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Research Article

The Existence of Positive Solutions for Fractional Differential Equations with Sign Changing Nonlinearities

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We investigate the existence of at least two positive solutions to eigenvalue problems of fractional differential equations with sign changing nonlinearities in more generalized boundary conditions. Our analysis relies on the Avery-Peterson fixed point theorem in a cone. Some examples are given for the illustration of main results.

1. Introduction

The theory of fractional differential equations has become an important aspect of differential equations (see [1–8]). Boundary value problems of fractional differential equations have been investigated in many papers (see [9–46]). The existence of positive solutions to boundary value problems of fractional differential equations has been studied by many authors when nonlinearities are positive (see [9–24]). There are a few papers to study the existence of positive solutions of semipositone fractional differential equations. For example, using the Krasnoselskii fixed point theorem, Yuan et al. [9] discussed the existence of positive solutions for the singular positone and semipositone two-point boundary value problems

$$\begin{gathered} D_{0^+}^{\alpha}u(t) = \mu a(t)f(t,u(t)), \\ u(0) = u'(0) = u(1) = u'(1) = 0, \quad 3 < \alpha \leq 4, \end{gathered} \tag{1.1}$$

where $\mu > 0$, a and f are continuous. In [10], Wang et al. studied the existence of positive solutions for the singular semipositone two-point boundary value problems

$$D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \tag{1.2}$$

$$u(0) = u'(0) = u(1) = 0, \quad 2 < \alpha \le 3, \tag{1.3}$$

where $\lambda > 0$, f is continuous.

In [11], using Krasnoselskii fixed point theorem, Goodrich discussed the existence of at least one positive solutions for the system of fractional boundary value problems

$$\begin{gathered} -D_{0^+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f(y_1(t), y_2(t)), \qquad -D_{0^+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g(y_1(t), y_2(t)), \\ y_1^{(i)}(0) = y_2^{(i)}(0) = 0, \quad 0 \le i \le n-2, \qquad D_{0^+}^{\alpha} y_1(t)|_{t=1} = \phi_1(y), \qquad D_{0^+}^{\alpha} y_2(t)|_{t=1} = \phi_2(y), \end{gathered} \tag{1.4}$$

where a_1, a_2, f, and g are nonnegative for $t \in [0,1]$.

Motivated by the excellent results mentioned above, in this paper, we investigate the existence of at least two positive solutions for the problem

$$\begin{gathered} D_{0^+}^{\alpha} y(t) + \lambda f(t, y(t)) = 0, \quad t \in [0,1], \\ y^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ D_{0^+}^{\beta} y(t)|_{t=1} = h(y), \end{gathered} \tag{1.5}$$

where $\lambda > 0$, $\alpha \in (n-1, n]$, $n \ge 3$, $1 \le \beta \le n-2 < \alpha-1$, $f \in C([0,1] \times \mathbb{R}^+, [-M, \infty))$, $M > 0$, $h \in C\ (C(\mathbb{R}^+), \mathbb{R}^+)$, $\mathbb{R}^+ = [0, \infty)$. The main tool is the Avery-Peterson theorem. To the best of our knowledge, this is the first paper dealing with eigenvalue problems of fractional differential equations with sign changing nonlinearities involving more general boundary conditions. Our results improve some of the earlier work presented in [10, 17, 46].

2. Preliminaries

For the convenience of the readers, we present here some necessary definitions and lemmas from fractional calculus theory. For more details see [1, 2].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractiontal derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s) ds, \tag{2.2}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$, where $n = [\alpha]+1$.

Lemma 2.3. *Assume $f \in C[0,1]$, $q \geq p \geq 0$, then*

$$D_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{q-p} f(t). \tag{2.3}$$

Lemma 2.4. *Assume $\alpha > 0$, then*

(1) *If $\lambda > -1$, $\lambda \neq \alpha - i$, $i = 1,2,\ldots,[\alpha]+1$, $t > 0$, then*

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}. \tag{2.4}$$

(2) $D_{0^+}^{\alpha} t^{\alpha-i} = 0$, $i = 1,2,\ldots,n$.

(3) $D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$, *for a.e.* $t \in [0,1]$, *where* $u \in L^1[0,1]$.

(4) $D_{0^+}^{\alpha} u(t) = 0$ *if and only if*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, where n is the least integer greater than or equal to α.

Lemma 2.5 (see[11]). *Given $g \in C[0,1]$, y is a solution of the problem*

$$\begin{gathered} D_{0^+}^{\alpha} y(t) + g(t) = 0, \quad t \in [0,1], \\ y^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\ D_{0^+}^{\beta} y(t)|_{t=1} = h(y), \end{gathered} \tag{2.6}$$

if and only if it satisfies

$$y(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1} h(y) + \int_0^1 G(t,s) g(s) ds, \tag{2.7}$$

where

$$G(t,s)=\begin{cases}\dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1.\end{cases}\tag{2.8}$$

Lemma 2.6 (see[11]). *$G(t,s)$ is continuous on $[0,1]\times[0,1]$ and*

$$0\le G(t,s)\le G(1,s),\quad t,s\in[0,1].\tag{2.9}$$

Lemma 2.7. *$G(t,s)\ge t^{\alpha-1}G(1,s)$, $t,s\in[0,1]$.*

Proof. For $s\le t$,

$$\begin{aligned}G(t,s)&=\frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\&=t^{\alpha-1}\frac{(1-s)^{\alpha-\beta-1}-(1-s/t)^{\alpha-1}}{\Gamma(\alpha)}\\&\ge t^{\alpha-1}\frac{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}=t^{\alpha-1}G(1,s).\end{aligned}\tag{2.10}$$

For $s>t$,

$$G(t,s)=\frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\ge t^{\alpha-1}\frac{(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}=t^{\alpha-1}G(1,s).\tag{2.11}$$

□

By simple calculation, we can get

$$\int_0^1 G(1,s)ds=\frac{\beta}{(\alpha-\beta)\Gamma(\alpha+1)},\qquad \int_{1/2}^1 G(1,s)ds=\frac{2^{\beta}\alpha-\alpha+\beta}{2^{\alpha}(\alpha-\beta)\Gamma(\alpha+1)}.\tag{2.12}$$

By Lemma 2.5, we can easily get the following lemma.

Lemma 2.8. *The boundary value problem*

$$\begin{aligned}&D_{0^+}^{\alpha}u(t)+1=0,\quad 0<t<1,\\&u^{(i)}(0)=D_{0^+}^{\beta}u(t)|_{t=1}=0,\quad 0\le i\le n-2,\end{aligned}\tag{2.13}$$

has a unique solution

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\alpha-\beta} - \frac{t}{\alpha}\right). \tag{2.14}$$

Obviously, u satisfies

$$\frac{\beta t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha+1)} \le u(t) \le \frac{t^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}, \quad t \in [0,1]. \tag{2.15}$$

Lemma 2.9. *$\tilde{y} \ge \lambda M u$ is a solution of the following problem:*

$$\begin{gathered} D_{0^+}^{\alpha}\tilde{y} + \lambda\left[f\left(t, \tilde{y} - \lambda M u\right) + M\right] = 0, \quad t \in [0,1], \\ \tilde{y}^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ D_{0^+}^{\beta}\tilde{y}(t)|_{t=1} = h(\tilde{y} - \lambda M u), \end{gathered} \tag{2.16}$$

if and only if $y = \tilde{y} - \lambda M u$ is a positive solution of (1.5).

Proof. In fact, if y is a positive solution of the problem (1.5), by Lemma 2.8, we get that y satisfies

$$\begin{gathered} D_{0^+}^{\alpha}(y + \lambda M u) + \lambda\left[f(t, y) + M\right] = 0, \quad t \in [0,1], \\ (y + \lambda M u)^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ D_{0^+}^{\beta}(y + \lambda M u)|_{t=1} = h(y). \end{gathered} \tag{2.17}$$

Take $\tilde{y} = y + \lambda M u$. Then $\tilde{y}$ satisfies (2.16) and $\tilde{y} \ge \lambda M u$.

On the other hand, if $\tilde{y}$ is a solution of (2.16) and $\tilde{y} \ge \lambda M u$. Take $y = \tilde{y} - \lambda M u$. By Lemma 2.8, we can easily get that y satisfies (1.5). Clearly, $y \ge 0$. □

Define functions $\tilde{h}$, $\tilde{f}$ and an operator $T : C[0,1] \to C[0,1]$ by

$$\begin{gathered} \tilde{h}(y) = h(\max\{y - \lambda M u, 0\}), \qquad \tilde{f}(t, y) = f(t, \max\{y - \lambda M u, 0\}) + M. \\ T y(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1}\tilde{h}(y) + \lambda \int_0^1 G(t,s)\tilde{f}(s, y(s))ds. \end{gathered} \tag{2.18}$$

Obviously, $y \ge \lambda M u$ is a fixed point of the operator T if and only if $y - \lambda M u$ is a positive solution of the problem (1.5).

Take $X = C[0,1]$ with norm $\|x\| = \max_{t\in[0,1]}|x(t)|$. Define a cone P by

$$P = \left\{ y \in C[0,1] \mid y(t) \ge t^{\alpha-1}\|y\|,\ t \in [0,1] \right\}. \tag{2.19}$$

Lemma 2.10. *$T : P \to P$ is a completely continuous operator.*

Proof. Take $y \in P$. By Lemmas 2.6 and 2.7, we get

$$Ty(t) \geq t^{\alpha-1}\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\widetilde{h}(y) + \lambda\int_0^1 G(1,s)\widetilde{f}(s,y(s))ds\right] \geq t^{\alpha-1}\|Ty\|. \tag{2.20}$$

So, $T : P \to P$. Let $\Omega \subset P$ be bounded. It follows from the continuity of h, f that there exist constants M_1 and M_2 such that $\widetilde{h}(y) \leq M_1$ and $\widetilde{f}(t,y) \leq M_2$ for $t \in [0,1]$, $y \in \Omega$. Thus,

$$\|Ty\| \leq \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}M_1 + \lambda M_2\int_0^1 G(1,s)ds. \tag{2.21}$$

That is $T(\Omega)$ is bounded. For $y \in \Omega$, $t_1, t_2 \in [0,1]$,

$$\left|Ty(t_1) - Ty(t_2)\right| \leq M_1\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\left|t_1^{\alpha-1} - t_2^{\alpha-1}\right| + \lambda M_2\int_0^1 |G(t_1,s) - G(t_2,s)|ds. \tag{2.22}$$

By the uniform continuity of $t^{\alpha-1}$ and $G(t,s)$, we get that $T(\Omega)$ is equicontinuous. Obviously, $T : P \to P$ is continuous. By the Arzela-Ascoli theorem, we get that $T : P \to P$ is completely continuous. □

Definition 2.11. A map ϕ is said to be a nonnegative, continuous, and concave functional on a cone P of a real Banach space E if and only if $\phi : P \to \mathbb{R}^+$ is continuous and

$$\phi(tx + (1-t)y) \geq t\phi(x) + (1-t)\phi(y), \tag{2.23}$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.12. A map Φ is said to be a nonnegative, continuous, and convex functional on a cone P of a real Banach space E iff $\Phi : P \to \mathbb{R}^+$ is continuous and

$$\Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y), \tag{2.24}$$

for all $x, y \in P$ and $t \in [0,1]$.

Let φ and Θ be nonnegative, continuous, and convex functional on P, Φ a nonnegative, continuous, and concave functional on P, and Ψ a nonnegative continuous functional on P. Then, for positive numbers a, b, c, and d, we define the following sets:

$$\begin{gathered} P(\varphi, d) = \{x \in P : \varphi(x) < d\}, \\ P(\varphi, \Phi, b, d) = \{x \in P : b \leq \Phi(x),\ \varphi(x) \leq d\}, \\ P(\varphi, \Theta, \Phi, b, c, d) = \{x \in P : b \leq \Phi(x),\ \Theta(x) \leq c,\ \varphi(x) \leq d\}, \\ R(\varphi, \Psi, a, d) = \{x \in P : a \leq \Psi(x),\ \varphi(x) \leq d\}. \end{gathered} \tag{2.25}$$

We will use the following fixed point theorem of Avery and Peterson to study the problem (1.5).

Theorem 2.13 (see [47]). *Let P be a cone in a real Banach space E. Let φ and Θ be nonnegative, continuous, and convex functionals on P, Φ a nonnegative, continuous, and concave functional on P, and Ψ a nonnegative continuous functional on P satisfying $\Psi(kx) \leq k\Psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers M and d,*

$$\Phi(x) \leq \Psi(x), \quad \|x\| \leq M\varphi(x) \tag{2.26}$$

for all $x \in \overline{P(\varphi, d)}$. Suppose that

$$T : \overline{P(\varphi, d)} \longrightarrow \overline{P(\varphi, d)} \tag{2.27}$$

is completely continuous and there exist positive numbers a, b, c with $a < b$, such that the following conditions are satisfied:

(S1) $\{x \in P(\varphi, \Theta, \Phi, b, c, d) : \Phi(x) > b\} \neq \emptyset$ *and* $\Phi(Tx) > b$ *for* $x \in P(\varphi, \Theta, \Phi, b, c, d)$;

(S2) $\Phi(Tx) > b$ *for* $x \in P(\varphi, \Phi, b, d)$ *with* $\Theta(Tx) > c$;

(S3) $0 \notin R(\varphi, \Psi, a, d)$ *and* $\Psi(Tx) < a$ *for* $x \in R(\varphi, \Psi, a, d)$ *with* $\Psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, such that

$$\begin{gathered} \varphi(x_i) \leq d, \quad \text{for } i = 1, 2, 3, \\ b < \Phi(x_1), \quad a < \Psi(x_2), \quad \Phi(x_2) < b, \quad \Psi(x_3) < a. \end{gathered} \tag{2.28}$$

3. Main Results

We define a concave function $\Phi(x) = \min_{t\in[1/2,1]}|x(t)|$ and convex functions $\Psi(x) = \Theta(x) = \varphi(x) = \|x\|$.

Theorem 3.1. *Assume that there exists a constant $0 < l < \Gamma(\alpha)/\Gamma(\alpha-\beta)$, such that $h(y) \leq l\|y\|$ for $y \in P$. In addition, suppose that there exist constants k, a, b, c, d with $k > 2^{2\alpha-1}\beta\Gamma(\alpha)/(\Gamma(\alpha) - \Gamma(\alpha-\beta)l)$, $[\Gamma(\alpha) - \Gamma(\alpha-\beta)l]\alpha M/\beta\Gamma(\alpha) < a < b - [\Gamma(\alpha) - \Gamma(\alpha-\beta)l]\alpha M/\beta\Gamma(\alpha) < 2^{\alpha-1}b < c < d$, such that the following conditions hold:*

(C1) $f(t, y) \leq d - M$, *for* $(t, y) \in [0, 1] \times [0, d]$;

(C2) $f(t, y) \geq kb - M$, *for* $(t, y) \in [1/2, 1] \times [b - [\Gamma(\alpha) - \Gamma(\alpha-\beta)l]\alpha M/\beta\Gamma(\alpha), c]$;

(C3) $f(t, y) \leq a - M$, *for* $(t, y) \in [0, 1] \times [0, a]$.

Then the problem (1.5) *has at least two positive solutions for*

$$\frac{2^{2\alpha-1}(\alpha-\beta)\Gamma(\alpha+1)}{k\left[\alpha(2^{\beta}-1)+\beta\right]} < \lambda < \frac{\alpha(\alpha-\beta)\left[\Gamma(\alpha)-\Gamma(\alpha-\beta)l\right]}{\beta}. \tag{3.1}$$

Proof. Take $y \in \overline{P(\varphi, d)}$. By $\| \max\{y - \lambda M u, 0\}\| \leq d$, (C1), Lemma 2.6, (2.12), and (3.1), we have

$$\begin{aligned} \|Ty\| &\leq \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} ld + \lambda d \int_0^1 G(1,s)ds \\ &= \left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} l + \lambda \int_0^1 G(1,s)ds\right] d \leq d. \end{aligned} \tag{3.2}$$

This means that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

It is easy to see that $\{y \in P(\varphi, \Theta, \Phi, b, c, d) : \Phi(y) > b\} \neq \emptyset$. $y \in P(\varphi, \Theta, \Phi, b, c, d)$ implies $\min_{t\in[1/2,1]} y(t) \geq b$, $\|y\| \leq c$. It follows from (2.15) and (3.1) that $\min_{t\in[1/2,1]}(y - \lambda M u) \geq b - \alpha[\Gamma(\alpha) - \Gamma(\alpha-\beta)l]M/\beta\Gamma(\alpha)$. By (C2), (2.12), (3.1), and Lemma 2.7, we get

$$\Phi(Ty) = \min_{t\in[1/2,1]} Ty(t) \geq \lambda \min_{t\in[1/2,1]} \int_0^1 G(t,s)\tilde{f}(s, y(s))ds \geq \left(\frac{1}{2}\right)^{\alpha-1} \lambda kb \int_{1/2}^1 G(1,s)ds > b. \tag{3.3}$$

So, the condition (S1) of Theorem 2.13 holds.

Take $y \in P(\varphi, \Phi, b, d)$ with $\Theta(Ty) > c$. By $Ty \in P$, we get

$$\min_{t\in[1/2,1]} Ty(t) \geq \min_{t\in[1/2,1]} t^{\alpha-1}\|Ty\| \geq \frac{1}{2^{\alpha-1}}\|Ty\| > \frac{1}{2^{\alpha-1}} c > b. \tag{3.4}$$

Thus, (S2) holds.

By $a > 0$, we have $0 \notin R(\varphi, \Psi, a, d)$. Take $y \in R(\varphi, \Psi, a, d)$ with $\Psi(y) = a$. By (C3), we get

$$\begin{aligned} \Psi(Ty) = \|Ty\| &\leq \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} la + \lambda a \int_0^1 G(1,s)ds \\ &= \left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} l + \lambda \int_0^1 G(1,s)ds\right] a \leq a. \end{aligned} \tag{3.5}$$

By Theorem 2.13, we get that T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\varphi, d)}$ such that $\|y_i\| \leq d$, $i = 1, 2, 3$, and

$$b < \Phi(y_1), \quad a < \Psi(y_2), \quad \Phi(y_2) < b, \quad \Psi(y_3) < a. \tag{3.6}$$

If $y \in P$ and $\|y\| > a$, by (2.15) and (3.1), we have

$$y(t) \geq t^{\alpha-1}\|y\| > at^{\alpha-1} > \lambda M u(t). \tag{3.7}$$

Obviously, $\|y_1\| > b > a$ and $\|y_2\| > a$. So, $y_1 - \lambda M u$, $y_2 - \lambda M u$ are two positive solutions of (1.5). The proof is completed. □

4. Example

For convenience, we define the following notations:

$$[a,b] := \{x : x \in \mathbb{R}, a \le x \le b\}, \quad (a,b] := \{x : x \in \mathbb{R}, a < x \le b\}, \quad \text{for } a,b \in \mathbb{R},\ a < b. \tag{4.1}$$

Example 4.1. Consider the following boundary value problem:

$$\begin{gathered} D_{0^+}^{5/2} y(t) + \lambda f\left(t, y(t)\right) = 0, \quad t \in [0,1], \\ y(0) = y'(0) = 0, \\ y'(1) = h(y), \end{gathered} \tag{4.2}$$

where $h(y) = \int_0^1 y(s)dg(s)$, $g(s)$ is a bounded variation function on $[0,1]$ with $0 < \bigvee_0^1(g) \le 1 < 3/2$,

$$f(t,y) = \begin{cases} \sin\left(t - \dfrac{1}{2}\right)\pi - 1 - \sqrt{y}, & (t,y) \in [0,1] \times [0,6], \\ \sin\left(t - \dfrac{1}{2}\right)\pi - 1 + 601(y-6) - \sqrt{y}, & (t,y) \in [0,1] \times (6,7], \\ \sin\left(t - \dfrac{1}{2}\right)\pi + 600 - \sqrt{y}, & (t,y) \in [0,1] \times (7,900], \\ \sin\left(t - \dfrac{1}{2}\right)\pi + 570, & (t,y) \in [0,1] \times (900,+\infty). \end{cases} \tag{4.3}$$

Corresponding to the problem (1.5), we get that $\alpha = 5/2$, $\beta = 1$, $n = 3$, $h(y) \le \|y\| \bigvee_0^1(g)$. Take $l = 1$, $k = 50$, $M = a = 6$, $b = 12$, $c = 36$, $d = 620$.

By simple calculation, we can get that the conditions of Theorem 3.1 are satisfied. So, when $(9/35)\sqrt{\pi} < \lambda < (15/16)\sqrt{\pi}$, the problem (4.2) has at least two positive solutions.

Example 4.2. Consider the following boundary value problem:

$$\begin{gathered} D_{0^+}^{7/2} y(t) + \lambda f\left(t, y(t)\right) = 0, \quad t \in [0,1], \\ y(0) = y'(0) = y''(0) = 0, \\ D_{0^+}^{3/2} y(t)|_{t=1} = h(y), \end{gathered} \tag{4.4}$$

where $h(y) = \int_0^1 y(s)dg(s)$, $g(s)$ is a bounded variation function on $[0,1]$ with $0 < \bigvee_0^1(g) \leq (15/8)\sqrt{\pi} - 1$,

$$f(t,y) = \begin{cases} -\dfrac{3}{4}\cos\dfrac{\pi}{2}t - \dfrac{1}{4}\sqrt{y}, & (t,y) \in [0,1] \times [0,1], \\ -\dfrac{3}{4}\cos\dfrac{\pi}{2}t + 3205(y-1) - \dfrac{1}{4}\sqrt{y}, & (t,y) \in [0,1] \times (1,1.2], \\ -\dfrac{3}{4}\cos\dfrac{\pi}{2}t + 641 - \dfrac{1}{4}\sqrt{y}, & (t,y) \in [0,1] \times (1.2,12], \\ -\dfrac{3}{4}\cos\dfrac{\pi}{2}t + 641 - \dfrac{1}{2}\sqrt{3}, & (t,y) \in [0,1] \times (12,+\infty). \end{cases} \tag{4.5}$$

Corresponding to the problem (1.5), we get that $\alpha = 7/2$, $\beta = 3/2$, $n = 4$, $h(y) \leq \|y\|\bigvee_0^1(g)$. Take $l = (15/8)\sqrt{\pi} - 1$, $k = 320$, $M = a = 1$, $b = 2$, $c = 12$, $d = 642$.

Obviously, $f \in C([0,1] \times \mathbb{R}^+, [-1,\infty))$, $h(y) \leq l\|y\|$. By simple calculation, we can get that $k, l, a, b, c, d, \alpha, \beta, M$ satisfy $0 < l < \Gamma(\alpha)/\Gamma(\alpha-\beta)$, $k > 2^{2\alpha-1}\beta\Gamma(\alpha)/(\Gamma(\alpha) - \Gamma(\alpha-\beta)l)$, and

$$\frac{[\Gamma(\alpha) - \Gamma(\alpha-\beta)l]\alpha M}{\beta\Gamma(\alpha)} < a < b - \frac{[\Gamma(\alpha) - \Gamma(\alpha-\beta)l]\alpha M}{\beta\Gamma(\alpha)} < 2^{\alpha-1}b < c < d. \tag{4.6}$$

It is easy to see that $f(t,y) + 1 \leq 642$, for $(t,y) \in [0,1] \times [0,642]$, and $f(t,y) + 1 \leq 1$, for $(t,y) \in [0,1] \times [0,1]$. So, conditions (C1) and (C3) of Theorem 3.1 hold.

For $(t,y) \in [1/2,1] \times [2 - 56/45\sqrt{\pi}, 12]$, $f(t,y) + 1 \geq kb = 640$. Therefore, condition (C2) of Theorem 3.1 holds. So, for $(21/8(7\sqrt{2} - 2))\sqrt{\pi} < \lambda < 14/3$, the problem (4.4) has at least two positive solutions.

Specially, in Example 4.2, we take

$$g(t) = \begin{cases} 0, & t < \xi, \\ 1, & t \geq \xi, \end{cases} \tag{4.7}$$

where $0 < \xi < 1$ and all other conditions remain unchanged. Then $h(y) = y(\xi)$. Clearly, $\vee_0^1(g) = 1 < (15/8)\sqrt{\pi} - 1$. The problem (4.4) has at least two positive solutions for $(21/8(7\sqrt{2} - 2))\sqrt{\pi} < \lambda < 14/3$.

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