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Research Article

Some Algorithms for Finding Fixed Points and Solutions of Variational Inequalities

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We introduce new implicit and explicit algorithms for finding the fixed point of a k-strictly pseudocontractive mapping and for solving variational inequalities related to the Lipschitzian and strongly monotone operator in Hilbert spaces. We establish results on the strong convergence of the sequences generated by the proposed algorithms to a fixed point of a k-strictly pseudocontractive mapping. Such a point is also a solution of a variational inequality defined on the set of fixed points. As direct consequences, we obtain the unique minimum-norm fixed point of a k-strictly pseudocontractive mapping.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S. We recall that a mapping $T : C \to H$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - \mathrm{T})x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings T on C (i.e., $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudocontractive. Recently, many authors have been devoting their studies to the problems of finding fixed points for k-strictly pseudocontractive mappings; see [1–5] and the references therein.

Variational inequalities have been studied widely and are being used as a mathematical programming tool in modeling a wide class of problems arising in several branches of pure and applied sciences; see [6–8]. For general variational inequalities and extended general variational inequalities, we can refer to [9–13] and references therein.

A variational inequality (VI) is formulated as finding a point $\tilde{x}$ with the property

$$\tilde{x} \in C \text{ such that } \langle F\tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in C, \tag{1.2}$$

where $F : C \to H$ is a nonlinear mapping. It is well known that VI (1.2) is equivalent to the fixed point equation

$$\tilde{x} = P_C(I - \lambda F)\tilde{x}, \tag{1.3}$$

where $\lambda > 0$ and P_C is the the metric projection of H onto C, which assigns, to each $x \in H$, the unique point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}. \tag{1.4}$$

Therefore, fixed point algorithms can be applied to solve VI (1.2). It is also well known that if F is ρ-Lipschitzian and η-strongly monotone with constants $\rho,\ \eta > 0$ (i.e., there exist $\rho,\ \eta \geq 0$ such that $\|Fx - Fy\| \leq \rho\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2,\ x, y \in C$, resp.), then, for small enough $\lambda > 0$, the mapping $P_C(I - \lambda F)$ is a contractive mapping on C and so the sequence $\{x_n\}$ of Picard iterates, given by $x_n = P_C(I - \lambda F)x_{n-1}$ $(n \geq 1)$, converges strongly to the unique solution of VI (1.2).

This sort of VI (1.2) where F is ρ-Lipschitzian and η-strongly monotone and where solutions are sought from the set of fixed points of a nonexpansive mapping is originated from Yamada [14], who provided the hybrid method for solving VI (1.2). In order to find solutions of ceratin variational inequality problems defined on the set of fixed points of nonexpansive mappings, several iterative algorithms were studied by many authors; see [15–24] and the references therein.

In this paper, we investigate the following variational inequality (VI) as a special form of VI (1.2), where the constraint set is the fixed points of a k-strictly pseudocontractive mapping T: finding a point $\tilde{x}$ with property

$$\tilde{x} \in \mathrm{Fix}(T) \text{ such that } \langle (\mu F - \gamma V)\tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T), \tag{1.5}$$

where $T : C \to C$ is a k-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$ for some $k \in [0, 1)$, $F : C \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho,\ \eta > 0$, and $V : C \to H$ is an l-Lipschitzian mapping with constant $l \geq 0$ and $\mu, \gamma > 0$. Indeed, variational inequalities of form (1.5) cover several topics recently considered in the literature, including monotone inclusions, convex optimization, and quadratic minimization over fixed point sets; see [2–4, 15, 18, 19, 21, 22, 24] and the references therein. For some iterative methods and some results related to our approach about VI (1.5), we can refer to [25–31] and references therein.

The main purpose of the present paper is to further study the hierarchical fixed point approach to the VI of form (1.5). First, we introduce new implicit and explicit algorithms for finding the fixed point of the k-strictly pseudocontractive mapping T. Then, we establish results on the strong convergence of the sequences generated by the proposed algorithms to a fixed point of the mapping T, which is also a solution of VI (1.5) defined on the set of fixed points of T. As direct consequences, we obtain the unique minimum-norm fixed point of T. Namely, we find the unique solution of the quadratic minimization problem: $\|\tilde{x}\|^2 = \min\{\|x\|^2 : x \in \mathrm{Fix}(T)\}$.

2. Preliminaries and Lemmas

Throughout this paper, when $\{x_n\}$ is a sequence in H, $x_n \to x$ (resp., $x_n \rightharpoonup x$) denotes strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that $f : C \to H$ is called a contractive mapping with constant $\alpha \in (0,1)$ if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$, for all $x, y \in C$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

P_C is called the *metric projection* of H to C. It is well known that P_C is nonexpansive and that, for $x \in H$,

$$z = P_C x \Longleftrightarrow \langle x - z, y - z\rangle \le 0, \quad \forall y \in C. \tag{2.2}$$

In a Hilbert space H, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \quad \forall x, y \in H. \tag{2.3}$$

We need the following lemmas for the proof of our main results.

Lemma 2.1. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.4}$$

Lemma 2.2 (see [32]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \lambda_n)s_n + \lambda_n\delta_n, \quad \forall n \ge 0, \tag{2.5}$$

where $\{\lambda_n\} \subset (0,1)$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(iii) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=0}^{\infty} \lambda_n\delta_n < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (see [33]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ a sequence in $[0,1]$ that satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1. \tag{2.6}$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n) z_n$ for all $n \geq 0$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.7}$$

Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.4 (Demiclosedness principle [34]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and $\{(I-S)x_n\}$ converges strongly to y, then $(I-S)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(S)$.*

Lemma 2.5 (see [14, 16]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $G : C \to H$ is monotone and weakly continuous along segments, that is, $G(x+ty) \to G(x)$ weakly as $t \to 0$. Then, the variational inequality*

$$\tilde{x} \in C, \quad \langle G\tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in C, \tag{2.8}$$

is equivalent to the dual variational inequality

$$\tilde{x} \in C, \quad \langle Gp, p - \tilde{x} \rangle \geq 0, \quad \forall p \in C. \tag{2.9}$$

Lemma 2.6 (see [5]). *Let H be a real Hilbert space and C a closed convex subset of H. If T is a k-strictly pseudocontractive mapping on C, then the fixed point set $\mathrm{Fix}(T)$ is closed convex, so that the projection $P_{\mathrm{Fix}(T)}$ is well defined.*

Lemma 2.7 (see [5]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to H$ a k- strictly pseudocontractive mapping. Define a mapping $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.*

The following lemma can be easily proven, and, therefore, we omit the proof.

Lemma 2.8. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$ and $F : H \to H$ a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho,\ \eta > 0$. Then, for $0 \leq \gamma l < \mu\eta$,*

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y \rangle \geq (\mu\eta - \gamma l) \|x - y\|^2, \quad \forall x, y \in C. \tag{2.10}$$

That is, $\mu F - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

The following lemma is an improvement of Lemma 2.9 in [4] (see also [14]).

Lemma 2.9. *Let H be a real Hilbert space H. Let $F : H \to H$ be a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho,\ \eta > 0$. Let $0 < \mu < 2\eta/\rho^2$ and $0 < t < \xi \le 1$. Then, $R := \rho I - t\mu F : H \to H$ is a contractive mapping with constant $\xi - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)} < 1$.*

Proof. First we show that $I - \mu F$ is strictly contractive. In fact, by applying the ρ-Lipschitz continuity and η-strongly monotonicity of F and (2.3), we obtain, for $x, y \in H$,

$$\begin{aligned}\|(I-\mu F)x-(I-\mu F)y\|^2 &= \|(x-y)-\mu(Fx-Fy)\|^2\\ &= \|x-y\|^2-2\mu\langle Fx-Fy,x-y\rangle+\mu^2\|Fx-Fy\|^2\\ &\le \|x-y\|^2-2\mu\eta\|x-y\|^2+\mu^2\rho^2\|x-y\|^2\\ &= \left(1-\mu\left(2\eta-\mu\rho^2\right)\right)\|x-y\|^2,\end{aligned} \tag{2.11}$$

and so

$$\|(I-\mu F)x-(I-\mu F)y\| \le \sqrt{1-\mu(2\eta-\mu\rho^2)}\,\|x-y\|. \tag{2.12}$$

Now, noting that $R := \xi I - t\mu F = (\xi - t)I - t(\mu F - I)$, from (2.12), we have, for $x, y \in H$,

$$\begin{aligned}\|Rx-Ry\| &= \|(\xi-t)(x-y)-t((\mu F-I)x-(\mu F-I)y)\|\\ &\le (\xi-t)\|x-y\|+t\|(\mu F-I)x-(\mu F-I)y\|\\ &\le (\xi-t)\|x-y\|+t\sqrt{1-\mu(2\eta-\mu\rho^2)}\,\|x-y\|\\ &= \left(\xi-t\left(1-\sqrt{1-\mu(2\eta-\mu\rho^2)}\right)\right)\|x-y\|\\ &= (\xi-t\tau)\|x-y\|.\end{aligned} \tag{2.13}$$

Hence, R is a contractive mapping with constant $\xi - t\tau$. □

3. Iterative Algorithms

Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \ne \emptyset$ for some $0 \le k < 1$, $F : C \to H$ a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho,\ \eta > 0$, and $V : C \to H$ an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \mu < 2\eta/\rho^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)} < 1$. Let $S : C \to C$ be a mapping defined by $Sx = kx + (1 - k)Tx$ and P_C a metric projection of H onto C.

In this section, we introduce the following algorithm that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = SP_C[t\gamma Vx_t + (I - t\mu F)x_t]. \tag{3.1}$$

We prove the strong convergence of $\{x_t\}$ as $t \to 0$ to a fixed point $\tilde{x}$ of T, which is a solution of the following variational inequality:

$$\langle (\mu F - \gamma V)\tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{3.2}$$

We also propose the following explicit algorithm, which generates a sequence in an explicit way:

$$x_{n+1} = \beta_n x_n + (1-\beta_n) SP_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F)x_n], \quad \forall n \geq 0, \tag{3.3}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $x_0 \in C$ is an arbitrary initial guess, and we establish the strong convergence of this sequence to a fixed point $\tilde{x}$ of T, which is also a solution of the variational inequality (3.2).

3.1. Strong Convergence of the Implicit Algorithm

Now, for $t \in (0,1)$, consider a mapping $Q_t : C \to C$ defined by

$$Q_t x = SP_C[t\gamma V x + (I - t\mu F)x], \quad \forall x \in C. \tag{3.4}$$

It is easy to see that Q_t is a contractive mapping with constant $1 - t(\tau - \gamma l)$. Indeed, by Lemma 2.9, we have

$$\begin{aligned} \|Q_t x - Q_t y\| &\leq t\gamma \|Vx - Vy\| + \|(I - t\mu F)x - (I - t\mu F)y\| \\ &\leq t\gamma l \|x - y\| + (1 - t\tau)\|x - y\| \\ &= (1 - t(\tau - \gamma l))\|x - y\|. \end{aligned} \tag{3.5}$$

Hence, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summarize the basic properties of $\{x_t\}$.

Proposition 3.1. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *for some $0 \leq k < 1$, $F : C \to H$ a ρ-Lipschitzain and η-strongly monotone mapping with constants ρ, $\eta > 0$, and $V : C \to H$ an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\rho^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)} < 1$. Let $S : C \to C$ be a mapping defined by $Sx = kx + (1-k)Tx$ and P_C a metric projection of H onto C. Let $\{x_t\}$ be defined via* (3.1)*. Then,*

(i) *$\{x_t\}$ is bounded for $t \in (0,1)$,*

(ii) $\lim_{t \to 0}\|x_t - Sx_t\| = 0$,

(iii) *x_t defines a continuous path from $(0,1)$ in C.*

Proof. (i) Let $p \in \mathrm{Fix}(T)$. Observing $\mathrm{Fix}(T) = \mathrm{Fix}(S)$ by Lemma 2.7, we have

$$\begin{aligned}\|x_t - p\| &= \left\|SP_C\left[t\gamma V x_t + (I - t\mu F)x_t\right] - SP_C p\right\| \\ &\le \left\|t(\gamma V x_t - \mu F p) + (I - t\mu F)x_t - (I - t\mu F)p\right\| \\ &\le (1 - t\tau)\|x_t - p\| + t\left(\gamma l\|x_t - p\| + \|(\gamma V - \mu F)p\|\right).\end{aligned} \tag{3.6}$$

So, it follows that

$$\|x_t - p\| \le \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}. \tag{3.7}$$

Hence, $\{x_t\}$ is bounded and so are $\{Vx_t\}$, $\{Sx_t\}$, and $\{Fx_t\}$.

(ii) By the boundedness of $\{Vx_t\}$ and $\{Fx_t\}$ in (i), we have

$$\begin{aligned}\|x_t - Sx_t\| &= \left\|SP_C\left[t\gamma V x_t + (I - t\mu F)x_t\right] - SP_C x_t\right\| \\ &\le t\left\|(\gamma V - \mu F)x_t\right\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0.\end{aligned} \tag{3.8}$$

(iii) Let $t,\ t_0 \in (0,1)$, and calculate

$$\begin{aligned}\|x_t - x_{t_0}\| &= \left\|SP_C\left[t\gamma V x_t + (I - t\mu F)x_t\right] - SP_C\left[t_0\gamma V x_{t_0} + (I - t_0\mu F)x_{t_0}\right]\right\| \\ &\le \left\|(t - t_0)\gamma V x_t + t_0\gamma(Vx_t - Vx_{t_0}) - (t - t_0)\mu F x_t + (I - t_0\mu F)x_t - (I - t_0\mu F)x_{t_0}\right\| \\ &\le |t - t_0|\gamma\|Vx_t\| + t_0\gamma l\|x_t - x_{t_0}\| + |t - t_0|\left\|\mu F x_t\right\| + (1 - t_0\tau)\|x_t - x_{t_0}\|.\end{aligned} \tag{3.9}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\gamma\|Vx_t\| + \mu\|Fx_t\|}{t_0(\tau - \gamma l)}|t - t_0|. \tag{3.10}$$

This shows that x_t is locally Lipschitzian and hence continuous. □

We establish the strong convergence of the net $\{x_t\}$ as $t \rightarrow 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.2. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \rightarrow C$ be a k-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \ne \emptyset$ for some $0 \le k < 1$, $F : C \rightarrow H$ a ρ-Lipschitzain and η-strongly monotone mapping with constants ρ, $\eta > 0$, and $V : C \rightarrow H$ an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \mu < 2\eta/\rho^2$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)} < 1$. Let $S : C \rightarrow C$ be a mapping defined by $Sx = kx + (1-k)Tx$ and P_C a metric projection of H onto C. The net $\{x_t\}$ defined via (3.1) converges strongly to a fixed point $\tilde{x}$ of T as $t \rightarrow 0$, which solves the variational inequality (3.2), or, equivalently, one has $P_{F(T)}(I - \mu F + \gamma V)\tilde{x} = \tilde{x}$.*

Proof. We first show the uniqueness of a solution of the variational inequality (3.2), which is indeed a consequence of the strong monotonicity of $\mu F - \gamma V$. In fact, noting that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau \Leftrightarrow \rho \ge \eta$, it follows from Lemma 2.8 that

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \ge (\mu\eta - \gamma l)\|x - y\|^2. \tag{3.11}$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau \le \mu\eta$. Suppose that $\tilde{x} \in \mathrm{Fix}(T)$ and $\hat{x} \in \mathrm{Fix}(T)$ both are solutions to (3.2). Then, we have

$$\begin{gathered} \langle (\mu F - \gamma V)\tilde{x}, \hat{x} - \tilde{x}\rangle \ge 0, \\ \langle (\mu F - \gamma V)\hat{x}, \tilde{x} - \hat{x}\rangle \ge 0. \end{gathered} \tag{3.12}$$

Adding up (3.12) yields

$$\langle (\mu F - \gamma V)\tilde{x} - (\mu F - \gamma V)\hat{x}, \tilde{x} - \hat{x}\rangle \le 0. \tag{3.13}$$

The strong monotonicity of $\mu F - \gamma V$ implies that $\tilde{x} = \hat{x}$ and the uniqueness is proved.

Next, we prove that $x_t \to \tilde{x}$ as $t \to 0$. To this end, set $y_t = P_C[t\gamma V x_t + (I - t\mu F)x_t]$ for all $t \in (0,1)$. Then, observing $\mathrm{Fix}(T) = \mathrm{Fix}(S)$ by Lemma 2.7, we have $x_t = Sy_t$ and for any $p \in \mathrm{Fix}(T)$

$$\|x_t - p\| \le \|y_t - p\|. \tag{3.14}$$

Also it follows that

$$\begin{aligned} \|y_t - x_t\| &= \|x_t - P_C[t\gamma V x_t + (I - t\mu F)x_t]\| \\ &\le t[\gamma\|x_t\| + \mu\|F x_t\|] \longrightarrow 0 \quad \text{as } t \longrightarrow 0. \end{aligned} \tag{3.15}$$

Since P_C is the metric projection from H onto C, we have, for given $p \in \mathrm{Fix}(T)$,

$$\begin{aligned} \|y_t - p\|^2 &= \langle P_C[t\gamma V x_t + (I - t\mu F)x_t] - (t\gamma V x_t + (I - t\mu F)x_t), y_t - p\rangle \\ &\quad + \langle t\gamma V x_t + (I - t\mu F)x_t - p, y_t - p\rangle \\ &\le \langle t\gamma V x_t + (I - t\mu F)x_t - p, y_t - p\rangle \\ &= \langle t\gamma V x_t - t\mu F p, y_t - p\rangle + \langle (I - t\mu F)x_t - (I - t\mu F)p, y_t - p\rangle \\ &\le (1 - t\tau)\|x_t - p\|\|y_t - p\| + t\langle \gamma V x_t - \mu F p, y_t - p\rangle. \end{aligned} \tag{3.16}$$

It follows that

$$\begin{aligned} \|y_t - p\|^2 &\le \frac{1}{\tau}\langle \gamma V x_t - \mu F p, y_t - p\rangle \\ &= \frac{1}{\tau}(\gamma\langle V x_t - V p, y_t - p\rangle + \langle (\gamma V - \mu F)p, y_t - p\rangle) \\ &\le \frac{1}{\tau}(\gamma l\|x_t - p\|\|y_t - p\| + \langle (\gamma V - \mu F)p, x_t - p\rangle). \end{aligned} \tag{3.17}$$

By (3.14), this implies

$$\left\| y_t - p \right\|^2 \le \frac{1}{\tau - \gamma l}\langle (\gamma V - \mu F)p, y_t - p\rangle = \frac{1}{\tau - \gamma l}\langle (\mu F - \gamma V)p, p - y_t\rangle, \tag{3.18}$$

and hence

$$\left\| x_t - p \right\|^2 \le \frac{1}{\tau - \gamma l}\langle (\mu F - \gamma V)p, p - y_t\rangle. \tag{3.19}$$

Since $\{x_t\}$ is bounded as $t \to 0$ (by Proposition 3.1 (i)), we see that if $\{t_n\}$ is a subsequence in $(0,1)$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup \tilde{x}$, then, from (3.15), we have $y_{t_n} \rightharpoonup \tilde{x}$. By Proposition 3.1 (ii), $\lim_{n\to\infty}(I - S)x_{t_n} = 0$. By Lemmas 2.4 and 2.7, $\tilde{x} \in \mathrm{Fix}(T)$. Therefore, we can substitute $\tilde{x}$ for p in (3.19) to obtain

$$\| x_{t_n} - \tilde{x} \|^2 \le \frac{1}{\tau - \gamma l}\langle (\mu F - \gamma V)\tilde{x}, \tilde{x} - y_{t_n}\rangle. \tag{3.20}$$

Consequently, the weak convergence of $\{y_{t_n}\}$ to $\tilde{x}$ yields that $\{x_{t_n}\} \to \tilde{x}$ strongly. Now we show that $\tilde{x}$ solves the variational inequality (3.2). Again, observe (3.19) and take the limit as $n \to \infty$ to obtain

$$\left\| \tilde{x} - p \right\|^2 \le \frac{1}{\tau - \gamma l}\langle (\mu F - \gamma V)p, p - \tilde{x}\rangle, \quad \forall p \in \mathrm{Fix}(T). \tag{3.21}$$

Hence $\tilde{x}$ solves the following variational inequality:

$$\langle (\mu F - \gamma V)p, p - \tilde{x}\rangle \ge 0, \quad \forall p \in \mathrm{Fix}(T), \tag{3.22}$$

or the equivalent dual variational inequality (see Lemmas 2.5 and 2.8)

$$\langle (\mu F - \gamma V)\tilde{x}, p - \tilde{x}\rangle \ge 0, \quad \forall p \in \mathrm{Fix}(T). \tag{3.23}$$

Moreover, if $\{t_j\}$ is another subsequence in $(0,1)$ such that $t_j \to 0$ and $x_{t_j} \rightharpoonup \hat{x}$, then we also have $y_{t_j} \rightharpoonup \hat{x}$ from (3.15). By the same argument, we can show that $\hat{x} \in \mathrm{Fix}(T)$ and $\hat{x}$ solves the variational inequality (3.2); hence $\hat{x} = \tilde{x}$ by uniqueness. In sum, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\tilde{x}$. Therefore $x_t \to \tilde{x}$ as $t \to 0$.

The variational inequality (3.2) can be rewritten as

$$\langle (I - \mu F + \gamma V)\tilde{x} - \tilde{x}, p - \tilde{x}\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T). \tag{3.24}$$

By reminding the reader of (2.2) and Lemma 2.6, this is equivalent to the fixed point equation

$$P_{F(T)}(I-\mu F+\gamma V)\tilde{x}=\tilde{x}. \tag{3.25}$$

□

From Theorem 3.2, we can deduce the following result.

Corollary 3.3. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *for some $0 \leq k < 1$. For each $t \in (0,1)$, let the net $\{x_t\}$ be defined by*

$$x_t = SP_C(1-t)x_t, \quad \forall t \in (0,1), \tag{3.26}$$

where $S : C \to C$ is a mapping defined by $Sx = kx + (1-k)Tx$ and P_C is a metric projection of H onto C. Then the net $\{x_t\}$ defined via (3.26) *converges strongly, as $t \to 0$, to the minimum-norm point $\tilde{x} \in \mathrm{Fix}(T)$.*

Proof. In (3.19) with $F \equiv I$, $V \equiv 0$, $l = 0$, $\mu = 1$, and $\tau = 1$, letting $t \to 0$ yields

$$\|\tilde{x}-p\|^2 \leq \langle p, p-\tilde{x}\rangle, \quad \forall p \in \mathrm{Fix}(T). \tag{3.27}$$

Equivalently,

$$\langle \tilde{x}, p-\tilde{x}\rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{3.28}$$

This obviously implies that

$$\|\tilde{x}\|^2 \leq \langle p, \tilde{x}\rangle \leq \|p\|\|\tilde{x}\|, \quad \forall p \in \mathrm{Fix}(T). \tag{3.29}$$

It turns out that $\|\tilde{x}\| \leq \|p\|$ for all $p \in \mathrm{Fix}(T)$. Therefore, $\tilde{x}$ is minimum-norm point of $\mathrm{Fix}(T)$.

□

3.2. Strong Convergence of the Explicit Algorithm

Now, using Theorem 3.2, we show the strong convergence of the sequence generated by the explicit algorithm (3.3) to a fixed point $\tilde{x}$ of T, which is also a solution of the variational inequality (3.2).

Theorem 3.4. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *for some $0 \leq k < 1$, $F : C \to H$ a ρ-Lipschitzain and η-strongly monotone mapping with constants ρ, $\eta > 0$, and $V : C \to H$ an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\rho^2$ and $0 < \gamma l < \tau$, where $\tau = 1-\sqrt{1-\mu(2\eta-\mu\rho^2)} < 1$. Let $S : C \to H$ be a mapping defined by $Sx = kx + (1-k)Tx$ and P_C a metric projection of H onto C. For any given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by the explicit algorithm* (3.3)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then, $\{x_n\}$ converges strongly to $\widetilde{x} \in \mathrm{Fix}(T)$, which is the unique solution of the variational inequality (3.2).

Proof. First, from condition (C1), without loss of generality, we assume that $\alpha_n(\tau-\gamma l) < 1$ and $\alpha_n < (1-\beta_n)$ for $n \geq 0$. From now, we put $y_n = SP_C u_n$ and $u_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F)x_n$.

We divide the proof several steps.

Step 1. We show that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \|(\gamma V - \mu F)p\|/(\tau - \gamma l)\}$ for all $n \geq 0$ and all $p \in \mathrm{Fix}(T) = \mathrm{Fix}(S)$. Indeed, let $p \in \mathrm{Fix}(T)$. Then, from Lemma 2.9, we have

$$\begin{aligned}
\|y_n - p\| &= \|SP_C u_n - SP_C p\| \\
&\leq \|\alpha_n(\gamma V x_n - \mu F p) + \beta_n(x_n - p) \\
&\qquad + ((1-\beta_n)I - \alpha_n\mu F)x_n - ((1-\beta_n)I - \alpha_n\mu F)p\| \\
&\leq (1 - \beta_n - \alpha_n\tau)\|x_n - p\| + \beta_n\|x_n - p\| + \alpha_n\|\gamma V x_n - \mu F p\| \\
&\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n(\|\gamma V x_n - \gamma V p\| + \|(\gamma V - \mu F)p\|) \\
&\leq (1 - (\tau - \gamma l)\alpha_n)\|x_n - p\| + \alpha_n\|(\gamma V - \mu F)p\|.
\end{aligned} \tag{3.30}$$

Thus, it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\
&\leq \beta_n\|x_n - p\| + (1-\beta_n)(1 - \alpha_n(\tau - \gamma l))\|x_n - p\| + (1-\beta_n)\alpha_n\|(\gamma V - \mu F)p\| \\
&= (1 - \alpha_n(1-\beta_n)(\tau - \gamma l))\|x_n - p\| + \alpha_n(1-\beta_n)(\tau - \gamma l)\frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\right\}.
\end{aligned} \tag{3.31}$$

Using an induction, we have $\|x_n - p\| \leq \max\{\|x_0 - p\|, \|(\gamma V - \mu F)p\|/(\tau - \gamma l)\}$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{u_n\}$, $\{Vx_n\}$, $\{Sx_n\}$, and $\{Fx_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Indeed, from (3.3), we observe

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|SP_C u_{n+1} - SP_C u_n\| \\
&\leq \|\alpha_{n+1}\gamma V x_{n+1} + (I - \alpha_{n+1}\mu F)x_{n+1} - (\alpha_n\gamma V x_n + (I - \alpha_n\mu F)x_n)\| \\
&= \|\alpha_{n+1}\gamma(V x_{n+1} - V x_n) + \gamma(\alpha_{n+1} - \alpha_n)V x_n \\
&\qquad + (I - \alpha_{n+1}\mu F)x_{n+1} - (I - \alpha_{n+1}\mu F)x_n + \mu(\alpha_{n+1} - \alpha_n)F x_n\| \\
&\leq (1 - \alpha_{n+1}(\tau - \gamma l))\|x_{n+1} - x_n\| + M|\alpha_{n+1} - \alpha_n|,
\end{aligned} \tag{3.32}$$

where $M = \sup\{\gamma\|Vx_n\| + \mu\|Fx_n\| : n \geq 0\}$. Thus, it follows that

$$\left\|y_{n+1} - y_n\right\| - \|x_{n+1} - x_n\| \leq -\alpha_{n+1}(\tau - \gamma l)\|x_{n+1} - x_n\| + M|\alpha_{n+1} - \alpha_n|, \tag{3.33}$$

which implies, from condition (C1), that

$$\limsup_{n\to\infty}\left(\left\|y_{n+1} - y_n\right\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{3.34}$$

Hence, by Lemma 2.3, we have

$$\lim_{n\to\infty}\left\|y_n - x_n\right\| = 0. \tag{3.35}$$

Consequently, from condition (C3), it follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\left\|y_n - x_n\right\| = 0. \tag{3.36}$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. Indeed, we have

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sx_n\| \\
&= \|x_{n+1} - x_n\| + \left\|\beta_n(x_n - Sx_n) + (1 - \beta_n)(y_n - Sx_n)\right\| \\
&\leq \|x_{n+1} - x_n\| + \beta_n\|x_n - Sx_n\| + (1 - \beta_n)\left\|y_n - SP_Cx_n\right\| \\
&\leq \|x_n - x_{n+1}\| + \beta_n\|x_n - Sx_n\| + (1 - \beta_n)\alpha_n\left\|(\gamma V - \mu F)x_n\right\|,
\end{aligned} \tag{3.37}$$

that is,

$$\|x_n - Sx_n\| \leq \frac{1}{1 - \beta_n}\|x_{n+1} - x_n\| + \alpha_n\left\|(\gamma V - \mu F)x_n\right\|. \tag{3.38}$$

This together with conditions (C1) and (C3) and Step 2 implies

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.39}$$

Step 4. We show that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, u_n - \tilde{x}\rangle \leq 0, \tag{3.40}$$

where $\tilde{x} = \lim_{t\to 0} x_t$ with x_t being defined by (3.1). (We note that, from Theorem 3.2, $\tilde{x} \in \mathrm{Fix}(T)$ and $\tilde{x}$ is the unique solution of the variational inequality (3.2)). To show this, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j\to\infty}\left\langle(\gamma V - \mu F)\tilde{x}, x_{n_j} - \tilde{x}\right\rangle = \limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_n - \tilde{x}\rangle. \tag{3.41}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$, which converges weakly to w. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$. Since $\|x_n - Sx_n\| \to 0$ by Step 3, we obtain $w = Sw$ by virtue of Lemma 2.4. From Lemma 2.7, we have $w \in \mathrm{Fix}(T)$. Therefore, from (3.2), it follows that

$$\begin{aligned} \limsup_{n\to\infty}\langle(\gamma V-\mu F)\tilde{x}, x_n-\tilde{x}\rangle &= \lim_{j\to\infty}\left\langle(\gamma V-\mu F)\tilde{x}, x_{n_j}-\tilde{x}\right\rangle \\ &= \langle(\gamma V-\mu F)\tilde{x}, w-\tilde{x}\rangle \le 0. \end{aligned} \tag{3.42}$$

We notice that, by condition (C1),

$$\|u_n - x_n\| \le \alpha_n\|(\mu F-\gamma V)x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.43}$$

Hence, from (3.42), we obtain

$$\limsup_{n\to\infty}\langle(\gamma V-\mu F)\tilde{x}, u_n-\tilde{x}\rangle \le 0. \tag{3.44}$$

Step 5. We show that $\lim_{n\to\infty}\|x_n-\tilde{x}\| = 0$, where $\tilde{x} = \lim_{t\to 0} x_t$ with x_t being defined by (3.1), and $\tilde{x}$ is the unique solution of the variational inequality (3.2). Indeed, we observe that

$$\|u_n-\tilde{x}\| \le \|x_n-\tilde{x}\| + \alpha_n\|(\gamma V-\mu F)x_n\|. \tag{3.45}$$

Therefore, from the convexity of $\|\cdot\|^2$, (3.3), and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1}-\tilde{x}\|^2 &\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\|y_n-\tilde{x}\|^2 \\ &\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\|u_n-\tilde{x}\|^2 \\ &\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\|(I-\alpha_n\mu F)x_n-(I-\alpha_n\mu F)\tilde{x}\|^2 \\ &\quad + 2(1-\beta_n)\alpha_n\langle\gamma V x_n-\mu F\tilde{x}, u_n-\tilde{x}\rangle \\ &\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)(1-\alpha_n\tau)^2\|x_n-\tilde{x}\|^2 \\ &\quad + 2(1-\beta_n)\alpha_n[\langle\gamma V x_n-\gamma V\tilde{x}, u_n-\tilde{x}\rangle + \langle(\gamma V-\mu F)\tilde{x}, u_n-\tilde{x}\rangle] \\ &\le \left[\beta_n + (1-\beta_n)\left(1-2\alpha_n\tau+\alpha_n^2\tau^2\right)\right]\|x_n-\tilde{x}\|^2 + 2(1-\beta_n)\alpha_n\gamma l\|x_n-\tilde{x}\|\|u_n-\tilde{x}\| \\ &\quad + 2(1-\beta_n)\alpha_n\langle\gamma V\tilde{x}-\mu F\tilde{x}, u_n-\tilde{x}\rangle \\ &\le \left[1-(1-\beta_n)\left(2\alpha_n\tau-\alpha_n^2\tau^2\right)\right]\|x_n-\tilde{x}\|^2 + 2(1-\beta_n)\alpha_n\gamma l\|x_n-\tilde{x}\|^2 \\ &\quad + 2(1-\beta_n)\alpha_n^2\gamma l\|x_n-\tilde{x}\|\|(\gamma V-\mu F)x_n\| + 2(1-\beta_n)\alpha_n\langle(\gamma V-\mu F)\tilde{x}, u_n-\tilde{x}\rangle \\ &\le [1-2(1-\beta_n)(\tau-\gamma l)\alpha_n]\|x_n-\tilde{x}\|^2 + \alpha_n^2\tau^2\|x_n-\tilde{x}\|^2 \\ &\quad + 2\alpha_n^2\gamma l\|x_n-\tilde{x}\|\|(\gamma V-\mu F)x_n\| + 2(1-\beta_n)\alpha_n\langle(\gamma V-\mu F)\tilde{x}, u_n-\tilde{x}\rangle \\ &\le [1-2(1-b)(\tau-\gamma l)\alpha_n]\|x_n-\tilde{x}\|^2 + \alpha_n\left(\alpha_n\tau^2M_1^2 + 2\alpha_n\gamma l M_1 M_2\right) \\ &\quad + 2(1-\beta_n)\alpha_n\langle(\gamma V-\mu F)\tilde{x}, u_n-\tilde{x}\rangle \\ &= (1-\lambda_n)\|x_n-\tilde{x}\|^2 + \lambda_n\delta_n, \end{aligned} \tag{3.46}$$

where $0 < b = \limsup_{n\to\infty} \beta_n < 1$, $M_1 = \sup\{\|x_n - \tilde{x}\| : n \geq 0\}$, $M_2 = \sup\{\|\gamma V x_n - \mu F x_n\| : n \geq 0\}$, $\lambda_n = 2(1-b)(\tau - \gamma l)\alpha_n$, and

$$\delta_n = \frac{1}{(1-b)(\tau-\gamma l)} \left(\frac{\alpha_n \tau^2 M_1^2}{2} + \alpha_n \gamma l M_1 M_2 + (1-\beta_n) \langle (\gamma V - \mu F)\tilde{x}, u_n - \tilde{x} \rangle \right). \tag{3.47}$$

From conditions (C1), (C2), and (C3) and Step 4, it is easy to see that $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence, by Lemma 2.2, we conclude that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

From Theorem 3.4, we can also deduce the following result.

Corollary 3.5. *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$ for some $0 \leq k < 1$. For each $x_0 \in C$, let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S P_C (1-\alpha_n) x_n, \quad \forall n \geq 0, \tag{3.48}$$

where $S : C \to C$ is a mapping defined by $Sx = kx + (1-k)Tx$ and P_C is a metric projection of H onto C. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (C1), (C2), and (C3) in Theorem 3.4, then the sequence $\{x_n\}$ defined via (3.48) converges strongly, as $n \to \infty$, to the minimum-norm point $\tilde{x} \in \mathrm{Fix}(T)$.

Proof. VI (3.2) is reduced to the inequality

$$\langle \tilde{x}, p - \tilde{x} \rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{3.49}$$

This is equivalent to $\|\tilde{x}\|^2 \leq \langle p, \tilde{x} \rangle \leq \|p\|\|\tilde{x}\|$ for all $p \in \mathrm{Fix}(T)$. It turns out that $\|\tilde{x}\| \leq \|p\|$ for all $p \in \mathrm{Fix}(T)$ and $\tilde{x}$ is the minimum-norm point of $\mathrm{Fix}(T)$. □

Remark 3.6. We point out that our algorithms (3.1) and (3.3) are new ones different from those in the literature (see [2–4, 15, 18, 21, 22] and references therein).

4. Conclusion and Future Directions

In this paper, we have introduced new implicit and explicit algorithms for finding fixed points of a k-strictly pseudocontractive mapping and for solving a certain variational inequality and have established strong convergence of the proposed algorithms to a fixed point of the mapping, which is a solution of a certain variational inequality, where the constraint set is the fixed points of the mapping. As direct consequences, we have considered the quadratic minimization problem on the set of fixed points of the mapping.

In forthcoming studies, we will consider implicit and explicit algorithms for solving some variational inequalities, where the constraint set is the common set of the set of fixed points of the mapping and the set of solutions of the equilibrium problem.

We hope that the ideas and techniques of this paper may stimulate further research in this field.

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