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Research Article

The Local Strong and Weak Solutions for a Generalized Novikov Equation

Meng Wu and Yue Zhong

Department of Applied Mathematics, Southwestern University of Finance and Economics, Chengdu 610074, China

Correspondence should be addressed to Meng Wu, wumeng@swufe.edu.cn

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The Kato theorem for abstract differential equations is applied to establish the local well-posedness of the strong solution for a nonlinear generalized Novikov equation in space $C([0,T),H^s(R)) \cap C^1([0,T),H^{s-1}(R))$ with $s > (3/2)$. The existence of weak solutions for the equation in lower-order Sobolev space $H^s(R)$ with $1 \leq s \leq (3/2)$ is acquired.

1. Introduction

The Novikov equation with cubic nonlinearities takes the form

$$u_t - u_{txx} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx}, \tag{1.1}$$

which was derived by Vladimir Novikov in a symmetry classification of nonlocal partial differential equations [1]. Using the perturbed symmetry approach, Novikov was able to isolate (1.1) and investigate its symmetries. A scalar Lax pair for it was discovered in [1, 2] and was shown to be related to a negative flow in the Sawada-Kotera hierarchy. Many conserved quantities were found as well as a bi-Hamiltonian structure. The scattering theory was employed by Hone et al. [3] to find nonsmooth explicit soliton solutions with multiple peaks for (1.1). This multiple peak property is common with the Camassa-Holm and Degasperis-Procesi equations (see [4–10]). Ni and Zhou [11] proved that the Novikov equation associated with initial value is locally well-posedness in Sobolev space H^s with $s > (3/2)$ by using the abstract Kato theorem. Two results about the persistence properties of the strong solution for (1.1) were established. It is shown in [12] that the local well-posedness for the periodic Cauchy problem of the Novikov equation in Sobolev space H^s with $s > (5/2)$.

The orbit invariants are employed to get the existence of periodic global strong solution if the Sobolev index $s \geq 3$ and a sign condition holds. For analytic initial data, the existence and uniqueness of analytic solutions for (1.1) are also obtained in [12].

In this paper, motivated by the work in [7, 13], we study the following generalized Novikov equation:

$$u_t - u_{txx} + 4u^2 u_x = 3u u_x u_{xx} + u^2 u_{xxx} + \beta \partial_x \left[(u_x)^N \right], \tag{1.2}$$

where $N \geq 1$ is a natural number. In fact, (1.1) has the property

$$\int_R \left(u^2 + u_x^2 \right) dx = \text{constant}. \tag{1.3}$$

Due to the term $\beta \partial_x [(u_x)^N]$ appearing in (1.2), the conservation law (1.3) for (1.2) is not valid. This brings us a difficulty to obtain the bounded estimates for the solution of (1.2). However, we will overcome this difficulty to investigate the local existence and uniqueness of the solution to (1.2) subject to initial value $u_0(x) \in H^s(R)$ with $s > (3/2)$. Meanwhile, a sufficient condition is presented to guarantee the existence of local weak solution for (1.2).

The main tasks of this work are two-fold. Firstly, by using the Kato theorem for abstract differential equations, we establish the local existence and uniqueness of solutions for the (1.2) with any β and arbitrary positive integer N in space $C([0,T), H^s(R)) \bigcap C^1([0,T), H^{s-1}(R))$ with $s > (3/2)$. Secondly, it is shown that there exist local weak solutions in lower-order Sobolev space $H^s(R)$ with $1 \leq s \leq (3/2)$. The ideas of proving the second result come from those presented in Li and Olver [8].

2. Main Results

Firstly, some notations are presented as follows.

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty) \times R$ is denoted by C_0^∞. $L^p = L^p(R)(1 \leq p < +\infty)$ is the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R \setminus e} |h(t,x)|$. For any real number s, $H^s = H^s(R)$ denotes the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1 + |\xi|^2 \right)^s \left| \widehat{h}(t,\xi) \right|^2 d\xi \right)^{1/2} < \infty, \tag{2.1}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x) dx$.

For $T > 0$ and nonnegative number s, $C([0,T); H^s(R))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1 - \partial_x^2)^{1/2}$.

The Cauchy problem for (1.2) is written in the form

$$\begin{gathered} u_t - u_{txx} = -\frac{4}{3}\left(u^3\right)_x + \frac{1}{3}\partial_x^3 u^3 - 2\partial_x\left(u u_x^2\right) + u u_x u_{xx} + \beta \partial_x \left[(u_x)^N \right], \\ u(0,x) = u_0(x), \end{gathered} \tag{2.2}$$

which is equivalent to

$$u_t + u^2 u_x = \Lambda^{-2}\left[-3u^2 u_x - \frac{3}{2}\partial_x\left(u u_x^2\right) - \frac{1}{2}u_x^3 + \beta\partial_x\left[(u_x)^N\right]\right],$$
$$u(0,x) = u_0(x). \tag{2.3}$$

Now, we state our main results.

Theorem 2.1. *Let $u_0(x) \in H^s(R)$ with $s > (3/2)$. Then problem* (2.2) *or problem* (2.3) *has a unique solution $u(t,x) \in C([0,T); H^s(R)) \bigcap C^1([0,T); H^{s-1}(R))$, where $T > 0$ depends on $\|u_0\|_{H^s(R)}$.*

Theorem 2.2. *Suppose that $u_0(x) \in H^s$ with $1 \le s \le (3/2)$ and $\|u_{0x}\|_{L^\infty} < \infty$. Then there exists a $T > 0$ such that* (1.2) *subject to initial value $u_0(x)$ has a weak solution $u(t,x) \in L^2([0,T], H^s)$ in the sense of distribution and $u_x \in L^\infty([0,T] \times R)$.*

3. Local Well-Posedness

Consider the abstract quasilinear evolution equation

$$\frac{dv}{dt} + A(v)v = f(v), \quad t \ge 0, \qquad v(0) = v_0. \tag{3.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y,X)$ be the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. We state the following conditions in which ρ_1, ρ_2, ρ_3, and ρ_4 are constants depending on $\max\{\|y\|_Y, \|z\|_Y\}$.

(I) $A(y) \in L(Y,X)$ for $y \in X$ with

$$\left\|(A(y) - A(z))w\right\|_X \le \rho_1\|y - z\|_X\|w\|_Y, \quad y,z,w \in Y, \tag{3.2}$$

and $A(y) \in G(X,1,\beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(II) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\left\|(B(y) - B(z))w\right\|_X \le \rho_2\|y - z\|_Y\|w\|_X, \quad y,z \in Y,\ w \in X. \tag{3.3}$$

(III) $f : Y \to Y$ extends to a map from X into X, is bounded on bounded sets in Y, and satisfies

$$\left\|f(y) - f(z)\right\|_Y \le \rho_3\|y - z\|_Y, \quad y,z \in Y,$$
$$\left\|f(y) - f(z)\right\|_X \le \rho_4\|y - z\|_X, \quad y,z \in Y. \tag{3.4}$$

Kato Theorem (see [14])

Assume that (I), (II), and (III) hold. If $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to problem (3.1) such that

$$v = v(\cdot, v_0) \in C([0,T);Y) \bigcap C^1([0,T);X). \tag{3.5}$$

Moreover, the map $v_0 \to v(\cdot, v_0)$ is a continuous map from Y to the space

$$C([0,T);Y) \bigcap C^1([0,T);X). \tag{3.6}$$

For problem (2.3), we set $A(u) = u^2\partial_x$, $Y = H^s(R)$, $X = H^{s-1}(R)$, $\Lambda = (1-\partial_x^2)^{1/2}$,

$$f(u) = \Lambda^{-2}\left[-3u^2u_x - \frac{3}{2}\partial_x\left(uu_x^2\right) - \frac{1}{2}u_x^3 + \beta\partial_x\left[(u_x)^N\right]\right], \tag{3.7}$$

and $Q = \Lambda$. In order to prove Theorem 2.1, we only need to check that $A(u)$ and $f(u)$ satisfy assumptions (I)–(III).

Lemma 3.1 (Ni and Zhou [11]). *The operator $A(u) = u^2\partial_x$ with $u \in H^s(R)$, $s > (3/2)$ belongs to $G(H^{s-1}, 1, \beta)$.*

Lemma 3.2 (Ni and Zhou [11]). *Let $A(u) = u^2\partial_x$ with $u \in H^s$ and $s > (3/2)$. Then $A(u) \in L(H^s, H^{s-1})$ for all $u \in H^s$. Moreover,*

$$\|(A(u) - A(z))w\|_{H^{s-1}} \le \rho_1\|u - z\|_{H^{s-1}}\|w\|_{H^s}, \quad u, z, w \in H^s(R). \tag{3.8}$$

Lemma 3.3 (Ni and Zhou [11]). *For $s > (3/2)$, $u, z \in H^s$ and $w \in H^{s-1}$, it holds that $B(u) = [\Lambda, u^2\partial_x]\Lambda^{-1} \in L(H^{s-1})$ for $u \in H^s$ and*

$$\|(B(u) - B(z))w\|_{H^{s-1}} \le \rho_2\|u - z\|_{H^s}\|w\|_{H^{s-1}}. \tag{3.9}$$

Lemma 3.4. *Let r and q be real numbers such that $-r < q \le r$. Then*

$$\begin{aligned} &\|uv\|_{H^q} \le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r > \frac{1}{2}, \\ &\|uv\|_{H^{r+q-1/2}} \le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r < \frac{1}{2}, \\ &\|uv\|_{H^{r_1}} \le c\|u\|_{L^\infty}\|v\|_{H^{r_1}}, \quad \text{if } r_1 \le 0. \end{aligned} \tag{3.10}$$

The above first two inequalities of this lemma can be found in [14, 15], and the third inequality can be found in [7].

Lemma 3.5. *Letting* $u, z \in H^s$ *with* $s > (3/2)$, *then* $f(u)$ *is bounded on bounded sets in* H^s *and satisfies*

$$\left\| f(u) - f(z) \right\|_{H^s} \le \rho_3 \|u - z\|_{H^s}, \tag{3.11}$$

$$\left\| f(u) - f(z) \right\|_{H^{s-1}} \le \rho_4 \|u - z\|_{H^{s-1}}. \tag{3.12}$$

Proof. Using the algebra property of the space H^{s_0} with $s_0 > (1/2)$ and $s - 1 > (1/2)$, we have

$$\begin{aligned}
&\left\| \Lambda^{-2}\left[\partial_x\left(uu_x^2\right) - \partial_x\left(zz_x^2\right)\right] \right\|_{H^s} \\
&\quad \le c\left\| uu_x^2 - zz_x^2 \right\|_{H^{s-1}} \\
&\quad \le c\|u - z\|_{H^{s-1}} \left\| u_x^2 \right\|_{H^{s-1}} + \|z\|_{H^{s-1}} \left\| u_x^2 - z_x^2 \right\|_{H^{s-1}} \\
&\quad \le c\|u - z\|_{H^s} \|u\|_{H^s}^2 + \|z\|_{H^{s-1}} \|u - z\|_{H^{s-1}} \|u + z\|_{H^{s-1}} \\
&\quad \le c\|u - z\|_{H^s} \left(\|u\|_{H^s}^2 + \|z\|_{H^{s-1}} (\|u\|_{H^s} + \|z\|_{H^s}) \right), \\
&\left\| \Lambda^{-2}\partial_x\left[(u_x)^N - (z_x)^N\right] \right\|_{H^s} \\
&\quad \le c\left\| (u_x)^N - (z_x)^N \right\|_{H^{s-1}} \\
&\quad \le c\|u_x - z_x\|_{H^{s-1}} \sum_{j=0}^{N-1} \|u_x\|_{H^{s-1}}^{N-j} \|z_x\|_{H^{s-1}}^{j} \le c\|u - z\|_{H^s} \sum_{j=0}^{N-1} \|u\|_{H^s}^{N-j} \|z\|_{H^s}^{j}.
\end{aligned} \tag{3.13}$$

It follows from Lemma 3.4 and $s - 1 > (1/2)$ that

$$\begin{aligned}
&\left\| \Lambda^{-2}\left[\partial_x\left(uu_x^2\right) - \partial_x\left(zz_x^2\right)\right] \right\|_{H^{s-1}} \\
&\quad \le c\left\| uu_x^2 - zz_x^2 \right\|_{H^{s-2}} \\
&\quad \le c\left\| (u - z)u_x^2 \right\|_{H^{s-2}} + \left\| z\left(u_x^2 - z_x^2\right) \right\|_{H^{s-2}} \\
&\quad \le c\|u - z\|_{H^{s-2}} \left\| u_x^2 \right\|_{H^{s-1}} + \|z\|_{H^{s-1}} \left\| u_x^2 - z_x^2 \right\|_{H^{s-2}} \\
&\quad \le c\|u - z\|_{H^{s-1}} \|u_x\|_{H^{s-1}}^2 + \|z\|_{H^{s-1}} \|u_x - z_x\|_{H^{s-2}} \|u_x + z_x\|_{H^{s-1}} \\
&\quad \le c\|u - z\|_{H^{s-1}} \left(\|u\|_{H^s}^2 + \|z\|_{H^{s-1}} (\|u\|_{H^s} + \|z\|_{H^s}) \right), \\
&\left\| \Lambda^{-2}\partial_x\left[(u_x)^N - (z_x)^N\right] \right\|_{H^{s-1}} \\
&\quad \le c\left\| (u_x)^N - (z_x)^N \right\|_{H^{s-2}} \le c\|u_x - z_x\|_{H^{s-2}} \left\| \sum_{j=0}^{N-1} u_x^{N-j} z_x^{j} \right\|_{H^{s-1}} \\
&\quad \le c\|u - z\|_{H^{s-1}} \sum_{j=0}^{N-1} \|u_x\|_{H^{s-1}}^{N-j} \|z_x\|_{H^{s-1}}^{j} \le c\|u - z\|_{H^{s-1}} \sum_{j=0}^{N-1} \|u\|_{H^s}^{N-j} \|z\|_{H^s}^{j}.
\end{aligned} \tag{3.14}$$

From (3.7), and (3.13), we know that (3.11) is valid, while inequality (3.12) follows from (3.14). □

Proof of Theorem 2.1. Using the Kato Theorem, Lemmas 3.1–3.3 and 3.5, we know that system (2.2) or problem (2.3) has a unique solution

$$u(t,x) \in C([0,T);H^s(R)) \bigcap C^1\left([0,T);H^{s-1}(R)\right). \tag{3.15}$$

□

4. Existence of Weak Solutions

For $s \geq 2$, using the first equation of system (1.3) derives

$$\frac{d}{dt}\int_R \left(u^2 + u_x^2 + 2\beta\int_0^t u_x^{N+1} d\tau\right) dx = 0, \tag{4.1}$$

from which we have the conservation law

$$\int_R \left(u^2 + u_x^2 + 2\beta\int_0^t u_x^{N+1} d\tau\right) dx = \int_R \left(u_0^2 + u_{0x}^2\right) dx. \tag{4.2}$$

Lemma 4.1 (Kato and Ponce [15]). *If $r \geq 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover*

$$\|uv\|_r \leq c(\|u\|_{L^\infty}\|v\|_r + \|u\|_r\|v\|_{L^\infty}), \tag{4.3}$$

where c is a constant depending only on r.

Lemma 4.2 (Kato and Ponce [15]). *Letting $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then*

$$\|[\Lambda^r, u]v\|_{L^2} \leq c\left(\|\partial_x u\|_{L^\infty}\left\|\Lambda^{r-1}v\right\|_{L^2} + \|\Lambda^r u\|_{L^2}\|v\|_{L^\infty}\right). \tag{4.4}$$

Lemma 4.3. *Let $s \geq (3/2)$ and the function $u(t,x)$ is a solution of problem (2.2) and the initial data $u_0(x) \in H^s$. Then the following results hold:*

$$\|u\|_{L^\infty} \leq \|u\|_{H^1} \leq c\|u_0\|_{H^1}\, e^{|\beta|\int_0^t \|u_x\|_{L^\infty}^{N-1} d\tau}. \tag{4.5}$$

For $q \in (0, s-1]$, there is a constant c such that

$$\int_R \left(\Lambda^{q+1}u\right)^2 dx \leq \int_R \left[\left(\Lambda^{q+1}u_0\right)^2\right] dx + c\int_0^t \|u\|_{H^{q+1}}^2 \left(\|u_x\|_{L^\infty}\|u\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^{N-1}\right) d\tau. \tag{4.6}$$

For $q \in [0, s-1]$, there is a constant c such that

$$\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}\|u\|_{H^1} + \|u\|_{L^\infty}\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^{N-1}\right). \tag{4.7}$$

Proof. The identity $\|u\|_{H^1}^2 = \int_R (u^2 + u_x^2)dx$, (4.2), and the Gronwall inequality result in (4.5).

Using $\partial_x^2 = -\Lambda^2 + 1$ and the Parseval equality gives rise to

$$\int_R \Lambda^q u \Lambda^q \partial_x^3\left(u^3\right)dx = -3\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}\left(u^2 u_x\right)dx + 3\int_R (\Lambda^q u)\Lambda^q\left(u^2 u_x\right)dx. \tag{4.8}$$

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the first equation of system (2.2) and integrating with respect to x by parts, we have the identity

$$\begin{aligned}
&\frac{1}{2}\int_R \left((\Lambda^q u)^2 + (\Lambda^q u_x)^2\right)dx \\
&\quad = -3\int_R \Lambda^q u \Lambda^q\left(u^2 u_x\right)dx \\
&\qquad - \int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}\left(u^2 u_x\right)dx + 2\int_R (\Lambda^q u_x)\Lambda^q\left(u u_x^2\right)dx \\
&\qquad + \int_R \Lambda^q u \Lambda^q (u u_x u_{xx})dx - \beta\int_R \Lambda^q u_x \Lambda^q\left[(u_x)^N\right]dx.
\end{aligned} \tag{4.9}$$

We will estimate the terms on the right-hand side of (4.9) separately. For the first term, by using the Cauchy-Schwartz inequality and Lemmas 4.1 and 4.2, we have

$$\begin{aligned}
&\left|\int_R (\Lambda^q u)\Lambda^q\left(u^2 u_x\right)dx\right| \\
&\quad = \left|\int_R (\Lambda^q u)\left[\Lambda^q\left(u^2 u_x\right) - u^2\Lambda^q u_x\right]dx + \int_R (\Lambda^q u)u^2\Lambda^q u_x dx\right| \\
&\quad \le c\|u\|_{H^q}(2\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|u\|_{H^q} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}\|u\|_{H^q}) + \|u\|_{L^\infty}\|u_x\|_{L^\infty}\|\Lambda^q u\|_{L^2}^2 \\
&\quad \le c\|u\|_{H^q}^2\|u\|_{L^\infty}\|u_x\|_{L^\infty}.
\end{aligned} \tag{4.10}$$

Using the above estimate to the second term yields

$$\left|\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}\left(u^2 u_x\right)dx\right| \le c\|u\|_{H^{q+1}}^2\|u\|_{L^\infty}\|u_x\|_{L^\infty}. \tag{4.11}$$

For the third term, using the Cauchy-Schwartz inequality and Lemma 4.2, we obtain

$$\begin{aligned}
&\left|\int_R (\Lambda^q u_x)\Lambda^q\left(u u_x^2\right)dx\right| \\
&\quad \le \|\Lambda^q u_x\|_{L^2}\left\|\Lambda^q\left(u u_x^2\right)\right\|_{L^2} \le c\|u\|_{H^{q+1}}(\|u u_x\|_{L^\infty}\|u_x\|_{H^q} + \|u_x\|_{L^\infty}\|u u_x\|_{H^q}) \\
&\quad \le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}\|u\|_{L^\infty},
\end{aligned} \tag{4.12}$$

in which we have used $\|u u_x\|_{H^q} \le c\|(u^2)_x\|_{H^q} \le c\|u\|_{L^\infty}\|u\|_{H^{q+1}}$.

For the fouth term in (4.9), using $u(u_x^2)_x = (uu_x^2)_x - u_x u_x^2$ results in

$$\left|\int_R (\Lambda^q u)\Lambda^q (uu_x u_{xx})dx\right| \le \frac{1}{2}\left|\int_R \Lambda^q u_x \Lambda^q\left(uu_x^2\right)dx\right| + \frac{1}{2}\left|\int_R \Lambda^q u \Lambda^q\left[u_x u_x^2\right]dx\right| = K_1 + K_2. \tag{4.13}$$

For K_1, it follows from (4.12) that

$$K_1 \le c\|u\|_{H^{q+1}}^2 \|u_x\|_{L^\infty}\|u\|_{L^\infty}. \tag{4.14}$$

For K_2, applying Lemma 4.2 derives

$$\begin{aligned} K_2 &\le c\|u\|_{H^q}\left\|u_x u_x^2\right\|_{H^q} \\ &\le c\|u\|_{H^q}\left(\|u_x\|_{L^\infty}\left\|u_x^2\right\|_{H^q} + \|u_x\|_{H^q}\left\|u_x^2\right\|_{L^\infty}\right) \\ &\le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}^2. \end{aligned} \tag{4.15}$$

For the last term in (4.9), using Lemma 4.1 repeatedly results in

$$\left|\int_R \Lambda^q u_x \Lambda^q (u_x)^N dx\right| \le c\|u_x\|_{H^q}\left\|u_x^N\right\|_{H^q} \le c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}^{N-1}. \tag{4.16}$$

It follows from (4.10)–(4.16) that there exists a constant c such that

$$\frac{1}{2}\frac{d}{dt}\int_R\left[(\Lambda^q u)^2 + (\Lambda^q u_x)^2\right]dx \le c\|u\|_{H^{q+1}}^2\left(\|u_x\|_{L^\infty}\|u\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^{N-1}\right). \tag{4.17}$$

Integrating both sides of the above inequality with respect to t results in inequality (4.6).

To estimate the norm of u_t, we apply the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation of system (2.2) to obtain the equation

$$u_t = \left(1-\partial_x^2\right)^{-1}\left[-\frac{4}{3}\left(u^3\right)_x + \frac{1}{3}\partial_x^3\left(u^3\right) - 2\partial_x\left(uu_x^2\right) + uu_x u_{xx} + \beta\partial_x\left[(u_x)^N\right]\right]. \tag{4.18}$$

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of (4.18) for $q \in [0, s-1]$ gives rise to

$$\int_R (\Lambda^q u_t)^2 dx = \int_R (\Lambda^q u_t)\Lambda^{q-2}\left[\partial_x\left(-\frac{4}{3}u^3 + \frac{1}{3}\partial_x^2 u^3 - 2uu_x^2 + \beta(u_x)^N\right) + uu_x u_{xx}\right]d\tau. \tag{4.19}$$

For the right-hand of (4.19), we have

$$\begin{aligned}
&\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q \partial_x\left(-\frac{4}{3}u^3-2uu_x^2\right)dx \\
&\quad\le c\|u_t\|_{H^q}\left(\int_R \left(1+\xi^2\right)^{q-1}\times\left[\int_R\left[-\frac{4}{3}\widehat{u^2}(\xi-\eta)\hat{u}(\eta)-2\widehat{uu_x}(\xi-\eta)\widehat{u_x}(\eta)\right]d\eta\right]^2\right)^{1/2} \\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\|u\|_{L^\infty}.
\end{aligned} \tag{4.20}$$

Since

$$\begin{aligned}
\int (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x^2\left(u^2u_x\right)dx &= -\int(\Lambda^q u_t)\Lambda^q\left(u^2u_x\right)dx \\
&\quad+\int(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\left(u^2u_x\right)dx,
\end{aligned} \tag{4.21}$$

using Lemma 4.2, $\|u^2u_x\|_{H^q}\le c\|(u^3)_x\|_{H^q}\le c\|u\|_{L^\infty}^2\|u\|_{H^{q+1}}$, and $\|u\|_{L^\infty}\le\|u\|_{H^1}$, we have

$$\begin{aligned}
&\int(\Lambda^q u_t)\Lambda^q\left(u^2u_x\right)dx \le c\|u_t\|_{H^q}\left\|u^2u_x\right\|_{H^q}\le c\|u_t\|_{H^q}\|u\|_{L^\infty}\|u\|_{H^1}\|u\|_{H^{q+1}}, \\
&\left|\int(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\left(u^2u_x\right)dx\right|\le c\|u_t\|_{H^q}\|u\|_{L^\infty}\|u\|_{H^1}\|u\|_{H^{q+1}}.
\end{aligned} \tag{4.22}$$

Using the Cauchy-Schwartz inequality and Lemmas 4.1 and 4.2 yields

$$\left|\int(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(u_x^N\right)dx\right|\le c\|u_t\|_{H^q}\|u_x\|_{L^\infty}^{N-1}\|u\|_{H^{q+1}}, \tag{4.23}$$

$$\begin{aligned}
&\left|\int_R(\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_xu_{xx})dx\right| \\
&\quad\le c\|u_t\|_{H^q}\|uu_xu_{xx}\|_{H^{q-2}} \\
&\quad\le c\|u_t\|_{H^q}\left\|u(u_x^2)_x\right\|_{H^{q-2}}\le c\|u_t\|_{H^q}\left\|\left[u\left(u_x^2\right)\right]_x-(u)_xu_x^2\right\|_{H^{q-2}} \\
&\quad\le c\|u_t\|_{H^q}\left(\left\|uu_x^2\right\|_{H^{q-1}}+\left\|u_xu_x^2\right\|_{H^{q-2}}\right)\le c\|u_t\|_{H^q}\left(\left\|uu_x^2\right\|_{H^q}+\left\|u_xu_x^2\right\|_{H^q}\right) \\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2\right),
\end{aligned} \tag{4.24}$$

in which we have used inequality (4.15).

Applying (4.20)–(4.24) into (4.19) yields the inequality

$$\|u_t\|_{H^q}\le c\|u\|_{H^{q+1}}\left(\|u\|_{L^\infty}\|u\|_{H^1}+\|u\|_{L^\infty}\|u_x\|_{L^\infty}+\|u_x\|_{L^\infty}^2+\|u_x\|_{L^\infty}^{N-1}\right) \tag{4.25}$$

for a constant $c>0$. This completes the proof of Lemma 4.3. □

Defining

$$\phi(x)=\begin{cases} e^{1/(x^2-1)}, & |x|<1,\\ 0, & |x|\geq 1,\end{cases} \tag{4.26}$$

and setting $\phi_\varepsilon(x)=\varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ with $0<\varepsilon<(1/4)$ and $u_{\varepsilon 0}=\phi_\varepsilon \star u_0$, we know that $u_{\varepsilon 0}\in C^\infty$ for any $u_0\in H^s(R)$ and $s>0$.

It follows from Theorem 2.1 that for each ε the Cauchy problem

$$\begin{aligned} u_t-u_{txx} &= -\frac{4}{3}\left(u^3\right)_x+\frac{1}{3}\partial_x^3 u^3-2\partial_x\left(uu_x^2\right)+uu_xu_{xx}+\beta\partial_x\left[(u_x)^N\right]\\ &= -\frac{4}{3}(u^3)_x+\frac{1}{3}\partial_x^3u^3-\frac{3}{2}\partial_x\left(uu_x^2\right)-\frac{1}{2}u_x^3+\beta\partial_x\left[(u_x)^N\right],\\ u(0,x)&=u_{\varepsilon 0}(x),\quad x\in R, \end{aligned} \tag{4.27}$$

has a unique solution $u_\varepsilon(t,x)\in C^\infty([0,T);H^\infty)$.

Lemma 4.4. *Under the assumptions of problem* (4.27), *the following estimates hold for any* ε *with* $0<\varepsilon<(1/4)$ *and* $s>0$:

$$\begin{gathered} \|u_{\varepsilon 0x}\|_{L^\infty}\leq c_1\|u_{0x}\|_{L^\infty},\\ \|u_{\varepsilon 0}\|_{H^q}\leq c_1,\quad \text{if } q\leq s,\\ \|u_{\varepsilon 0}\|_{H^q}\leq c_1\varepsilon^{(s-q)/4},\quad \text{if } q>s,\\ \|u_{\varepsilon 0}-u_0\|_{H^q}\leq c_1\varepsilon^{(s-q)/4},\quad \text{if } q\leq s,\\ \|u_{\varepsilon 0}-u_0\|_{H^s}=o(1), \end{gathered} \tag{4.28}$$

where c_1 *is a constant independent of* ε.

The proof of this Lemma can be found in Lai and Wu [7].

Lemma 4.5. *If* $u_0(x)\in H^s(R)$ *with* $s\in[1,(3/2)]$ *such that* $\|u_{0x}\|_{L^\infty}<\infty$, *and* $u_{\varepsilon 0}$ *is defined as in system* (4.27). *Then there exist two positive constants* T *and* c, *which are independent of* ε, *such that the solution* u_ε *of problem* (4.27) *satisfies* $\|u_{\varepsilon x}\|_{L^\infty}\leq c$ *for any* $t\in[0,T)$.

Proof. Using notation $u=u_\varepsilon$ and differentiating both sides of the first equation of problem (4.27) with respect to x give rise to

$$u_{tx}+\frac{1}{2}uu_x^2+u^2u_{xx}=u^3-\beta(u_x)^N-\Lambda^{-2}\left[u^3+\frac{3}{2}uu_x^2+\frac{1}{2}\left(u_x^3\right)_x-\beta(u_x)^N\right]. \tag{4.29}$$

Integrating by parts leads to

$$\int_R u^2 u_{xx}(u_x)^{p+1}dx = -\frac{1}{p+1}\int_R u u_x^{2p+3}dx, \quad \text{integer } p > 0, \tag{4.30}$$

from which we obtain

$$\int_R \left(\frac{1}{2}u u_x^2 + u^2 u_{xx}\right)(u_x)^{p+1}dx = \frac{p-1}{2(p+1)}\int_R u u_x^{2p+3}dx. \tag{4.31}$$

Multiplying the above equation by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\begin{aligned} &\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx + \frac{p-1}{2p+2}\int_R u(u_x)^{2p+3}dx \\ &\quad = \int_R (u_x)^{2p+1}\left(u^3 - \beta u_x^N\right)dx \\ &\qquad - \int_R (u_x)^{2p+1}\Lambda^{-2}\left[u^3 + \frac{3}{2}u u_x^2 + \frac{1}{2}\left(u_x^3\right)_x - \beta u_x^N\right]dx. \end{aligned} \tag{4.32}$$

Applying the Hölder's inequality yields

$$\begin{aligned} &\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx \\ &\quad \le \left(\left(\int_R \left|u^3\right|^{2p+2}dx\right)^{1/(2p+2)} + \beta\left(\int_R \left|u_x^N\right|^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)}\right) \\ &\qquad \times \left(\int_R |u_x|^{2p+2}dx\right)^{(2p+1)/(2p+2)} + \left|\frac{p-1}{2p+2}\right| \|u u_x\|_{L^\infty}\int_R |u_x|^{2p+2}dx \end{aligned} \tag{4.33}$$

or

$$\begin{aligned} &\frac{d}{dt}\left(\int_R (u_x)^{2p+2}dx\right)^{1/(2p+2)} \\ &\quad \le \left(\int_R \left|u^3\right|^{2p+2}dx\right)^{1/(2p+2)} + \beta\left(\int_R \left|u_x^N\right|^{2p+2}dx\right)^{1/(2p+2)} + \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)} \\ &\qquad + \left|\frac{p-1}{2p+2}\right| \|u u_x\|_{L^\infty}\left(\int_R |u_x|^{2p+2}dx\right)^{1/(2p+2)}, \end{aligned} \tag{4.34}$$

where

$$G = \Lambda^{-2}\left[u^3 + \frac{3}{2}uu_x^2 + \frac{1}{2}\left(u_x^3\right)_x - \beta u_x^N\right]. \tag{4.35}$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \bigcap L^2$, integrating both sides of the inequality (4.34) with respect to t and taking the limit as $p \to \infty$ result in the estimate

$$\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + \int_0^t c\left(\|u\|_{L^\infty}^3 + \beta\|u_x\|_{L^\infty}^N + \|G\|_{L^\infty} + \|u\|_{L^\infty}\|u_x\|_{L^\infty}^2\right)d\tau. \tag{4.36}$$

Using the algebra property of $H^{s_0}(R)$ with $s_0 > (1/2)$ yields ($\|u_\varepsilon\|_{H^{(1/2)+}}$ means that there exists a sufficiently small $\delta > 0$ such that $\|u_\varepsilon\|_{H^{(1/2)+}} = \|u_\varepsilon\|_{H^{(1/2)+\delta}}$):

$$\begin{aligned}
\|G\|_{L^\infty} &\le c\|G\|_{H^{(1/2)+}} \\
&\le c\left\|\Lambda^{-2}\left[u^3 + \frac{3}{2}uu_x^2 + \frac{1}{2}\left(u_x^3\right)_x - \beta u_x^N\right]\right\|_{H^{(1/2)+}} \\
&\le c\left(\|u\|_{H^1}^3 + \left\|\Lambda^{-2}\left(uu_x^2\right)\right\|_{H^{(1/2)+}} + \left\|\Lambda^{-2}\left(u_x^3\right)_x\right\|_{H^{(1/2)+}} + \left\|\Lambda^{-2}\left(u_x^N\right)\right\|_{H^{(1/2)+}}\right) \\
&\le c\left(\|u\|_{H^1}^3 + \left\|uu_x^2\right\|_{H^{-1}} + \left\|u_x^3\right\|_{H^{-(1/2)+}} + \left\|u_x^N\right\|_{H^0}\right) \\
&\le c\left(\|u\|_{H^1}^3 + \|uu_x\|_{H^{-1}}\|u_x\|_{L^\infty} + \left\|u_x^3\right\|_{H^0} + \|u_x\|_{L^\infty}^{N-1}\|u\|_{H^0}\right) \\
&\le \left(\|u\|_{H^1}^3 + \|u\|_{H^1}^2\|u_x\|_{L^\infty} + \|u\|_{H^1}\|u_x\|_{L^\infty}^2 + \|u\|_{H^1}\|u_x\|_{L^\infty}^{N-1}\right),
\end{aligned} \tag{4.37}$$

in which Lemma 3.4 is used. From Lemma 4.3, we get

$$\int_0^t \|G\|_{L^\infty}d\tau \le c\int_0^t e^{c\int_0^\tau \|u_x\|_{L^\infty}^{N-1}d\xi}\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^{N-1}\right)d\tau. \tag{4.38}$$

Using $\|u\|_{L^\infty} \le \|u\|_{H^1}$, from (4.36) and (4.38), it has

$$\|u_x\|_{L^\infty} \le \|u_{0x}\|_{L^\infty} + c\int_0^t e^{c\int_0^t \|u_x\|_{L^\infty}^{N-1}d\tau}\left[1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^{N-1} + \|u_x\|_{L^\infty}^N\right]d\tau. \tag{4.39}$$

From Lemma 4.4, it follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c\int_0^t e^{c\int_0^t \|W\|_{L^\infty}^{N-1}d\tau}\left[1 + \|W\|_{L^\infty} + \|W\|_{L^\infty}^2 + \|W\|_{L^\infty}^{N-1} + \|W\|_{L^\infty}^N\right]d\tau \tag{4.40}$$

has a unique solution $W \in C[0, T]$. Using the theorem presented at page 51 in [8] yields that there are constants $T > 0$ and $c > 0$ independent of ε such that $\|u_x\|_{L^\infty} \le W(t)$ for arbitrary $t \in [0, T]$, which leads to the conclusion of Lemma 4.5.

Using Lemmas 4.3 and 4.5, notation $u_\varepsilon = u$, and Gronwall's inequality results in the inequalities

$$\|u_\varepsilon\|_{H^q} \le C_T e^{C_T}, \quad \|u_{\varepsilon t}\|_{H^r} \le C_T e^{C_T}, \tag{4.41}$$

where $q \in (0, s]$, $r \in (0, s-1]$, and C_T depends on T. It follows from Aubin's compactness theorem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}$ and their temporal derivatives $\{u_{\varepsilon_n t}\}$ are weakly convergent to a function $u(t,x)$ and its derivative u_t in $L^2([0,T], H^s)$ and $L^2([0,T], H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0,T], H^q(-R_1, R_1))$ for $q \in [0, s)$ and $\{u_{\varepsilon_n t}\}$ converges to u_t strongly in the space $L^2([0,T], H^r(-R_1, R_1))$ for $r \in [0, s-1]$. Thus, we can prove the existence of a weak solution to (1.2). □

Proof of Theorem 2.2. From Lemma 4.5, we know that $\{u_{\varepsilon_n x}\}$ ($\varepsilon_n \to 0$) is bounded in the space L^∞. Thus, the sequences $\{u_{\varepsilon_n}\}$ and $\{u^N_{\varepsilon_n x}\}$ are weakly convergent to u and u^N_x in $L^2[0,T]$, $H^r(-R,R)$ for any $r \in [0, s-1)$, respectively. Therefore, u satisfies the equation

$$-\int_0^T \int_R u(g_t - g_{xxt})dx\,dt = \int_0^T \int_R \left[\left(\frac{4}{3}u^3 + \frac{3}{2}uu_x^2\right)g_x - \frac{1}{2}u_x^3 g(x) - \frac{1}{3}u^3 g_{xxx} - \beta(u_x)^N g_x\right]dx\,dt \tag{4.42}$$

with $u(0,x) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0,T] \times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T] \times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0,T] \times R)$. It derives from the $\{u_{\varepsilon_n x}\}$ weakly convergent to u_x in $L^2([0,T] \times R)$ that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0,T] \times R)$. □

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