2}u + g(x, u(x), \nabla u(x)) = f(x) \quad \text{a.e. } x \in \Omega \tag{3.18}$$

is true.

From (3.18), we know that $f(x) = -\operatorname{div}[(C(x) + |\nabla u|^2)^{(p-2)/2} \nabla u] + \varepsilon|u|^{q-2}u + g(x, u(x), \nabla u(x)) \in L^{p'}(\Omega)$. By using Green's formula, we have that for any $v \in W^{1,p}(\Omega)$,

$$\begin{aligned} &\int_\Gamma \left\langle \vartheta, \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u \right\rangle v|_\Gamma d\Gamma(x) \\ &\quad= \int_\Omega \operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u\right] v dx + \int_\Omega \left\langle \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u, \nabla v \right\rangle dx. \end{aligned} \tag{3.19}$$

Then $-\langle \vartheta, (C(x)+|\nabla u|^2)^{(p-2)/2}\nabla u\rangle \in W^{-1/p,p'}(\Gamma) = (W^{1/p,p}(\Gamma))^*$, where $W^{1/p,p}(\Gamma)$ is the space of the traces of $W^{1,p}(\Omega)$.

Combining with the results of Lemma 3.5, (3.18), and (3.19), we have

$$\begin{aligned}
&\int_\Gamma \left\langle \vartheta, \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u \right\rangle v|_\Gamma d\Gamma(x) \\
&\quad = \left(v, \varepsilon|u(x)|^{q-2}u(x) + g(x,u(x),\nabla u(x)) - f\right) + \left(v, B_p u - \varepsilon|u(x)|^{q-2}u(x)\right) \\
&\quad = (v, B_p u + Fu - f) = (v, -\partial\Phi_p(u)) = (v, -K^*SKu) = -(Kv, SKu) \\
&\quad = -\int_\Gamma \beta_x(u|_\Gamma(x))\, v|_\Gamma(x) d\Gamma(x).
\end{aligned} \tag{3.20}$$

Then

$$-\left\langle \vartheta, \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u \right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma. \tag{3.21}$$

From (3.19) and (3.21) we know that u is the solution of (1.4).

This completes the proof. □

3.3. *Iterative Construction of the Solution of* (1.4)

Lemma 3.8 (Wei and Zhou [16]). *Suppose that X is a real smooth and uniformly convex Banach space and $A : X \to 2^{X^*}$ is a maximal monotone operator with $A^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative scheme:*

$$\begin{gathered}
x_1 \in X, \qquad r_1 > 0, \\
x_{n+1} = J^{-1}\left[\beta_n J x_1 + (1-\beta_n) J (J + r_n A)^{-1} J x_n\right], \quad \forall n \geq 1.
\end{gathered} \tag{3.22}$$

If $\{r_n\} \subset (0,+\infty)$ with $\lim_{n\to\infty} r_n = +\infty$, $\{\beta_n\} \subset [0,1]$ with $\sum_{n=1}^{\infty}\beta_n = +\infty$, and $\beta_n \to 0$ as $n \to \infty$, then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}(0)}(x_1)$, where $\Pi_{A^{-1}(0)}$ is the generalized projection operator from X onto $A^{-1}(0)$.

Definition 3.9. Define the mapping $C_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by

$$\begin{aligned}
(v, C_{p,q}u) = &\int_\Omega \left\langle \left(C(x)+|\nabla u|^2\right)^{(p-2)/2}\nabla u, \nabla v \right\rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2}u(x)v(x)dx \\
&- \int_\Omega f(x)v(x)dx + \int_\Omega g(x,u(x),\nabla u(x))v(x)dx + (v, \partial\Phi_p(u))
\end{aligned} \tag{3.23}$$

for any $u, v \in W^{1,p}(\Omega)$, where $f \in L^{p'}(\Omega)$ is the same as that in (1.4).

Similarly to Lemma 3.4, we know that $C_{p,q}$ is also a maximal monotone operator. Moreover, we can easily get the following result.

Lemma 3.10. *$u \in W^{1,p}(\Omega)$ is the solution of (1.4) if and only if $u \in W^{1,p}(\Omega)$ is the zero point of $C_{p,q}$.*

Proof. Let $u(x)$ be the solution of (1.4), then for all $v \in W^{1,p}(\Omega)$, by using Green's formula and Lemma 3.5, we have

$$\begin{aligned}
(v, C_{p,q}u) &= \int_\Omega \left\langle \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u, \nabla v \right\rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) v(x) dx \\
&\quad - \int_\Omega f(x) v(x) dx + \int_\Omega g(x, u(x), \nabla u(x)) v(x) dx + (v, \partial\Phi_p(u)) \\
&= -\int_\Omega \operatorname{div}\left[\left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u\right] v dx \\
&\quad + \int_\Gamma \left\langle \vartheta, \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u \right\rangle v|_\Gamma d\Gamma(x) \\
&\quad + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) v(x) dx - \int_\Omega f(x) v(x) dx \\
&\quad + \int_\Omega g(x, u(x), \nabla u(x)) v(x) dx + (v, K^* S K u) \\
&= \int_\Gamma \left\langle \vartheta, \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u \right\rangle v|_\Gamma d\Gamma(x) + (v, K^* S K u) \\
&= -\int_\Gamma \beta_x(u|_\Gamma) v|_\Gamma d\Gamma(x) + \int_\Gamma \beta_x(u|_\Gamma) v|_\Gamma d\Gamma(x) = 0.
\end{aligned} \tag{3.24}$$

Thus $u \in C_{p,q}^{-1}(0)$.

If $u \in C_{p,q}^{-1}(0)$, then for all $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
0 = &\int_\Omega \left\langle \left(C(x) + |\nabla u|^2\right)^{(p-2)/2} \nabla u, \nabla \varphi \right\rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) \varphi(x) dx \\
&- \int_\Omega f \varphi dx + \int_\Omega g(x, u(x), \nabla u(x)) \varphi dx,
\end{aligned} \tag{3.25}$$

which implies the result $-\operatorname{div}[(C(x) + |\nabla u|^2)^{(p-2)/2} \nabla u] + \varepsilon |u(x)|^{q-2} u(x) + g(x, u(x), \nabla u(x)) = f(x)$, a.e. $x \in \Omega$ is true.

Copying the last part of Theorem 3.7, we can obtain (3.21), which implies that u is the solution of (1.4).

This completes the proof. □

Remark 3.11. From Lemma 3.10 we can see that $C_{p,q}^{-1}(0) \neq \emptyset$. It is a good example to show that the assumption that $A^{-1}(0) \neq \emptyset$ in Lemma 3.8 is valid.

Lemma 3.12 (Takahashi [17]). *Let X be a Banach space and J a duality mapping defined on X. If J is single valued, then X is smooth.*

Based on the facts of Lemmas 3.8, 3.10, and 3.12, we can construct an iterative sequence to approximate strongly to the solution of (1.4).

Theorem 3.13. *Let $\{u_n(x)\}$ be a sequence generated by the following iterative scheme:*

$$\begin{gathered} u_1(x) \in W^{1,p}(\Omega), \quad r_1 > 0, \text{ chosen arbitrarily}, \\ u_{n+1}(x) = J^{-1}\left[\beta_n J u_1(x) + (1-\beta_n) J\left(J + r_n C_{p,q}\right)^{-1} J u_n(x)\right], \quad \forall n \geq 1. \end{gathered} \tag{3.26}$$

If $\{r_n\} \subset (0,+\infty)$ with $\lim_{n\to\infty} r_n = +\infty$, $\{\beta_n\} \subset [0,1]$ with $\sum_{n=1}^{\infty} \beta_n = +\infty$, and $\beta_n \to 0$ as $n \to \infty$, then $\{u_n(x)\}$ converges strongly to $\Pi_{C_{p,q}^{-1}(0)}(u_1(x))$.

Remark 3.14. Theorem 3.13 not only tells us that the sequence $\{u_n(x)\}$ generated by (3.26) converges strongly to the solution of (1.4), but also tells us that the unique solution of (1.4) is the generalized projection of the initial function $u_1(x)$ onto $C_{p,q}^{-1}(0)$.

Remark 3.15. Compared to the work done in [8], we may find that different techniques are employed to show the existence and uniqueness of the solution of the desired equation because the nonlinear item $g(x,u(x),\nabla u(x))$ is involved.

Remark 3.16. We can get the following special cases in our paper.

Corollary 3.17. *For $f \in L^2(\Omega)$, the following equation has a unique solution in $H^1(\Omega)$:*

$$\begin{gathered} -\Delta u + \varepsilon |u|^{q-2} u + g(x,u(x),\nabla u(x)) = f(x), \quad \text{a.e. in } \Omega, \\ -\frac{\partial u}{\partial \vartheta} \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma. \end{gathered} \tag{3.27}$$

Corollary 3.18. *Let $\{u_n(x)\}$ be a sequence generated by*

$$\begin{gathered} u_1(x) \in H^1(\Omega), \quad r_1 > 0, \text{ chosen arbitrarily}, \\ u_{n+1}(x) = \beta_n u_1(x) + (1-\beta_n)\left(I + r_n C_{2,q}\right)^{-1} u_n(x), \quad \forall n \geq 1, \end{gathered} \tag{3.28}$$

where $C_{2,q}$ is a special case of $C_{p,q}$ defined in Definition 3.9 if $p \equiv 2$.

If $\{r_n\} \subset (0,+\infty)$ with $\lim_{n\to\infty} r_n = +\infty$, $\{\beta_n\} \subset [0,1]$ with $\sum_{n=1}^{\infty} \beta_n = +\infty$, and $\beta_n \to 0$ as $n \to \infty$, then $\{u_n(x)\}$ converges strongly to $P_{C_{2,q}^{-1}(0)}(u_1(x))$, where $P_{C_{2,q}^{-1}(0)}(u_1(x))$ denotes the metric projection from $H^1(\Omega)$ onto $C_{2,q}^{-1}(0)$. And, $P_{C_{2,q}^{-1}(0)}(u_1(x))$ is the unique solution of (3.27).

Corollary 3.19. *For $f \in L^{p'}(\Omega)$, the following equation has a unique solution in $W^{1,p}(\Omega)$:*

$$\begin{gathered} -\Delta_p u + \varepsilon |u|^{q-2} u + g(x,u(x),\nabla u(x)) = f(x), \quad \text{a.e. in } \Omega, \\ -\left\langle \vartheta, |\nabla u|^{p-2} \nabla u \right\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma. \end{gathered} \tag{3.29}$$

Corollary 3.20. *Let* $\{u_n(x)\}$ *be a sequence generated by* (3.26), *where* $C_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *is defined by*

$$(v, C_{p,q}u) = \int_\Omega \left\langle |\nabla u|^{p-2}\nabla u, \nabla v \right\rangle dx + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) v(x) dx \\ - \int_\Omega f(x) v(x) dx + \int_\Omega g\,(x, u(x), \nabla u(x)) v(x) dx + \left(v, \partial \Phi_p(u)\right), \tag{3.30}$$

for $u(x), v(x) \in W^{1,p}(\Omega)$.

Then under the conditions of Theorem 3.13, we know that $\{u_n(x)\}$ converges strongly to both the unique solution of (3.29) and the zero point of $C_{p,q}$.

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