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Research Article

Inequalities for the Polar Derivative of a Polynomial

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For a polynomial $p(z)$ of degree n, we consider an operator D_α which map a polynomial $p(z)$ into $D_\alpha p(z) := (\alpha - z)p'(z) + np(z)$ with respect to α. It was proved by Liman et al. (2010) that if $p(z)$ has no zeros in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$, $|zD_\alpha p(z) + n\beta((|\alpha|-1)/2)p(z)| \leq (n/2)\{[|\alpha + \beta((|\alpha|-1)/2)| + |z + \beta((|\alpha|-1)/2)|]\max_{|z|=1}|p(z)| - [|\alpha + \beta((|\alpha|-1)/2)| - |z + \beta((|\alpha|-1)/2)|]\min_{|z|=1}|p(z)|\}$. In this paper we extend the above inequality for the polynomials having no zeros in $|z| < k$, where $k \leq 1$. Our result generalizes certain well-known polynomial inequalities.

1. Introduction and Statement of Results

According to a result well known as Bernstein's inequality on the derivative of a polynomial $p(z)$ of degree n, we have

$$\max_{|z|=1}|p'(z)| \leq n \max_{|z|=1}|p(z)|. \tag{1.1}$$

The result is best possible, and equality holds for a polynomial having all its zeros at the origin (see [1, 2]).

The inequality (1.1) can be sharpened, by considering the class of polynomials having no zeros in $|z| < 1$.

In fact, P. Erdös conjectured, and later Lax [3] proved that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.2}$$

As a refinement of (1.2), Aziz and Dawood [4] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then

$$\max_{|z|=1}|p'(z)| \le \frac{n}{2}\left\{\max_{|z|=1}|p(z)| - \min_{|z|=1}|p(z)|\right\}. \tag{1.3}$$

As an improvement of (1.3), Dewan and Hans [5] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for any β with $|\beta| \le 1$ and $|z| = 1$,

$$\begin{aligned}\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \le \frac{n}{2}\Bigg\{&\left(\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right)\max_{|z|=1}|p(z)| \\ &-\left(\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)\min_{|z|=1}|p(z)|\Bigg\}.\end{aligned} \tag{1.4}$$

Let α be a complex number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \tag{1.5}$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and that $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right] = p'(z). \tag{1.6}$$

As an extension to (1.1) for the polar derivative $D_\alpha p(z)$, Aziz and Shah [6] proved that if $p(z)$ is a polynomial of degree n, then for every α with $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \le n|\alpha|\max_{|z|=1}|p(z)|. \tag{1.7}$$

As a refinement and extension of (1.7), Aziz and Mohammad Shah [7] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then, for every α with $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_\alpha p(z)| \le \frac{n}{2}\left\{(|\alpha|+1)\max_{|z|=1}|p(z)| - (|\alpha|-1)\min_{|z|=1}|p(z)|\right\}. \tag{1.8}$$

Recently Dewan et al. [8] generalized (1.8) to the polynomial of the form $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \le t \le n$ and proved that if $p(z) = a_0 + \sum_{\nu=t}^{n} a_\nu z^\nu$, $1 \le t \le n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \ge 1$, then for $|\alpha| \ge 1$

$$\max_{|z|=1}|D_\alpha p(z)| \le \frac{n}{1+s_0}\left\{(|\alpha|+s_0)\max_{|z|=1}|p(z)| - (|\alpha|-1)\min_{|z|=k}|p(z)|\right\}, \tag{1.9}$$

where $s_0 = k^{t+1}\{(((t/n)(|a_t|/(|a_0|-m)))k^{t-1}+1)/(((t/n)(|a_t|/(|a_0|-m)))k^{t+1}+1)\}$, and $m = \min_{|z|=k}|p(z)|$.

As a generalization of (1.9), Bidkham et al. [9] proved that if $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \ge 1$, then for $0 < r \le R \le k$ and $|\alpha| \ge R$

$$\max_{|z|=R} |D_\alpha p(z)| \le \frac{n}{1+s_0'} \left\{ \left(\frac{|\alpha|}{R} + s_0' \right) \exp\left\{ n \int_r^R A_t dt \right\} \max_{|z|=r} |p(z)| + s_0' + 1 - \left(\frac{|\alpha|}{R} + s_0' \right) \exp\left\{ n \int_r^R A_t dt \right\} \min_{|z|=k} |p(z)| \right\}, \tag{1.10}$$

where

$$A_t = \frac{(\mu/n)(|a_\mu|/(|a_0|-m))k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + (\mu/n)(|a_\mu|/(|a_0|-m))(k^{\mu+1}t^\mu + k^{2\mu}t)}, \quad s_0' = \left(\frac{k}{R}\right)^{\mu+1} \left\{ \frac{(\mu/n)(|a_\mu|/(|a_0|-m))Rk^{\mu-1} + 1}{(\mu/n)(|a_\mu|/(R(|a_0|-m)))k^{\mu+1} + 1} \right\}, \quad m = \min_{|z|=k} |p(z)|. \tag{1.11}$$

As an improvement and generalization to (1.8) and (1.4), Liman et al. [10] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then, for all α, β with $|\alpha| \ge 1$, $|\beta| \le 1$ and $|z| = 1$,

$$\left| zD_\alpha p(z) + n\beta \frac{|\alpha|-1}{2} p(z) \right| \le \frac{n}{2} \left\{ \left(\left| \alpha + \beta \frac{|\alpha|-1}{2} \right| + \left| z + \beta \frac{|\alpha|-1}{2} \right| \right) \max_{|z|=1} |p(z)| - \left(\left| \alpha + \beta \frac{|\alpha|-1}{2} \right| - \left| z + \beta \frac{|\alpha|-1}{2} \right| \right) \min_{|z|=1} |p(z)| \right\}. \tag{1.12}$$

In this paper, we obtain the following extension of (1.12).

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n that does not vanish in $|z| < k$, $k \le 1$, then, for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \ge k$, $|\beta| \le 1$ and $|z| = 1$, we have*

$$\left| zD_\alpha p(z) + n\beta \frac{|\alpha|-k}{1+k} p(z) \right| \le \frac{n}{2} \left\{ \left(k^{-n} \left| \alpha + \beta \frac{|\alpha|-k}{1+k} \right| + \left| z + \beta \frac{|\alpha|-k}{1+k} \right| \right) \max_{|z|=1} |p(z)| - \left(k^{-n} \left| \alpha + \beta \frac{|\alpha|-k}{1+k} \right| - \left| z + \beta \frac{|\alpha|-k}{1+k} \right| \right) \min_{|z|=k} |p(z)| \right\}. \tag{1.13}$$

If we take $k = 1$ in Theorem 1.1, then (1.13) reduces to (1.12).

Theorem 1.1 simplifies to the following result by taking $\beta = 0$.

Corollary 1.2. *Let $p(z)$ be a polynomial of degree n does not vanish in $|z| < k$, $k \leq 1$, then for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have*

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \leq \frac{n}{2}\left\{\left(k^{-n}|\alpha|+1\right)\max_{|z|=1}\left|p(z)\right| - \left(k^{-n}|\alpha|-1\right)\min_{|z|=k}\left|p(z)\right|\right\}. \tag{1.14}$$

If we take $k = 1$ in Corollary 1.2, then (1.14) reduce to (1.8).

Dividing two sides of inequality (1.13) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following generalization of the inequality (1.4).

Corollary 1.3. *Let $p(z)$ be a polynomial of degree n, having no zeros in $|z| < k$, $k \leq 1$, then, for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\begin{aligned}\left|zp'(z) + \frac{n\beta}{1+k}p(z)\right| \leq \frac{n}{2}\Bigg\{&\left(k^{-n}\left|1+\frac{\beta}{1+k}\right| + \left|\frac{\beta}{1+k}\right|\right)\max_{|z|=1}\left|p(z)\right| \\ &-\left(k^{-n}\left|1+\frac{\beta}{1+k}\right| - \left|\frac{\beta}{1+k}\right|\right)\min_{|z|=k}\left|p(z)\right|\Bigg\}.\end{aligned} \tag{1.15}$$

Taking $\beta = 0$ and $k = 1$ in Corollary 1.3, (1.15) reduces to (1.3).

2. Lemmas

For proof of the theorem, we need the following lemmas. The first lemma is due to Laguerre [11, 12].

Lemma 2.1. *If all the zeros of an nth degree polynomial $p(z)$ lie in a circular region C, and w is any zero of $D_\alpha p(z)$, then at most one of the points w and α may lie outside C.*

Lemma 2.2. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \leq k$, $k \leq 1$, then on $|z| = 1$*

$$\left|p'(z)\right| \geq \frac{n}{1+k}\left|p(z)\right|. \tag{2.1}$$

This lemma is due to Malik [13].

Lemma 2.3. *Let $p(z)$ be a polynomial of degree n and have no zero in $|z| < k$, $k \geq 1$, then on $|z| = 1$*

$$k\left|p'(z)\right| \leq \left|q'(z)\right|, \tag{2.2}$$

where $q(z) = z^n\overline{p(1/\overline{z})}$.

The above lemma is due to Chan and Malik [14].

Lemma 2.4. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \le k$, $k \le 1$, then on $|z| = 1$*

$$\left|q'(z)\right| \le k\left|p'(z)\right|, \tag{2.3}$$

where $q(z) = z^n\overline{p(1/\overline{z})}$.

Proof. Since $p(z)$ has all its zeros in $|z| \le k$, $k \le 1$; therefore, $q(z)$ has no zero in $|z| < 1/k$, $1/k \ge 1$. Now applying Lemma 2.3 to the polynomial $q(z)$ and the result follows. □

Lemma 2.5. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \le k$, $k \le 1$, then for all real or complex number α with $|\alpha| \ge k$ and $|z| = 1$, we have*

$$\left|D_\alpha p(z)\right| \ge n\frac{|\alpha| - k}{1 + k}\left|p(z)\right|. \tag{2.4}$$

Proof. Let $q(z) = z^n\overline{p(1/\overline{z})}$, then $|q'(z)| = |np(z) - zp'(z)|$ on $|z| = 1$. Thus on $|z| = 1$

$$\begin{aligned} \left|D_\alpha p(z)\right| &= \left|np(z) + (\alpha - z)p'(z)\right| \\ &= \left|\alpha p'(z) + np(z) - zp'(z)\right| \\ &\ge \left|\alpha p'(z)\right| - \left|np(z) - zp'(z)\right|, \end{aligned} \tag{2.5}$$

which implies that

$$\left|D_\alpha p(z)\right| \ge |\alpha|\left|p'(z)\right| - \left|q'(z)\right|. \tag{2.6}$$

Combining (2.3) and (2.6), we get the following:

$$\left|D_\alpha p(z)\right| \ge (|\alpha| - k)\left|p'(z)\right|, \tag{2.7}$$

along with Lemma 2.2, which gives the following:

$$\left|D_\alpha p(z)\right| \ge n\frac{|\alpha| - k}{1 + k}\left|p(z)\right|. \tag{2.8}$$

□

Lemma 2.6. *Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$. Then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \ge k$, $|\beta| \le 1$ and $|z| = 1$, we have*

$$\left|zD_\alpha p(z) + n\beta\frac{|\alpha| - k}{1 + k}p(z)\right| \ge nk^{-n}\left|\alpha + \beta\frac{|\alpha| - k}{1 + k}\right|\min_{|z|=k}\left|p(z)\right|. \tag{2.9}$$

Proof. If $p(z)$ has a zero on $|z| = k$, then (2.9) is trivial. Therefore, we assume that $p(z)$ has all its zeros in $|z| < k$. Let $m = \min_{|z|=k}|p(z)|$, then $m > 0$ and $|p(z)| \ge m$ where $|z| = k$. Therefore, for $|\lambda| < 1$, it follows by Rouche's Theorem that the polynomial $G(z) = p(z) - \lambda m(z/k)^n$ has all its zeros in $|z| < k$. By using Lemma 2.1, $D_\alpha G(z) = D_\alpha p(z) - \alpha\lambda mn(z^{n-1}/k^n)$ has all its zeros in $|z| < k$, where $|\alpha| \geq k$. Applying Lemma 2.5 to the polynomial $G(z)$ yields

$$|zD_\alpha G(z)| \geq n\frac{|\alpha| - k}{1 + k}|G(z)|, \quad |z| = 1. \tag{2.10}$$

Since $zD_\alpha G(z)$ has all its zeros in $|z| < k \leq 1$, by using Rouche's Theorem, it can be easily verified from (2.10) that the polynomial

$$zD_\alpha G(z) + \beta n\frac{|\alpha| - k}{1 + k}G(z) \tag{2.11}$$

has all its zeros in $|z| < 1$, where $|\beta| < 1$.

Substituting for $G(z)$, we conclude that the polynomial

$$T(z) = \left(zD_\alpha p(z) + n\beta\frac{|\alpha| - k}{1 + k}p(z)\right) - \lambda mn\left(\frac{z}{k}\right)^n\left(\alpha + \beta\frac{|\alpha| - k}{1 + k}\right) \tag{2.12}$$

will have no zeros in $|z| \geq 1$. This implies for every β with $|\beta| < 1$ and $|z| \geq 1$,

$$\left|zD_\alpha p(z) + n\beta\frac{|\alpha| - k}{1 + k}p(z)\right| \geq nm\left|\frac{z}{k}\right|^n\left|\alpha + \beta\frac{|\alpha| - k}{1 + k}\right|. \tag{2.13}$$

If (2.13) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\left|z_0 D_\alpha p(z_0) + n\beta\frac{|\alpha| - k}{1 + k}p(z_0)\right| < nm\left|\frac{z_0}{k}\right|^n\left|\alpha + \beta\frac{|\alpha| - k}{1 + k}\right|. \tag{2.14}$$

Take

$$\lambda = \frac{z_0 D_\alpha p(z_0) + n\beta((|\alpha| - k)/(1 + k))p(z_0)}{nm(z_0/k)^n\left(\alpha + \beta((|\alpha| - k)/(1 + k))\right)}, \tag{2.15}$$

then $|\lambda| < 1$ and with this choice of λ, we have $T(z_0) = 0$ for $|z_0| \geq 1$, from (2.12). But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For β with $|\beta| = 1$, (2.13) follows by continuity. This completes the proof of Lemma 2.6. □

Lemma 2.7. *If $p(z)$ is a polynomial of degree n, then for all $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|\alpha| \geq k$, where $k \leq 1$, we have*

$$\left|zD_\alpha p(z) + n\beta\frac{|\alpha| - k}{1 + k}p(z)\right| \leq nk^{-n}\left|\alpha + \beta\frac{|\alpha| - k}{1 + k}\right|\max_{|z|=k}|p(z)|, \quad |z| = 1. \tag{2.16}$$

Proof. Let $M = \max_{|z|=k}|p(z)|$, if $|\lambda| < 1$, then $|\lambda p(z)| < |M(z/k)^n|$ for $|z| = k$. Therefore, it follows by Rouche's Theorem that the polynomial $G(z) = M(z/k)^n - \lambda p(z)$ has all its zeros in $|z|<k$. By using Lemma 2.1, $D_\alpha G(z)=\alpha Mn(z^{n-1}/k^n)-\lambda D_\alpha p(z)$ has all its zeros in $|z|<k$ for $|\alpha|\geq k$.

On applying Lemma 2.5 to the polynomial $G(z)$, we have

$$|zD_\alpha G(z)|\geq n\frac{|\alpha|-k}{1+k}|G(z)|,\quad |z|=1. \tag{2.17}$$

Now, using a similar argument as used in the proof of Lemma 2.6, the result follows. □

Lemma 2.8. *If $p(z)$ is a polynomial of degree n, then for all $\alpha,\ \beta\in\mathbb{C}$ with $|\beta|\leq 1$ and $|\alpha|\geq k$, where $k\leq 1$, we have*

$$\begin{aligned}&\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-k}{1+k}p(z)\right|+\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-k}{1+k}Q(z)\right|\\&\quad\leq n\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right|+\left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}\max_{|z|=1}|p(z)|,\quad |z|=1,\end{aligned} \tag{2.18}$$

where $Q(z)=(z/k)^n\overline{p(k^2/\overline{z})}$.

Proof. Let $M=\max_{|z|=k}|p(z)|$. For λ with $|\lambda|>1$, it follows by Rouche's Theorem that the polynomial $G(z)=p(z)-\lambda M$ has no zeros in $|z|<k$. Consequently the polynomial

$$H(z)=\left(\frac{z}{k}\right)^n\overline{G\left(\frac{k^2}{\overline{z}}\right)} \tag{2.19}$$

has all its zeros in $|z|\leq k$, also $|G(z)|=|H(z)|$ for $|z|=k$. Since all the zeros of $H(z)$ lie in $|z|\leq k$; therefore, for δ with $|\delta|>1$, by Rouche's Theorem all the zeros of $G(z)+\delta H(z)$ lie in $|z|\leq k$. Hence by Lemma 2.5 for every α with $|\alpha|\geq k$, and $|z|=1$, we have

$$n\frac{|\alpha|-k}{1+k}|G(z)+\delta H(z)|\leq|zD_\alpha(G(z)+\delta H(z))|. \tag{2.20}$$

On the other hand by Lemma 2.1, all the zeros of $D_\alpha(G(z)+\delta H(z))$ lie in $|z|<k\leq 1$, where $|\alpha|\geq k$. Therefore, for any β with $|\beta|\leq 1$, Rouche's Theorem implies that all the zeros of $zD_\alpha(G(z)+\delta H(z))+\beta n((|\alpha|-k)/(1+k))(G(z)+\delta H(z))$ lie in $|z|<1$. This means that the polynomial

$$T(z)=zD_\alpha G(z)+n\beta\frac{|\alpha|-k}{1+k}G(z)+\delta\left(zD_\alpha H(z)+n\beta\frac{|\alpha|-k}{1+k}H(z)\right) \tag{2.21}$$

will have no zeros in $|z|\geq 1$. Now using a similar argument as used in the proof of Lemma 2.6, we get for $|z|\geq 1$,

$$\left|zD_\alpha G(z)+n\beta\frac{|\alpha|-k}{1+k}G(z)\right|\leq\left|zD_\alpha H(z)+n\beta\frac{|\alpha|-k}{1+k}H(z)\right|. \tag{2.22}$$

Therefore by the equalities

$$H(z) = \left(\frac{z}{k}\right)^{n}\overline{G\left(\frac{k^2}{\overline{z}}\right)} = \left(\frac{z}{k}\right)^{n}\overline{p\left(\frac{k^2}{\overline{z}}\right)} - \overline{\lambda}M\left(\frac{z}{k}\right)^{n} = Q(z) - \overline{\lambda}M\left(\frac{z}{k}\right)^{n}, \tag{2.23}$$

or

$$H(z) = Q(z) - \overline{\lambda}M\left(\frac{z}{k}\right)^{n}, \tag{2.24}$$

and substitute for $G(z)$ and $H(z)$ in (2.22), we get the following:

$$\begin{aligned} &\left|\left(zD_{\alpha}p(z) + n\beta\frac{|\alpha| - k}{1+k}p(z)\right) - \lambda nM\left(z + \beta\frac{|\alpha| - k}{1+k}\right)\right| \\ &\quad \le \left|\left(zD_{\alpha}Q(z) + n\beta\frac{|\alpha| - k}{1+k}Q(z)\right) - \overline{\lambda}nM\left(\frac{z}{k}\right)^{n}\left(\alpha + \beta\frac{|\alpha| - k}{1+k}\right)\right|. \end{aligned} \tag{2.25}$$

This implies that

$$\begin{aligned} &\left|zD_{\alpha}p(z) + n\beta\frac{|\alpha| - k}{1+k}p(z)\right| - \left|\lambda nM\left(z + \beta\frac{|\alpha| - k}{1+k}\right)\right| \\ &\quad \le \left|\left(zD_{\alpha}Q(z) + n\beta\frac{|\alpha| - k}{1+k}Q(z)\right) - \overline{\lambda}nM\left(\frac{z}{k}\right)^{n}\left(\alpha + \beta\frac{|\alpha| - k}{1+k}\right)\right|. \end{aligned} \tag{2.26}$$

As $|p(z)| = |Q(z)|$ for $|z| = k$, that is, $\max_{|z|=k}|p(z)| = \max_{|z|=k}|Q(z)| = M$, by Lemma 2.7 for $Q(z)$, we obtain the following:

$$\left|zD_{\alpha}Q(z) + n\beta\frac{|\alpha| - k}{1+k}Q(z)\right| < |\lambda|nMk^{-n}\left|\alpha + \beta\frac{|\alpha| - k}{1+k}\right|. \tag{2.27}$$

Thus, taking suitable choice of argument of λ, result is

$$\begin{aligned} &\left|\left(zD_{\alpha}Q(z) + n\beta\frac{|\alpha| - k}{1+k}Q(z)\right) - \overline{\lambda}nM\left(\frac{z}{k}\right)^{n}\left(\alpha + \beta\frac{|\alpha| - k}{1+k}\right)\right| \\ &\quad = |\lambda|nMk^{-n}\left|\alpha + \beta\frac{|\alpha| - k}{1+k}\right| - \left|zD_{\alpha}Q(z) + n\beta\frac{|\alpha| - k}{1+k}Q(z)\right|. \end{aligned} \tag{2.28}$$

By combining right hand side of (2.26) and (2.28) for $|z| = 1$ and $|\beta| \le 1$, we get that

$$\begin{aligned} &\left|zD_{\alpha}p(z) + n\beta\frac{|\alpha| - k}{1+k}p(z)\right| - \left|\lambda nM\left(z + \beta\frac{|\alpha| - k}{1+k}\right)\right| \\ &\quad \le |\lambda|\left|\alpha + \beta\frac{|\alpha| - k}{1+k}\right|nk^{-n}M - \left|zD_{\alpha}Q(z) + n\beta\frac{|\alpha| - k}{1+k}Q(z)\right|, \end{aligned} \tag{2.29}$$

That is,

$$\begin{aligned}&\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-k}{1+k}p(z)\right|+\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-k}{1+k}Q(z)\right|\\&\quad\le|\lambda|\left\{\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right|k^{-n}+\left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}nM.\end{aligned}\tag{2.30}$$

Taking $|\lambda|\to 1$, we have

$$\begin{aligned}&\left|zD_\alpha p(z)+n\beta\frac{|\alpha|-k}{1+k}p(z)\right|+\left|zD_\alpha Q(z)+n\beta\frac{|\alpha|-k}{1+k}Q(z)\right|\\&\quad\le\left\{\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right|k^{-n}+\left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}nM.\end{aligned}\tag{2.31}$$

Then, by applying the Principal Maximum Modulus for polynomial $p(z)$ when $k\le 1$, we get

$$\max_{|z|=k}|p(z)|\le\max_{|z|=1}|p(z)|.\tag{2.32}$$

This in conjunction with (2.31) gives the following result. □

Lemma 2.9. *Let $H(z)$ be a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, and $G(z)$ be a polynomial of degree not exceeding that of $H(z)$. If $|G(z)|\le|H(z)|$ for $|z|=k$, $k\le 1$, then for all $\alpha,\ \beta\in\mathbb{C}$ with $|\alpha|\ge k$, $|\beta|\le 1$ and $|z|=1$, we have*

$$\left|zD_\alpha G(z)+n\beta\left(\frac{|\alpha|-k}{1+k}\right)G(z)\right|\le\left|zD_\alpha H(z)+n\beta\left(\frac{|\alpha|-k}{1+k}\right)H(z)\right|.\tag{2.33}$$

Proof. Since $|\lambda G(z)|<|G(z)|\le|H(z)|$, for $|\lambda|<1$, and $|z|=k$, then by Rouche's Theorem $H(z)-\lambda G(z)$ and $H(z)$ have the same number of zeros in $|z|<k$. On the other hand by inequality $|G(z)\le|H(z)|$ for $|z|=k$, any zero of $H(z)$, that lies on $|z|=k$, is the zero of $G(z)$. Therefore, $H(z)-\lambda G(z)$ has all its zeros in the closed disk $|z|\le k$. Hence by Lemma 2.5, for all real or complex numbers α with $|\alpha|\ge k$ and $|z|=1$, we have

$$|zD_\alpha(H(z)-\lambda G(z))|\ge n\frac{|\alpha|-k}{1+k}|H(z)-\lambda G(z)|.\tag{2.34}$$

Now, consider a similar argument as used in the proof of Lemma 2.6, that for any value β with $|\beta|<1$, we have

$$\begin{aligned}|zD_\alpha(H(z)-\lambda G(z))|&\ge n\frac{|\alpha|-k}{1+k}|H(z)-\lambda G(z)|\\&>n|\beta|\frac{|\alpha|-k}{1+k}|H(z)-\lambda G(z)|,\end{aligned}\tag{2.35}$$

where $|z| = 1$, resulting in

$$T(z) = [zD_\alpha H(z) - \lambda z D_\alpha G(z)] + n\beta \frac{|\alpha| - k}{1+k}[H(z) - \lambda G(z)] \neq 0, \tag{2.36}$$

where $|z| = 1$.

That is,

$$T(z) = \left[zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1+k} H(z)\right] - \lambda \left[zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1+k} G(z)\right] \neq 0, \tag{2.37}$$

for $|z| = 1$.

We also conclude that

$$\left|zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1+k} H(z)\right| \geq \left|zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1+k} G(z)\right| \tag{2.38}$$

for $|z| = 1$.

If (2.38) is not true, then there is a point $z = z_0$ with $|z_0| = 1$ such that

$$\left|z_0 D_\alpha H(z_0) + n\beta \frac{|\alpha| - k}{1+k} H(z_0)\right| < \left|z_0 D_\alpha G(z_0) + n\beta \frac{|\alpha| - k}{1+k} G(z_0)\right|. \tag{2.39}$$

Take

$$\lambda = \frac{z_0 D_\alpha H(z_0) + n\beta((|\alpha| - k)/(1+k))H(z_0)}{z_0 D_\alpha G(z_0) + n\beta((|\alpha| - k)/(1+k))G(z_0)}, \tag{2.40}$$

then $|\lambda| < 1$ and with this choice of λ, we have from (2.37), $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| = 1$. For β with $|\beta| = 1$, (2.38) follows by continuity. This completes the proof. □

3. Proof of the Theorem

Proof of the Theorem 1.1. Under the assumption of Theorem 1.1, the polynomial $p(z) \neq 0$ in $|z| < k$, and thus if $m = \min_{|z|=k}|p(z)|$, then $m \leq |p(z)|$ for $|z| \leq k$. Now, for λ with $|\lambda| < 1$, we have

$$|\lambda m| < m \leq |p(z)|, \tag{3.1}$$

where $|z| = k$.

It follows by Rouche's Theorem that the polynomial $G(z) = p(z) - \lambda m$ has no zero in $|z| < k$. Therefore, the polynomial

$$H(z) = \left(\frac{z}{k}\right)^n \overline{G\left(\frac{k^2}{\overline{z}}\right)} = Q(z) - \overline{\lambda} m \left(\frac{z}{k}\right)^n, \tag{3.2}$$

will have all its zeros in $|z| \le k$, where $Q(z) = (z/k)^n \overline{p(k^2/\overline{z})}$. Also $|G(z)| = |H(z)|$ for $|z| = k$. Applying Lemma 2.9 for the polynomials $H(z)$ and $G(z)$, we have

$$\left| zD_\alpha G(z) + n\beta \frac{|\alpha| - k}{1+k} G(z) \right| \le \left| zD_\alpha H(z) + n\beta \frac{|\alpha| - k}{1+k} H(z) \right|, \tag{3.3}$$

where $|\alpha| \ge k,\ |\beta| \le 1$ and $|z| = 1$. Substituting for $G(z)$ and $H(z)$ in the above inequality, we conclude that for every $\alpha,\ \beta$, with $|\alpha| \ge k,\ |\beta| \le 1$, and $|z| = 1$

$$\begin{aligned} &\left| zD_\alpha p(z) - \lambda nmz + n\beta \frac{|\alpha| - k}{1+k} (p(z) - \lambda m) \right| \\ &\quad \le \left| zD_\alpha Q(z) - \overline{\lambda}\alpha nm \left(\frac{z}{k}\right)^n + n\beta \frac{|\alpha| - k}{1+k} \left(Q(z) - \overline{\lambda} m \left(\frac{z}{k}\right)^n \right) \right|, \end{aligned} \tag{3.4}$$

that is,

$$\begin{aligned} &\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - k}{1+k} p(z) - \lambda nm \left(z + \beta \frac{|\alpha| - k}{1+k} \right) \right| \\ &\quad \le \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1+k} Q(z) - \overline{\lambda} nm \left(\frac{z}{k}\right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1+k} \right) \right|. \end{aligned} \tag{3.5}$$

Since all the zeros of $Q(z)$ lie in $|z| \le k$ and $|p(z)| = |Q(z)|$ for $|z| = k$; therefore, by applying Lemma 2.6 to $Q(z)$, we have

$$\begin{aligned} \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1+k} Q(z) \right| &\ge nk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1+k} \right| \min_{|z|=k} |Q(z)| \\ &= nk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1+k} \right| m. \end{aligned} \tag{3.6}$$

Then, for an appropriate choice of the argument of λ, we have

$$\begin{aligned} \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1+k} Q(z) - \overline{\lambda} nm \left(\frac{z}{k}\right)^n \left(\alpha + \beta \frac{|\alpha| - k}{1+k} \right) \right| &= \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - k}{1+k} Q(z) \right| \\ &\quad - |\lambda| nmk^{-n} \left| \alpha + \beta \frac{|\alpha| - k}{1+k} \right|, \end{aligned} \tag{3.7}$$

where $|z| = 1$.

Then combining the right hand sides of (3.5) and (3.7), we can rewrite (3.5) as

$$\begin{aligned} &\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-k}{1+k}p(z)\right| - |\lambda| nm\left|z+\beta\frac{|\alpha|-k}{1+k}\right| \\ &\quad\le \left|zD_\alpha Q(z) + n\beta\frac{|\alpha|-k}{1+k}Q(z)\right| - |\lambda| nmk^{-n}\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right|, \end{aligned} \tag{3.8}$$

where $|z| = 1$.

Equivalently,

$$\begin{aligned} \left|zD_\alpha p(z) + n\beta\frac{|\alpha|-k}{1+k}p(z)\right| &\le \left|zD_\alpha Q(z) + n\beta\frac{|\alpha|-k}{1+k}Q(z)\right| \\ &\quad - |\lambda| nm\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right| - \left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}. \end{aligned} \tag{3.9}$$

As $|\lambda| \to 1$ we have

$$\begin{aligned} \left|zD_\alpha p(z) + n\beta\frac{|\alpha|-k}{1+k}p(z)\right| &\le \left|zD_\alpha Q(z) + n\beta\frac{|\alpha|-k}{1+k}Q(z)\right| \\ &\quad - nm\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right| - \left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}. \end{aligned} \tag{3.10}$$

It implies for every real or complex number β with $|\beta| \le 1$ and $|z| = 1$,

$$\begin{aligned} 2\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-k}{1+k}p(z)\right| &\le \left|zD_\alpha p(z) + n\beta\frac{|\alpha|-k}{1+k}p(z)\right| \\ &\quad + \left|zD_\alpha Q(z) + n\beta\frac{|\alpha|-k}{1+k}Q(z)\right| \\ &\quad - nm\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right| - \left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}. \end{aligned} \tag{3.11}$$

This in conjunction with Lemma 2.8 gives for $|\beta| \le 1$ and $|z| = 1$,

$$\begin{aligned} 2\left|zD_\alpha p(z) + n\beta\frac{|\alpha|-k}{1+k}p(z)\right| &\le n\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right| + \left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}\max_{|z|=1}|p(z)| \\ &\quad - n\left\{k^{-n}\left|\alpha+\beta\frac{|\alpha|-k}{1+k}\right| - \left|z+\beta\frac{|\alpha|-k}{1+k}\right|\right\}\min_{|z|=k}|p(z)|. \end{aligned} \tag{3.12}$$

The proof is complete. □

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