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*Research Article*

# A Fixed Point Approach to the Stability of a Cauchy-Jensen Functional Equation

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We find out the general solution of a generalized Cauchy-Jensen functional equation and prove its stability. In fact, we investigate the existence of a Cauchy-Jensen mapping related to the generalized Cauchy-Jensen functional equation and prove its uniqueness. In the last section of this paper, we treat a fixed point approach to the stability of the Cauchy-Jensen functional equation.

## 1. Introduction

In 1940, Ulam [1] gave a wide-range talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphisms.

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Rassias [3] gave a generalization of Hyers's result. Many authors investigated solutions or stability of various functional equations (see [4–7]).

Let $X$ be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

In this paper, let $X$ and $Y$ be two real vector spaces.

*Definition 1.1.* A mapping $f : X \times X \to Y$ is called a *Cauchy-Jensen mapping* if $f$ satisfies the system of equations:

$$\begin{aligned} f(x+y,z) &= f(x,z)+f(y,z), \\ 2f\left(x,\frac{y+z}{2}\right) &= f(x,y)+f(x,z). \end{aligned} \tag{1.1}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x,y) := axy+bx$ is a solution of (1.1).

For a mappings $f : X \times X \to Y$, consider the functional equation:

$$nf\left(\sum_{i=1}^{n} x_i, \frac{1}{n}\sum_{j=1}^{n} y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} f(x_i,y_j), \tag{1.2}$$

where $n$ is a fixed integer greater than 1. In 2006, the authors [8] solved the functional equation:

$$2f\left(x+y,\frac{z+w}{2}\right) = f(x,z)+f(x,w)+f(y,z)+f(y,w), \tag{1.3}$$

which is a special case of (1.2) for $n = 2$.

In this paper, we find out the general solution and we prove the generalized Hyers-Ulam stability of the functional equation (1.2).

## 2. General Solution of (1.2)

The following lemma ia a well-known fact (see, e.g., [6]).

**Lemma 2.1.** *A mapping $g : X \to Y$ satisfies Jensen's functional equation:*

$$2g\left(\frac{y+z}{2}\right) = g(y)+g(z) \tag{2.1}$$

*for all $y, z \in X$ if and only if it satisfies the generalized Jensen's functional equation:*

$$ng\left(\frac{y_1+\cdots+y_n}{n}\right) = g(y_1)+\cdots+g(y_n) \tag{2.2}$$

*for all $y_1,\ldots,y_n \in X$.*

**Theorem 2.2.** *A mapping $f : X \times X \to Y$ satisfies* (1.1) *if and only if it satisfies* (1.2).

*Proof.* If $f$ satisfies (1.1), then we get

$$nf\left(\sum_{i=1}^{n}x_i, \frac{1}{n}\sum_{j=1}^{n}y_j\right) = n\sum_{i=1}^{n}f\left(x_i, \frac{1}{n}\sum_{j=1}^{n}y_j\right), \tag{2.3}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Hence, we obtain that $f$ satisfies (1.2) by Lemma 2.1.

Conversely, assume that $f$ satisfies (1.2). Letting $x_1 = \cdots = x_n = 0$ and $y_1 = \cdots = y_n = z$ in (1.2), we get $f(0, z) = 0$ for all $z \in X$. Putting $x_1 = x, x_2 = y, x_3 = \cdots = x_n = 0$, and $y_1 = \cdots = y_n = z$ in (1.2), we have

$$f(x + y, z) = f(x, z) + f(y, z) \tag{2.4}$$

for all $x, y, z \in X$. Setting $x_1 = x$ and $x_2 = \cdots = x_n = 0$ in (1.2), we obtain that

$$nf\left(x, \frac{1}{n}\sum_{j=1}^{n}y_j\right) = \sum_{j=1}^{n}f(x, y_j) \tag{2.5}$$

for all $x, y_1, \ldots, y_n \in X$. By Lemma 2.1, we see that

$$2f\left(x, \frac{y+z}{2}\right) = f(x, y) + f(x, z), \tag{2.6}$$

for all $x, y, z \in X$. □

## 3. Stability of (1.3) Using the Alternative of Fixed Point

In this section, let $Y$ be a real Banach space. We investigate the stability of functional equation (1.3) using the alternative of fixed point. Before proceeding the proof, we will state the theorem which is the alternative of fixed point.

**Theorem 3.1** (The alternative of fixed point [9]). *Suppose that one is given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant $L$. Then, for each given $x \in \Omega$, either*

$$d\left(T^n x, T^{n+1}x\right) = \infty \quad \forall n \geq 0, \tag{3.1}$$

*Or there exists a positive integer $n_0$ such that*

(i) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;*

(ii) *the sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;*

(iii) *$y^*$ is the unique fixed point of $T$ in the set $\Delta = \{y \in \Omega \mid d(T^{n_0}x, y) < \infty\}$;*

(iv) *$d(y, y^*) \leq 1/(1 - L)d(y, Ty)$ for all $y \in \Delta$.*

*From now on, let $\Omega$ be the set of all mappings $g : X \times X \to Y$ satisfying $g(0, 0) = 0$.*

**Lemma 3.2.** *Let $\psi : X \times X \to [0, \infty)$ be a function. Consider the generalized metric $d$ on $\Omega$ given by*

$$d(g,h) = d_\psi(g,h) := \inf S_\psi(g,h), \tag{3.2}$$

*where $S_\psi(g,h) := \{K \in [0,\infty] \mid \|g(x,y) - h(x,y)\| \le K\psi(x,y) \text{ forall } x,y \in X\}$ for all $g,h \in \Omega$. Then, $(\Omega, d)$ is complete.*

*Proof.* Let $\{g_n\}$ be a Cauchy sequence in $(\Omega, d)$. Then, given $\varepsilon > 0$, there exists $N$ such that $d(g_n, g_k) < \varepsilon$ if $n, k \ge N$. Let $n, k \ge N$. Since $d(g_n, g_k) = \inf S_\psi(g_n, g_k) < \varepsilon$, there exists $K \in [0, \varepsilon)$ such that

$$\|g_n(x,y) - g_k(x,y)\| \le K\psi(x,y) \le \varepsilon\psi(x,y) \tag{3.3}$$

for all $x, y \in X$. So, for each $x, y \in X$, $\{g_n(x,y)\}$ is a Cauchy sequence in $Y$. Since $Y$ is complete, for each $x, y \in X$, there exists $g(x,y) \in Y$ such that $g_n(x,y) \to g(x,y)$ as $n \to \infty$. So $g(0,0) = \lim_{n\to\infty} g_n(0,0) = 0$. Thus, we have $g \in \Omega$. Taking the limit as $k \to \infty$ in (3.3), we obtain that

$$\begin{aligned} n \ge N &\Longrightarrow \|g_n(x,y) - g(x,y)\| \le \varepsilon\psi(x,y), \quad \forall x, y \in X \\ &\Longrightarrow \varepsilon \in S_\psi(g_n, g) \\ &\Longrightarrow d(g_n, g) = \inf S_\psi(g_n, g) \le \varepsilon. \end{aligned} \tag{3.4}$$

Hence, $g_n \to g \in \Omega$ as $n \to \infty$. □

Using an idea of Cădariu and Radu (see [10] and also [4] where applications of different fixed point theorems to the theory of the Hyers-Ulam stability can be found), we will prove the generalized Hyers-Ulam stability of (1.3).

**Theorem 3.3.** *Let $L \in (0,1)$ and $\varphi$ satisfy*

$$\varphi(x,y,z,w) \le 6L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{3}, \frac{w}{3}\right) \tag{3.5}$$

*for all $x, y, z, w \in X$. Suppose that a mapping $f : X \times X \to Y$ fulfils $f(0,0) = 0$ and the functional inequality:*

$$\left\|2f\left(x+y, \frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w)\right\| \le \varphi(x,y,z,w) \tag{3.6}$$

*for all $x, y, z, w \in X$. Then, there exists a unique mapping $F : X \times X \to Y$ satisfying* (1.3) *such that*

$$\|f(x,y) - F(x,y)\| \le \frac{L}{1-L}\psi(x,y), \tag{3.7}$$

*where* $\psi : X \times X \to [0,\infty)$ *is a function given by*

$$\psi(x,y) := \varphi(x,x,y,-y) + 2\varphi(x,x,-y,y) + \varphi(x,x,y,y) + \varphi(x,x,-y,3y) + \frac{1}{2}\varphi(x,x,3y,3y) \tag{3.8}$$

*for all* $x,y \in X$.

*Proof.* By a similar method to the proof of Theorem 2.3 in [11], we have the inequality:

$$(\|6f(x,y) - f(2x,3y)\|) \le \varphi(x,x,y,-y) + 2\varphi(x,x,-y,y) + \varphi(x,x,y,y) + \varphi(x,x,-y,3y) + \frac{1}{2}\varphi(x,x,3y,3y) \tag{3.9}$$

for all $x,y \in X$. By (3.5), we get

$$\|6f(x,y) - f(2x,3y)\| \le \psi(x,y) \le 6L\psi\left(\frac{x}{2},\frac{y}{3}\right) \tag{3.10}$$

for all $x,y \in X$. Consider the generalized metric $d$ on $\Omega$ given by

$$d(g,h) = d_\psi(g,h) := \inf S_\psi(g,h) \tag{3.11}$$

for all $g,h \in \Omega$. Then, we obtain

$$d(f,Tf) \le L < \infty. \tag{3.12}$$

By Lemma 3.2, the generalized metric space $(\Omega,d)$ is complete. Now, we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x,y) := \frac{1}{6}g(2x,3y) \tag{3.13}$$

for all $g \in \Omega$ and all $x,y \in X$. Observe that, for all $g,h \in \Omega$,

$$\begin{aligned} &K' \in S_\psi(g,h), \quad K' < K \\ &\Longrightarrow \|g(x,y) - h(x,y)\| \le K'\psi(x,y) \le K\psi(x,y) \quad \forall x,y \in X \\ &\Longrightarrow K \in S_\psi(g,h). \end{aligned} \tag{3.14}$$

Let $g, h \in \Omega$, $K \in [0, \infty]$ and $d(g, h) < K$. Then, there is a $K' \in S_\psi(g, h)$ such that $K' < K$. By the above observation, we gain $K \in S_\psi(g, h)$. So, we get $\|g(x, y) - h(x, y)\| \le K\psi(x, y)$ for all $x, y \in X$. Thus, we have

$$\left\| \frac{1}{6} g(2x, 3y) - \frac{1}{6} h(2x, 3y) \right\| \le \frac{1}{6} K\psi(2x, 3y) \tag{3.15}$$

for all $x, y \in X$. By (3.5), we obtain that

$$\left\| \frac{1}{6} g(2x, 3y) - \frac{1}{6} h(2x, 3y) \right\| \le LK\psi(x, y) \tag{3.16}$$

for all $x, y \in X$. Hence, $d(Tg, Th) \le LK$. Therefore, we obtain that

$$d(Tg, Th) \le Ld(g, h) \tag{3.17}$$

for all $g, h \in \Omega$, that is, $T$ is a strictly contractive mapping of $\Omega$ with Lipschitz constant $L$. Applying the alternative of fixed point, we see that there exists a fixed point $F$ of $T$ in $\Omega$ such that

$$F(x, y) = \lim_{n \to \infty} \frac{1}{6^n} f(2^n x, 3^n y) \tag{3.18}$$

for all $x, y \in X$. Replacing $x, y, z, w$ by $2^n x, 2^n y, 3^n z, 3^n w$ in (3.6), respectively, and dividing by $4^n$, we have

$$\begin{aligned}
&\|F(x + y, z - w) + F(x - y, z + w) - 2F(x, z) - 2F(y, w)\| \\
&= \lim_{n \to \infty} \frac{1}{6^n} \| f(2^n(x + y), 3^n(z - w)) + f(2^n(x - y), 3^n(z + w)) \\
&\qquad -2f(2^n x, 3^n z) - 2f(2^n y, 3^n w) \| \\
&\le \lim_{n \to \infty} \frac{1}{6^n} \varphi(2^n x, 2^n y, 3^n z, 3^n w)
\end{aligned} \tag{3.19}$$

for all $x, y, z, w \in X$. By (3.5), the mapping $F$ satisfies (1.3). By (3.5) and (3.10), we obtain that

$$\begin{aligned}
\left\| T^n f(x, y) - T^{n+1} f(x, y) \right\| &= \frac{1}{6^n} \left\| f(2^n x, 3^n y) - \frac{1}{6} f\left(2^{n+1} x, 3^{n+1} y\right) \right\| \\
&\le \frac{L}{6^n} \psi\left(2^{n-1} x, 3^{n-1} y\right) \le \cdots \le \frac{L}{6^n} (6L)^{n-1} \psi(x, y) \\
&= \frac{L^n}{6} \psi(x, y)
\end{aligned} \tag{3.20}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \leq L^n/6 < \infty$ for all $n \in \mathbb{N}$. By the fixed point alternative, there exists a natural number $n_0$ such that the mapping $F$ is the unique fixed point of $T$ in the set $\Delta = \{g \in \Omega \mid d(T^{n_0} f, g) < \infty\}$. So, we have $d(T^{n_0} f, F) < \infty$. Since

$$d(f, T^{n_0} f) \leq d(f, Tf) + d\left(Tf, T^2 f\right) + \cdots + d\left(T^{n_0-1} f, T^{n_0} f\right) < \infty, \tag{3.21}$$

we get $f \in \Delta$. Thus, we have $d(f, F) \leq d(f, T^{m_0} f) + d(T^{m_0} f, F) < \infty$. Hence, we obtain

$$\| f(x, y) - F(x, y) \| \leq K \psi(x, y) \tag{3.22}$$

for all $x, y \in X$ and a $K \in [0, \infty)$. Again, using the fixed point alternative, we have

$$d(f, F) \leq \frac{1}{1 - L} d(f, Tf). \tag{3.23}$$

By (3.12), we may conclude that

$$d(f, F) \leq \frac{L}{1 - L}, \tag{3.24}$$

which implies inequality (3.7). □

**Theorem 3.4.** $L \in (0, 1)$ *and* $\varphi$ *satisfy*

$$\varphi(x, y, z, w) \leq \frac{L}{6} \varphi(2x, 2y, 3z, 3w) \tag{3.25}$$

*for all* $x, y, z, w \in X$. *Suppose that a mapping* $f : X \times X \to Y$ *fulfils* $f(0, 0) = 0$ *and the functional inequality* (3.6). *Then, there exists a unique mapping* $F : X \times X \to Y$ *satisfying* (1.3) *such that*

$$\| f(x, y) - F(x, y) \| \leq \frac{1}{1 - L} \psi(x, y), \tag{3.26}$$

*where* $\psi : X \times X \to [0, \infty)$ *is a function given by*

$$\begin{aligned} \psi(x, y) := {} & \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{3}, -\frac{y}{3}\right) + 2\varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{y}{3}, \frac{y}{3}\right) + \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{3}, \frac{y}{3}\right) \\ & + \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{y}{3}, y\right) + \frac{1}{2} \varphi\left(\frac{x}{2}, \frac{x}{2}, y, y\right) \end{aligned} \tag{3.27}$$

*for all* $x, y \in X$.

*Proof.* By a similar method to the proof of Theorem 2.3 in [11], we have the inequality

$$\|6f(x,y) - f(2x,3y)\| \le \varphi(x,x,y,-y) + 2\varphi(x,x,-y,y) + \varphi(x,x,y,y) + \varphi(x,x,-y,3y) + \frac{1}{2}\varphi(x,x,3y,3y) \tag{3.28}$$

for all $x, y \in X$. So, we get

$$\left\| f(x,y) - 6f\left(\frac{x}{2},\frac{y}{3}\right)\right\| \le \psi(x,y) \tag{3.29}$$

for all $x, y \in X$. Consider the generalized metric $d$ on $\Omega$ given by

$$d(g,h) = d_{\psi}(g,h) := \inf S_{\psi}(g,h) \tag{3.30}$$

for all $g, h \in \Omega$. Then, we obtain

$$d(f,Tf) \le 1 < \infty. \tag{3.31}$$

By Lemma 3.2, the generalized metric space $(\Omega, d)$ is complete. Now, we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x,y) := 6g\left(\frac{x}{2},\frac{y}{3}\right) \tag{3.32}$$

for all $g \in \Omega$ and all $x, y \in X$. By the same argument as in the proof of Theorem 2.3 in [11], $T$ is a strictly contractive mapping of $\Omega$ with Lipschitz constant $L$. Applying the alternative of fixed point, we see that there exists a fixed point $F$ of $T$ in $\Omega$ such that

$$F(x,y) = \lim_{n\to\infty} 6^n f\left(\frac{x}{2^n},\frac{y}{3^n}\right) \tag{3.33}$$

for all $x, y \in X$. Replacing $x, y, z, w$ by $x/2^n, y/2^n, z/3^n, w/3^n$ in (3.6), respectively, and multiplying by $6^n$, we have

$$\begin{aligned} &\|F(x+y,z-w) + F(x-y,z+w) - 2F(x,z) - 2F(y,w)\| \\ &\quad = \lim_{n\to\infty} 6^n \left\| f\left(\frac{x+y}{2^n},\frac{z-w}{3^n}\right) + f\left(\frac{x-y}{2^n},\frac{z+w}{3^n}\right) - 2f\left(\frac{x}{2^n},\frac{z}{3^n}\right) - 2f\left(\frac{y}{2^n},\frac{w}{3^n}\right)\right\| \\ &\quad \le \lim_{n\to\infty} 6^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{3^n},\frac{w}{3^n}\right) \end{aligned} \tag{3.34}$$

for all $x, y, z, w \in X$. By (3.25), the mapping $F$ satisfies (1.3). By (3.25), we obtain that

$$\begin{aligned}
&\|T^n f(x,y) - T^{n+1} f(x,y) \\
&\quad = 6^n \left\| f\left(\frac{x}{2^n}, \frac{y}{3^n}\right) - 6f\left(\frac{x}{2^{n+1}}, \frac{y}{3^{n+1}}\right) \right\| \\
&\quad \leq 6^n \psi\left(\frac{x}{2^n}, \frac{y}{3^n}\right) \leq 6^{n-1} L\psi\left(\frac{x}{2^{n-1}}, \frac{y}{3^{n-1}}\right) \leq 6^{n-2} L^2 \psi\left(\frac{x}{2^{n-2}}, \frac{y}{3^{n-2}}\right) \leq \cdots \leq L^n \psi(x,y)
\end{aligned} \tag{3.35}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \leq L^n < \infty$ for all $n \in \mathbb{N}$. By the same reasoning as in the proof of Theorem 2.3 in [11], we have

$$d(f,F) \leq \frac{1}{1-L} d(f,Tf). \tag{3.36}$$

By (3.31), we may conclude that

$$d(f,F) \leq \frac{1}{1-L}, \tag{3.37}$$

which implies inequality (3.26). □

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