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Research Article

Existence of Almost Periodic Solutions to *N*th-Order Neutral Differential Equations with Piecewise Constant Arguments

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We present some conditions for the existence and uniqueness of almost periodic solutions of Nth-order neutral differential equations with piecewise constant arguments of the form $(x(t)+px(t-1))^{(N)} = qx([t])+f(t)$, here $[\cdot]$ is the greatest integer function, p and q are nonzero constants, N is a positive integer, and $f(t)$ is almost periodic.

1. Introduction

In this paper we study certain functional differential equations of neutral delay type with piecewise constant arguments of the form

$$\big(x(t) + px(t-1)\big)^{(N)} = qx([t]) + f(t), \tag{1.1}$$

here $[\cdot]$ is the greatest integer function, p and q are nonzero constants, N is a positive integer, and $f(t)$ is almost periodic. Throughout this paper, we use the following notations: $\mathbb{R}$ is the set of reals; $\mathbb{R}^+$ the set of positive reals; $\mathbb{Z}$ the set of integers; that is, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$; $\mathbb{Z}^+$ the set of positive integers; $\mathbb{C}$ denotes the set of complex numbers. A function $x : \mathbb{R} \to \mathbb{R}$ is called a solution of (1.1) if the following conditions are satisfied:

(i) x is continuous on $\mathbb{R}$;

(ii) the Nth-order derivative of $x(t) + p(t)x(t-1)$ exists on $\mathbb{R}$ except possibly at the points $t = n$, $n \in \mathbb{Z}$, where one-sided Nth-order derivatives of $x(t) + p(t)x(t-1)$ exist;

(iii) x satisfies (1.1) on each interval $(n, n+1)$ with integer $n \in \mathbb{Z}$.

Differential equations with piecewise constant arguments are usually referred to as a hybrid system, and could model certain harmonic oscillators with almost periodic forcing. For some excellent works in this field we refer the reader to [1–5] and references therein, and for a survey of work on differential equations with piecewise constant arguments we refer the reader to [6].

In paper [1, 2], Yuan and Li and He, respectively, studied the existence of almost periodic solutions for second-order equations involving the argument $2[(t+1)/2]$ in the unknown function. In paper [3], Seifert intensively studied the special case of (1.1) for $N = 2$ and $|p| < 1$ by using different methods. However, to the best of our knowledge, there are no results regarding the existence of almost periodic solutions for Nth-order neutral differential equations with piecewise constant arguments as (1.1) up to now.

Motivated by the ideas of Yuan [1] and Seifert [3], in this paper we will investigate the existence of almost periodic solutions to (1.1). Both the cases when $|p| < 1$ and $|p| > 1$ are considered.

2. The Main Results

We begin with some definitions, which can be found (or simply deduced from the theory) in any book, say [7], on almost periodic functions.

Definition 2.1. A set $K \subset \mathbb{R}$ is said to be relatively dense if there exists $L > 0$ such that $[a, a+L] \cap K \neq \emptyset$ for all $a \in \mathbb{R}$.

Definition 2.2. A bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ (resp., $\mathbb{C}$) is said to be almost periodic if the ε-translation set of f

$$T(f,\varepsilon) = \{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| < \varepsilon \ \forall t \in \mathbb{R}\} \tag{2.1}$$

is relatively dense for each $\varepsilon > 0$. We denote the set of all such function f by $\mathrm{AP}(\mathbb{R},\mathbb{R})$ (resp., $\mathrm{AP}(\mathbb{R},\mathbb{C})$).

Definition 2.3. A sequence $x : \mathbb{Z} \to \mathbb{R}^k$ (resp., $\mathbb{C}^k$), $k \in \mathbb{Z}, k > 0$, denoted by $\{x_n\}$, is called an almost periodic sequence if the ε-translation set of $\{x_n\}$

$$T(\{x_n\},\varepsilon) = \{\tau \in \mathbb{Z} : |x_{n+\tau} - x_n| < \varepsilon \ \forall n \in \mathbb{Z}\} \tag{2.2}$$

is relatively dense for each $\varepsilon > 0$, here $|\cdot|$ is any convenient norm in $\mathbb{R}^k$ (resp., $\mathbb{C}^k$). We denote the set of all such sequences $\{x_n\}$ by $\mathrm{APS}(\mathbb{Z},\mathbb{R}^k)$ (resp., $\mathrm{APS}(\mathbb{Z},\mathbb{C}^k)$).

Proposition 2.4. $\{x_n\} = \{(x_{n1}, x_{n2}, \dots, x_{nk})\} \in \mathrm{APS}(\mathbb{Z},\mathbb{R}^k)$ *(resp.,* $\mathrm{APS}(\mathbb{Z},\mathbb{C}^k)$*) if and only if* $\{x_{ni}\} \in \mathrm{APS}(\mathbb{Z},\mathbb{R})$ *(resp.,* $\mathrm{APS}(\mathbb{Z},\mathbb{C})$*),* $i = 1, 2, \dots, k$.

Proposition 2.5. *Suppose that* $\{x_n\} \in \mathrm{APS}(\mathbb{Z},\mathbb{R})$, $f \in \mathrm{AP}(\mathbb{R},\mathbb{R})$*. Then the sets* $T(f,\varepsilon) \cap \mathbb{Z}$ *and* $T(\{x_n\},\varepsilon) \cap T(f,\varepsilon)$ *are relatively dense.*

Now one rewrites (1.1) as the following equivalent system

$$
\begin{aligned}
&\left(x(t)+px(t-1)\right)' = y_1(t), && (2.3_1)\\
&y_1'(t) = y_2(t), && (2.3_2)\\
&\vdots && \vdots\\
&y_{N-2}'(t) = y_{N-1}(t), && (2.3_{N-1})\\
&y_{N-1}'(t) = qx([t]) + f(t). && (2.3_N)
\end{aligned}
\tag{2.3}
$$

Let $(x(t), y_1(t), \dots, y_{N-1}(t))$ be solutions of system (2.3) on $\mathbb{R}$, for $n \le t < n+1$, $n \in \mathbb{Z}$, using (2.3_N) we obtain

$$
y_{N-1}(t) = y_{N-1}(n) + qx(n)(t-n) + \int_n^t f(t_1)dt_1, \tag{2.4}
$$

and using this with (2.3_{N-1}) we obtain

$$
y_{N-2}(t) = y_{N-2}(n) + y_{N-1}(n)(t-n) + \frac{1}{2}qx(n)(t-n)^2 + \int_n^t\int_n^{t_2} f(t_1)dt_1dt_2. \tag{2.5}
$$

Continuing this way, and, at last, we get

$$
\begin{aligned}
x(t) + px(t-1) = {} & x(n) + px(n-1) + y_1(n)(t-n) + \frac{1}{2}y_2(n)(t-n)^2 + \cdots \\
& + \frac{1}{(N-1)!}y_{N-1}(n)(t-n)^{N-1} + \frac{1}{N!}qx(n)(t-n)^N \\
& + \int_n^t\int_n^{t_N}\cdots\int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N.
\end{aligned}
\tag{2.6}
$$

Since $x(t)$ must be continuous at $n+1$, using these equations we get for $n \in \mathbb{Z}$,

$$
\begin{aligned}
x(n+1) &= \left(1-p+\frac{q}{N!}\right)x(n) + y_1(n) + \frac{1}{2!}y_2(n) + \cdots + \frac{1}{(N-1)!}y_{N-1}(n) + px(n-1) \\
&\quad + f_n^{(1)}, && (2.7_1)\\
y_1(n+1) &= \frac{q}{(N-1)!}x(n) + y_1(n) + y_2(n) + \frac{1}{2!}y_3(n) + \cdots + \frac{1}{(N-2)!}y_{N-1}(n) + f_n^{(2)}, && (2.7_2)\\
&\vdots && \vdots\\
y_{N-2}(n+1) &= \frac{q}{2}x(n) + y_{N-2}(n) + y_{N-1}(n) + f_n^{(N-1)}, && (2.7_{N-1})\\
y_{N-1}(n+1) &= qx(n) + y_{N-1}(n) + f_n^{(N)}, && (2.7_N)
\end{aligned}
\tag{2.7}
$$

where

$$f_n^{(1)} = \int_n^{n+1}\int_n^{t_N}\cdots\int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N,\ldots,\; f_n^{(N-1)} = \int_n^{n+1}\int_n^{t_2} f(t_1)dt_1dt_2,$$
$$f_n^{(N)} = \int_n^{n+1} f(t_1)dt_1. \tag{2.8}$$

Lemma 2.6. *If $f \in AP(\mathbb{R},\mathbb{R})$, then sequences $\{f_n^{(i)}\} \in APS(\mathbb{Z},\mathbb{R}), i = 1,2,\ldots,N$.*

Proof. We typically consider $\{f_n^{(1)}\}$ for all $\varepsilon > 0$ and $\tau \in T(f,\varepsilon)\cap\mathbb{Z}$, we have

$$\begin{aligned}\left|f_{n+\tau}^{(1)} - f_n^{(1)}\right| &= \left|\int_{n+\tau}^{n+\tau+1}\int_{n+\tau}^{t_N}\cdots\int_{n+\tau}^{t_2} f(t_1)dt_1dt_2\cdots dt_N - \int_n^{n+1}\int_n^{t_N}\cdots\int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N\right|\\ &\le \int_n^{n+1}\int_n^{t_N}\cdots\int_n^{t_2}\left|f(t_1+\tau) - f(t_1)\right|dt_1dt_2\cdots dt_N\\ &\le \frac{\varepsilon}{N!}.\end{aligned} \tag{2.9}$$

From Definition 2.3, it follows that $\{f_n^{(1)}\}$ is an almost periodic sequence. In a manner similar to the proof just completed, we know that $\{f_n^{(2)}\},\{f_n^{(3)}\},\ldots,\{f_n^{(N)}\}$ are also almost periodic sequences. This completes the proof of the lemma. □

Lemma 2.7. *The system of difference equations*

$$c_{n+1} = \left(1 - p + \frac{q}{N!}\right)c_n + d_n^{(1)} + \frac{1}{2!}d_n^{(2)} + \cdots + \frac{1}{(N-1)!}d_n^{(N-1)} + pc_{n-1} + f_n^{(1)}, \tag{2.10$_1$}$$

$$d_{n+1}^{(1)} = \frac{q}{(N-1)!}c_n + d_n^{(1)} + d_n^{(2)} + \frac{1}{2!}d_n^{(3)} + \cdots + \frac{1}{(N-2)!}d_n^{(N-1)} + f_n^{(2)}, \tag{2.10$_2$}$$

$$\vdots$$

$$d_{n+1}^{(N-2)} = \frac{q}{2}c_n + d_n^{(N-2)} + d_n^{(N-1)} + f_n^{(N-1)}, \tag{2.10$_{N-1}$}$$

$$d_{n+1}^{(N-1)} = qc_n + d_n^{(N-1)} + f_n^{(N)}, \tag{2.10$_N$}$$

(2.10)

has solutions on $\mathbb{Z}$; these are in fact uniquely determined by $c_0, c_{-1}, d_0^{(1)},\ldots,d_0^{(N-1)}$.

Proof. It is easy to check that $c_n, d_n^{(i)}$, $i = 1,2,\ldots,N-1$ are uniquely determined in term of c_0, c_{-1}, $d_0^{(1)}, d_0^{(2)},\ldots,d_0^{(N-1)}$ for $n \in \mathbb{Z}^+$. For $n = -1$, (2.10$_N$) uniquely determines $d_{-1}^{(N-1)}$, (2.10$_{N-1}$) uniquely determines $d_{-1}^{(N-2)},\ldots,$ (2.10$_2$) uniquely determines $d_{-1}^{(1)}$, and thus since $p \neq 0$, (2.10$_1$) uniquely determines c_{-2}. So $c_{-1}, c_{-2}, d_{-1}^{(1)}, d_{-1}^{(2)},\ldots,d_{-1}^{(N-1)}$ are determined. Continuing in this way, we establish the lemma. □

Lemma 2.8. *For any solution* $(c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$, $n \in \mathbb{Z}$, *of system* (2.10), *there exists a solution* $(x(t), y_1(t), y_2(t), \dots, y_{N-1}(t))$, $t \in R$, *of* (2.3) *such that* $x(n) = c_n$, $y_1(n) = d_n^{(1)}, \dots,$ $y_{N-1}(n) = d_n^{(N-1)}$, $n \in \mathbb{Z}$.

Proof. Define

$$w(t) = c_n + pc_{n-1} + d_n^{(1)}(t-n) + \frac{1}{2!}d_n^{(2)}(t-n)^2 + \cdots + \frac{1}{(N-1)!}d_n^{(N-1)}(t-n)^{N-1} + \frac{1}{N!}qc_n(t-n)^N + \int_n^t \int_n^{t_N} \cdots \int_n^{t_2} f(t_1)dt_1dt_1 \cdots dt_N, \tag{2.11}$$

for $n \le t < n+1$, $n \in \mathbb{Z}$. It can easily be verified that $w(t)$ is continuous on $\mathbb{R}$; we omit the details.

Define $x(t) = \varphi(t)$, $-1 \le t \le 0$, where $\varphi(t)$ is continuous, and $\varphi(0) = c_0$, $\varphi(-1) = c_{-1}$;

$$\begin{aligned} x(t) &= \frac{[w(t+1) - \varphi(t+1)]}{p}, \quad -2 \le t < -1, \\ x(t) &= \frac{[w(t+1) - x(t+1)]}{p}, \quad -3 \le t < -2. \end{aligned} \tag{2.12}$$

Continuing this way, we can define $x(t)$ for $t < 0$. Similarly, define

$$\begin{aligned} x(t) &= -p\varphi(t-1) + w(t), \quad 0 \le t < 1, \\ x(t) &= -px(t-1) + w(t), \quad 1 \le t < 2, \end{aligned} \tag{2.13}$$

continuing in this way $x(t)$ is defined for $t \ge 0$, and so $x(t)$ is defined for all $t \in \mathbb{R}$.

Next, define $y_1(t) = w'(t), y_2(t) = w''(t), \dots, y_{N-1}(t) = w^{(N-1)}(t)$, $t \neq n \in \mathbb{Z}$, and by the appropriate one-sided derivative of $w'(t), w''(t), \dots, w^{(N-1)}(t)$ at $n \in \mathbb{Z}$. It is easy to see that $y_1(t), y_2(t), \dots, y_{N-1}(t)$ are continuous on $\mathbb{R}$, and $(x(n), y_1(n), y_2(n), \dots, y_{N-1}(n)) = (c_n, d_n^{(1)}, d_n^{(2)}, \dots, d_n^{(N-1)})$ for $n \in \mathbb{Z}$; we omit the details. □

Next we express system (2.7) in terms of an equivalent system in $\mathbb{R}^{N+1}$ give by

$$v_{n+1} = Av_n + h_n, \tag{2.14}$$

where

$$A = \begin{pmatrix} 1-p+\dfrac{q}{N!} & 1 & \dfrac{1}{2!} & \cdots & \dfrac{1}{(N-1)!} & p \\ \dfrac{q}{(N-1)!} & 1 & 1 & \cdots & \dfrac{1}{(N-2)!} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{q}{2!} & 0 & 0 & \cdots & 1 & 0 \\ q & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.15}$$

$$v_n = (x(n), y_1(n), y_2(n), \ldots, y_{N-1}, x(n-1))^T, \qquad h_n = \left(f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(N)}, 0\right)^T.$$

Lemma 2.9. *Suppose that all eigenvalues of A are simple (denoted by $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$) and $|\lambda_i| \neq 1$, $1 \leq i \leq N+1$. Then system* (2.14) *has a unique almost periodic solution.*

Proof. From our hypotheses, there exists a $(N+1)\times(N+1)$ nonsingular matrix P such that $PAP^{-1} = \Lambda$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{N+1})$ and $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$ are the distinct eigenvalues of A. Define $\overline{v}_n = Pv_n$, then (2.14) becomes

$$\overline{v}_{n+1} = \Lambda \overline{v}_n + \overline{h}_n, \tag{2.16}$$

where $\overline{h}_n = Ph_n$.

For the sake of simplicity, we consider first the case $|\lambda_1| < 1$. Define

$$\overline{v}_{n1} = \sum_{m \leq n} \lambda_1^{n-m} \overline{h}_{(m-1)1}, \tag{2.17}$$

where $\overline{h}_n = (\overline{h}_{n1}, \overline{h}_{n2}, \ldots, \overline{h}_{n(N+1)})^T$, $n \in \mathbb{Z}$. Clearly $\{\overline{h}_{n1}\}$ is almost periodic, since $\overline{h}_n = Ph_n$, and $\{h_n\}$ is. For $\tau \in T(\{\overline{h}_{n1}\}, \varepsilon)$, we have

$$\begin{aligned} \left|\overline{v}_{(n+\tau)1} - \overline{v}_{n1}\right| &= \left| \sum_{m \leq n+\tau} \lambda_1^{n+\tau-m} \overline{h}_{(m-1)1} - \sum_{m \leq n} \lambda_1^{n-m} \overline{h}_{(m-1)1} \right| \\ &\qquad (\text{letting } m = m' + \tau, \text{then replacing } m' \text{ by } m) \\ &= \left| \sum_{m \leq n} \lambda_1^{n-m} \overline{h}_{(m+\tau-1)1} - \sum_{m \leq n} \lambda_1^{n-m} \overline{h}_{(m-1)1} \right| \\ &= \left| \sum_{m \leq n} \lambda_1^{n-m} \left(\overline{h}_{(m+\tau-1)1} - \overline{h}_{(m-1)1} \right) \right| \\ &\leq \frac{\varepsilon}{1 - |\lambda_1|}, \end{aligned} \tag{2.18}$$

this shows that $\{\overline{v}_{n1}\} \in \mathrm{APS}(\mathbb{Z}, \mathbb{C})$.

If $|\lambda_i| < 1$, $2 \le i \le N+1$, in a manner similar to the proof just completed for λ_1, we know that $\{\overline{v}_{ni}\} \in PAS(\mathbb{Z}, \mathbb{C})$, $2 \le i \le N+1$, and so $\{\overline{v}_n\} \in \mathrm{APS}(\mathbb{Z}, \mathbb{C}^{N+1})$. It follows easily that then $\{P^{-1}\overline{v}_n\} = \{v_n\} \in \mathrm{APS}(\mathbb{Z}, \mathbb{R}^{N+1})$ and our lemma follows.

Assume now $|\lambda_1| > 1$. Now define

$$\overline{v}_{n1} = \sum_{m \le n} \lambda_1^{m-n} \overline{h}_{(m-1)1}, \quad n \in \mathbb{Z}. \tag{2.19}$$

As before, the fact that $\{\overline{v}_{n1}\} \in \mathrm{APS}(\mathbb{Z}, \mathbb{C})$ follows easily from the fact that $\{\overline{h}_{n1}\} \in \mathrm{APS}(\mathbb{Z}, \mathbb{C})$. So in every possible case, we see that each component v_{ni}, $i = 1, 2, \ldots, N+1$, of v_n is almost periodic and so $\{v_n\} \in \mathrm{APS}(\mathbb{Z}, \mathbb{R}^{N+1})$.

The uniqueness of this almost periodic solution $\{v_n\}$ of (2.14) follows from the uniqueness of the solution $\overline{v}_n$ of (2.16) since $P^{-1}\overline{v}_n = v_n$, and the uniqueness of $\overline{v}_n$ of (2.16) follows, since if $\tilde{v}_n$ were a solution of (2.16) distinct from $\overline{v}_n$, $u_n = \overline{v}_n - \tilde{v}_n$ would also be almost periodic and solve $u_{n+1} = \Lambda u_n$, $n \in \mathbb{Z}$. But by our condition on Λ, it follows that each component of u_n must become unbounded either as $n \to \infty$ or as $n \to -\infty$, and that is impossible, since it must be almost periodic. This proves the lemma. □

Lemma 2.10. *Suppose that conditions of Lemma 2.9 hold, $w(t)$ is as defined in the proof of Lemma 2.8 with $(c_n, d_n^{(1)}, d_n^{(2)}, \ldots, d_n^{(N-1)})$ the unique first N components of the almost periodic solution of* (2.14) *given by Lemma 2.9, then $w(t)$ is almost periodic.*

Proof. For $\tau \in T(\{c_n\}, \varepsilon) \cap T(\{d_n^{(1)}\}, \varepsilon) \cap T(\{d_n^{(2)}\}, \varepsilon) \cap \cdots \cap T(\{d_n^{(N-1)}\}, \varepsilon) \cap T(f, \varepsilon)$,

$$\begin{aligned}
&|w(t+\tau) - w(t)| \\
&= \Bigg| (c_{n+\tau} - c_n) + p(c_{n+\tau-1} - c_{n-1}) + \left(d_{n+\tau}^{(1)} - d_n^{(1)}\right)(t-n) + \frac{1}{2!}\left(d_{n+\tau}^{(2)} - d_n^{(2)}\right)(t-n)^2 + \cdots \\
&\qquad + \frac{1}{(N-1)!}\left(d_{n+\tau}^{(N-1)} - d_n^{(N-1)}\right)(t-n)^{N-1} + \frac{q}{N!}(c_{n+\tau} - c_n)(t-n)^N \\
&\qquad + \int_{n+\tau}^{t+\tau}\int_{n+\tau}^{t_N}\cdots\int_{n+\tau}^{t_2} f(t_1)dt_1dt_2\cdots dt_N - \int_n^t\int_n^{t_N}\cdots\int_n^{t_2} f(t_1)dt_1dt_2\cdots dt_N \Bigg| \\
&\le \left(1 + |p| + \frac{|q|}{N!} + \sum_{i=0}^{N-1}\frac{1}{i!}\right)\varepsilon.
\end{aligned} \tag{2.20}$$

It follows from definition that $w(t)$ is almost periodic. □

Theorem 2.11. *Suppose that $|p| \ne 1$ and all eigenvalues of A in* (2.14) *are simple (denoted by $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$) and satisfy $|\lambda_i| \ne 1$, $1 \le i \le N+1$. Then* (1.1) *has a unique almost periodic solution $\overline{x}(t)$, which can, in fact be determined explicitly in terms of $w(t)$ as defined in the proof of Lemma 2.8.*

Proof. Consider the following.

Case 1 ($|p| < 1$). For each $m \in \mathbb{Z}^+$ define $x_m(t)$ as follows:

$$x_m(t) = w(t) - px_m(t-1), \quad t > -m, \tag{2.21}$$

$$x_m(t) = \phi(t), \quad t \leq -m, \tag{2.22}$$

here $w(t)$ is as defined in the proof of Lemma 2.8, and

$$\phi(t) = c_n + (c_{n+1} - c_n)(t-n), \quad n \leq t < n+1, \ n \in \mathbb{Z}, \tag{2.23}$$

where c_n is the first component of the solution v_n of (2.14) given by Lemma 2.9. Let $l \in \mathbb{Z}^+$, then from (2.21) we get

$$(-p)^l x_m(t-l) = (-p)^l w(t-l) + (-p)^{l+1} x_m(t-l-1), \quad t > -m. \tag{2.24}$$

It follows that

$$x_m(t) = \sum_{j=0}^{l-1} (-p)^j w(t-j) + (-p)^l x_m(t-l), \quad t > -m. \tag{2.25}$$

If $l > t+m$, $x_m(t-l) = \phi(t-l)$, and so for such l,

$$\left| x_m(t) - \sum_{j=0}^{l-1} (-p)^j w(t-j) \right| \leq \ |p|^l |\phi(t-l)|. \tag{2.26}$$

Let $l \to \infty$, we get

$$x_m(t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} (-p)^j w(t-j), & t > -m, \\ \phi(t), & t \leq -m. \end{cases} \tag{2.27}$$

Since $w(t)$ and $\phi(t)$ are uniformly continuous on $\mathbb{R}$, it follows that $\{x_m(t) : m \in \mathbb{Z}^+\}$ is equicontinuous on each interval $[-L, L]$, $L \in \mathbb{Z}^+$, and by the Ascoli-Arzelá Theorem, there exists a subsequence, which we again denote by $x_m(t)$, and a function $\overline{x}(t)$ such that $x_m(t) \to \overline{x}(t)$ uniformly on $[-L, L]$, and by a familiar diagonalization procedure, can find a subsequence, again denoted by $x_m(t)$ which is such that $x_m(t) \to \overline{x}(t)$ for each $t \in \mathbb{R}$. From (2.27) it follows that

$$x_m(t) = \sum_{j=0}^{\infty} (-p)^j w(t-j), \tag{2.28}$$

and so $\overline{x}(t)$ is almost periodic since $w(t-j)$ is almost periodic in t for each $j \geq 0$, and $|p| < 1$. From (2.21), letting $m \to \infty$, we get $\overline{x}(t) + p\overline{x}(t-1) = w(t)$, $t \in \mathbb{R}$, and since $w(t)$ solves (1.1), $\overline{x}(t)$ does also. The uniqueness of $\overline{x}(t)$ as an almost periodic solution of (1.1) follows from the uniqueness of the almost periodic solution $v_n : \mathbb{Z} \to \mathbb{R}^{N+1}$ of (2.14) given by Lemma 2.9, which determines the uniqueness of $w(t)$, and therefore from (2.21) the uniqueness of $\overline{x}(t)$.

Case 2 ($|p| > 1$). Rewriting (2.24) as

$$\left(\frac{-1}{p}\right)^l x_m(t-l) = \left(\frac{-1}{p}\right)^l w(t-l) + \left(\frac{-1}{p}\right)^{l+1} x_m(t-l-1), \quad t > -m, \tag{2.29}$$

we deduce in a similar manner that

$$x_m(t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \left(\frac{-1}{p}\right)^j w(t-j), & t > -m, \\ \phi(t), & t \leq -m. \end{cases} \tag{2.30}$$

The remainder of the proof is similar to that of Case 1, we omit the details. □

If $p = 0$, the system of difference equations (2.10) of Lemma 2.7 now becomes

$$\begin{aligned} c_{n+1} &= \left(1 + \frac{1}{N!}q\right)c_n + d_n^{(1)} + \frac{1}{2!}d_n^{(2)} + \cdots + \frac{1}{(N-1)!}d_n^{(N-1)} + f_n^{(1)}, \\ d_{n+1}^{(1)} &= \frac{1}{(N-1)!}qc_n + d_n^{(1)} + d_n^{(2)} + \frac{1}{2!}d_n^{(3)} + \cdots + \frac{1}{(N-2)!}d_n^{(N-1)} + f_n^{(2)}, \\ &\vdots \\ d_{n+1}^{(N-2)} &= \frac{q}{2}c_n + d_n^{(N-1)} + d_n^{(N-2)} + f_n^{(N-1)}, \\ d_{n+1}^{(N-1)} &= qc_n + d_n^{(N-1)} + f_n^{(N)}, \end{aligned} \tag{2.31}$$

and system (2.14) reduces to

$$v_{n+1}^* = A^* v_n^* + h_n^*, \tag{2.32}$$

where

$$A^* = \begin{pmatrix} 1 + \dfrac{q}{N!} & 1 & \dfrac{1}{2!} & \cdots & \dfrac{1}{(N-2)!} & \dfrac{1}{(N-1)!} \\ \dfrac{q}{(N-1)!} & 1 & 1 & \cdots & \dfrac{1}{(N-3)!} & \dfrac{1}{(N-2)!} \\ \dfrac{q}{(N-2)!} & 0 & 1 & \cdots & \dfrac{1}{(N-4)!} & \dfrac{1}{(N-3)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{q}{2!} & 0 & 0 & \cdots & 1 & 1 \\ q & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.33}$$

and $v_n^* = (x(n), y_1(n), y_2(n), \dots, y_{N-1})^T$, $h_n^* = (f_n^{(1)}, f_n^{(2)}, \dots, f_n^{(N)})^T$. Then we have the following theorem.

Theorem 2.12. *Let $p = 0$ and $q \neq (-1)^N N!$, if all eigenvalues of A^* in* (2.32) *are simple (denoted by $\lambda_1, \lambda_2, \dots, \lambda_N$) and satisfy $|\lambda_i| \neq 1, 1 \leq i \leq N$, then* (1.1) *has a unique almost periodic solution $\overline{x}(t)$.*

Proof. System (2.32) has a solution on $\mathbb{Z}$ since A^* is nonsingular because $q \neq (-1)^N N!$. The rest of the proof follows in the same way as the proof of Theorem 2.11 and is omitted. □

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