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Research Article

On Subclass of k-Uniformly Convex Functions of Complex Order Involving Multiplier Transformations

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We introduce a subclass of k-uniformly convex functions of order α with negative coefficients by using the multiplier transformations in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We obtain coefficient estimates, radii of convexity and close-to-convexity, extreme points, and integral means inequalities for the function f that belongs to the class $\mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$.

1. Introduction

Let $\mathcal{N}$ denote the class of functions of the form:

$$f(z)^{\beta} = z^{\beta} + \sum_{n=2}^{\infty} \beta a_n z^{\beta+n-1}, \quad \beta > 0, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ (see [1]). Also denote by $\mathcal{M}$ the subclass of $\mathcal{N}$ consisting of functions of the form:

$$f(z)^{\beta} = z^{\beta} - \sum_{n=2}^{\infty} \beta a_n z^{\beta+n-1}, \quad (a_n \geq 0,\ \beta > 0). \tag{1.2}$$

For any integer m, we define the multiplier transformations I_m^ℓ (see [2, 3]) of functions $f \in \mathcal{N}(n)$ by

$$\begin{aligned} I_m^\ell f(z)^\beta &= z^\beta - \sum_{n=2}^{\infty} \beta \left(\frac{\beta+\ell}{\beta+\ell+n-1} \right)^m a_n z^{\beta+n-1} \\ &= z^\beta - \sum_{n=2}^{\infty} \beta Q(n,\beta,\ell) a_n z^{\beta+n-1}, \quad (\ell \geq 0,\ z \in U), \end{aligned} \tag{1.3}$$

where $Q(n,\beta,\ell) = ((\beta+\ell)/(\beta+\ell+n-1))^m$.

A function $f \in \mathcal{M}$ is said to be in the class $\mathrm{USL}(\alpha,k)$ (k-uniformly starlike Functions of order α) if it satisfies the condition:

$$\mathrm{Re}\left\{ \frac{zf'(z)^\beta}{f(z)^\beta} - \alpha \right\} > k \left| \frac{zf'(z)^\beta}{f(z)^\beta} - 1 \right|, \quad (0 \leq \alpha < 1,\ k \geq 0),\ z \in U \tag{1.4}$$

and is said to be in the class $\mathrm{UCV}(\alpha,k)$ (k-uniformly convex Functions of order α) if it satisfies the condition:

$$\mathrm{Re}\left\{ 1 + \frac{zf''(z)^\beta}{f'(z)^\beta} - \alpha \right\} > k \left| \frac{zf''(z)^\beta}{f'(z)^\beta} \right|, \quad (0 \leq \alpha < 1,\ k \geq 0),\ z \in U. \tag{1.5}$$

Indeed it follows from (1.4) and (1.5) that

$$f \in \mathrm{UCV}(\alpha,k) \Longleftrightarrow zf' \in \mathrm{USL}(\alpha,k). \tag{1.6}$$

The interesting geometric properties of these function classes were extensively studied by Kanas et al., in [4, 5], motivated by Altintas et al. [6], Murugusundaramoorthy and Srivastava [7], and Murugusundaramoorthy and Magesh [8, 9], Atshan and Kulkarni [10] and Atshan and Buti [11].

Now, we define a new subclass of uniformly convex functions of complex order.

For $0 \leq \alpha < 1$, $k \geq 0$, $\upsilon \in \mathbb{C} \setminus \{0\}$, we let $\mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$ be the class of functions f satisfying (1.2) with the analytic criterion:

$$\mathrm{Re}\left\{ 1 + \frac{1}{\upsilon} \left(1 + \frac{z\left(I_m^\ell f(z)^\beta\right)''}{\left(I_m^\ell f(z)^\beta\right)'} - \alpha \right) \right\} > k \left| 1 + \frac{1}{\upsilon} \left(\frac{z\left(I_m^\ell f(z)^\beta\right)''}{\left(I_m^\ell f(z)^\beta\right)'} \right) \right|, \quad z \in U, \tag{1.7}$$

where $I_m^\ell f(z)^\beta$ is given by (1.3).

2. Main Results

First, we obtain the necessary and sufficient condition for functions f in the class $\mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$.

Theorem 2.1. *The necessary and sufficient condition for f of the form of (1.2) to be in the class $\mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$ is*

$$\sum_{n=2}^{\infty}(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)a_n \le (k-\alpha)+(1-k)(\beta+|\upsilon|), \tag{2.1}$$

where $0 \le \alpha < 1$, $k \ge 0$, $\upsilon \in \mathbb{C} \setminus \{0\}$.

Proof. Suppose that (2.1) is true for $z \in U$. Then

$$\operatorname{Re}\left\{1+\frac{1}{\upsilon}\left(1+\frac{z\left(I_m^{\ell}f(z)^{\beta}\right)''}{\left(I_m^{\ell}f(z)^{\beta}\right)'}-\alpha\right)\right\}-k\left|1+\frac{1}{\upsilon}\left(\frac{z\left(I_m^{\ell}f(z)^{\beta}\right)''}{\left(I_m^{\ell}f(z)^{\beta}\right)'}\right)\right|>0, \tag{2.2}$$

if

$$\begin{aligned}&1+\frac{1}{|\upsilon|}\left(\frac{(\beta-\alpha)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-\alpha-1)Q(n,\beta,\ell)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n|z|^{n-1}}\right)\\&\quad-k\left[1+\frac{1}{|\upsilon|}\left(\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)Q(n,\beta,\ell)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n|z|^{n-1}}\right)\right]>0,\end{aligned} \tag{2.3}$$

that is, if

$$\sum_{n=2}^{\infty}(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)a_n \le (k-\alpha)+(1-k)(\beta+|\upsilon|). \tag{2.4}$$

Conversely, assume that $f \in \mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$, then

$$\operatorname{Re}\left\{1+\frac{1}{\upsilon}\left(1+\frac{z\left(I_m^\ell f(z)^\beta\right)''}{\left(I_m^\ell f(z)^\beta\right)'}-\alpha\right)\right\} > k\left|1+\frac{1}{\upsilon}\left(\frac{z\left(I_m^\ell f(z)^\beta\right)''}{\left(I_m^\ell f(z)^\beta\right)'}\right)\right|,$$

$$\operatorname{Re}\left\{1+\frac{1}{\upsilon}\left(\frac{(\beta-\alpha)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-\alpha-1)Q(n,\beta,\ell)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n z^{n-1}}\right)\right\}$$
$$> k\left|1+\frac{1}{\upsilon}\left(\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)Q(n,\beta,\ell)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n z^{n-1}}\right)\right|. \tag{2.5}$$

Letting $z \to 1^-$ along the real axis, we have

$$1+\frac{1}{|\upsilon|}\left(\frac{(\beta-\alpha)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-\alpha-1)Q(n,\beta,\ell)a_n}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n}\right)$$
$$> k\left[1+\frac{1}{|\upsilon|}\left(\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)Q(n,\beta,\ell)a_n}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n}\right)\right]. \tag{2.6}$$

Hence, by maximum modulus theorem, the simple computation leads to the desired inequality

$$\sum_{n=2}^{\infty}(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)a_n \le (k-\alpha)+(1-k)(\beta+|\upsilon|), \tag{2.7}$$

which completes the proof. □

Corollary 2.2. *Let the function f defined by* (1.2) *belong to $\mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$. Then,*

$$a_n \le \frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}, \tag{2.8}$$

where $0 \le \alpha < 1$, $k \ge 0$, $\upsilon \in \mathbb{C}\setminus\{0\}$, with equality for

$$f(z)^\beta = z^\beta - \beta\frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}z^{\beta+n-1}. \tag{2.9}$$

3. Radii of Convexity and Close-to-Convexity

We obtain the radii of convexity and close-to-convexity results for f functions in the class $\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$ in the following theorems.

Theorem 3.1. *Let* $f\in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. *Then* f *is convex of order* $\delta(0\le\delta<1)$ *in the disk* $|z|<r=r_1(\alpha,\beta,k,\upsilon,n,\delta)$, *where*

$$r_1=\inf_{n\ge2}\left[\frac{(2-\delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(3-\delta-\beta-n)\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}\right]^{1/n-1}. \tag{3.1}$$

Proof. Let $f\in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Then by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}Q(n,\beta,\ell)a_n\le1. \tag{3.2}$$

For $0\le\delta<1$, we need to show that

$$\left|\frac{zf''(z)^{\beta}}{f'(z)^{\beta}}\right|\le1-\delta, \tag{3.3}$$

and we have to show that

$$\left|\frac{zf''(z)^{\beta}}{f'(z)^{\beta}}\right|\le\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)a_n|z|^{n-1}}\le1-\delta. \tag{3.4}$$

Hence,

$$\sum_{n=2}^{\infty}\frac{(\beta+n-1)(3-\delta-\beta-n)}{(2-\delta-\beta)}a_n|z|^{n-1}\le1. \tag{3.5}$$

This is enough to consider

$$|z|^{n-1}\le\frac{(2-\delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(3-\delta-\beta-n)\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}. \tag{3.6}$$

Therefore,

$$|z|\le\left\{\frac{(2-\delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(3-\delta-\beta-n)\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}\right\}^{1/n-1}. \tag{3.7}$$

Setting $z=r_1(\alpha,\beta,k,\upsilon,n,\delta)$ in (3.7), we get the radius of convexity, which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let* $f \in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. *Then* f *is close-to-convex of order* $\delta(0 \le \delta < 1)$ *in the disk* $|z| < r = r_2(\alpha,\beta,k,\upsilon,n,\delta)$, *where*

$$r_2 = \inf_{n\ge 2}\left[\frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}\right]^{1/n-1}. \tag{3.8}$$

Proof. Let $f \in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Then by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}Q(n,\beta,\ell)a_n \le 1. \tag{3.9}$$

For $0 \le \delta < 1$, we need to show that

$$\left|\frac{f'(z)^{\beta}}{z^{\beta-1}}-1\right| \le 1-\delta, \tag{3.10}$$

and we have to show that

$$\left|\frac{f'(z)^{\beta}}{z^{\beta-1}}-1\right| \le (\beta-1)+\sum_{n=2}^{\infty}\beta(\beta+n-1)a_n|z|^{n-1} \le 1-\ \delta. \tag{3.11}$$

Hence,

$$\sum_{n=2}^{\infty}\frac{\beta(\beta+n-1)}{(2-\ \delta-\beta)}a_n|z|^{n-1} \le 1. \tag{3.12}$$

This is enough to consider

$$|z|^{n-1} \le \frac{(2-\ \delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{\beta\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}. \tag{3.13}$$

Therefore,

$$|z| \le \left\{\frac{(2-\ \delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{\beta\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}\right\}^{1/n-1}. \tag{3.14}$$

Setting $z = r_2(\alpha,\beta,k,\upsilon,n,\delta)$ in (3.14), we get the radius of close-to-convexity, which completes the proof of Theorem 3.2. □

4. Extreme Points

The extreme points of the class $\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$ are given by the following theorem.

Theorem 4.1. *Let*

$$f_1(z)^\beta = z^\beta,$$
$$f_n(z)^\beta = z^\beta - \beta\frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)]Q(n,\beta,\ell)}z^{\beta+n-1}, \tag{4.1}$$

for $n = 2, 3, 4, \ldots$.

Then, $f\in \mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$ *if and only if it can be expressed in the form:*

$$f(z)^\beta = \sum_{n=1}^{\infty}\Upsilon_n f_n(z)^\beta, \tag{4.2}$$

where $\Upsilon_n \geq 0$ *and*

$$\sum_{n=1}^{\infty}\Upsilon_n = 1. \tag{4.3}$$

Proof. Suppose that f can be expressed as in (4.2). Our goal is to show that $f\in \mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$. By (4.2), we have that

$$\begin{aligned} f(z)^\beta &= \sum_{n=1}^{\infty}\Upsilon_n f_n(z)^\beta = \Upsilon_1 f_1(z)^\beta + \sum_{n=2}^{\infty}\Upsilon_n f_n(z)^\beta \\ &= \Upsilon_1 f_1(z)^\beta \\ &\quad+ \sum_{n=2}^{\infty}\Upsilon_n\left(z^\beta - \beta\frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)]Q(n,\beta,\ell)}z^{\beta+n-1}\right) \\ &= \sum_{n=1}^{\infty}\Upsilon_n z^\beta - \sum_{n=2}^{\infty}\beta\Upsilon_n\frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)]Q(n,\beta,\ell)}z^{\beta+n-1} \\ &= z^\beta - \sum_{n=2}^{\infty}\beta\frac{\Upsilon_n[(k-\alpha)+(1-k)(\beta+|\upsilon|)]}{(\beta+n-1)[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)]Q(n,\beta,\ell)}z^{\beta+n-1}. \end{aligned} \tag{4.4}$$

Now,

$$\begin{aligned} &\sum_{n=2}^{\infty}\frac{(\beta+n-1)[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)]Q(n,\beta,\ell)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)} \\ &\qquad\times\frac{\Upsilon_n[(k-\alpha)+(1-k)(\beta+|\upsilon|)]}{(\beta+n-1)[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)]Q(n,\beta,\ell)} \\ &\quad= \sum_{n=2}^{\infty}\Upsilon_n = 1-\Upsilon_1 \leq 1. \end{aligned} \tag{4.5}$$

Thus, $f\in \mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$.

Conversely, assume that $f\in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Since

$$a_n \le \frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)} \quad (n\ge 2), \tag{4.6}$$

we can set

$$\begin{gathered}\Upsilon_n = \frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}a_n \quad (n\ge 2),\\ \Upsilon_1 = 1-\sum_{n=2}^{\infty}\Upsilon_n.\end{gathered} \tag{4.7}$$

Then,

$$\begin{aligned} f(z)^{\beta} &= z^{\beta} - \sum_{n=2}^{\infty}\beta a_n z^{\beta+n-1} \\ &= z^{\beta} - \sum_{n=2}^{\infty}\beta\frac{\Upsilon_n\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}z^{\beta+n-1} \\ &= z^{\beta} - \sum_{n=2}^{\infty}\Upsilon_n\left(z^{\beta}-f_n(z)^{\beta}\right) \\ &= z^{\beta}\left(1-\sum_{n=2}^{\infty}\Upsilon_n\right) + \sum_{n=2}^{\infty}\Upsilon_n f_n(z)^{\beta} \\ &= \Upsilon_1 f_1(z)^{\beta} + \sum_{n=2}^{\infty}\Upsilon_n f_n(z)^{\beta} \\ &= \sum_{n=1}^{\infty}\Upsilon_n f_n(z)^{\beta}. \end{aligned} \tag{4.8}$$

This completes the proof of Theorem 4.1. □

5. Integral Means

In order to find the integral means inequality and to verify the Silverman Conjuncture [12] for $f\in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$, we need the following definition of subordination and subordination result according to Littlewood [13].

Definition 5.1 (see [13]). Let f and g be analytic in U. Then, we say that the function f is subordinate to g if there exists a Schwarz function w, analytic in U with $w(0)=0$, $|w(z)|<1$ such that $f(z)=g(w(z))$ $(z\in U)$. We denote this subordination $f \prec g$ or $f(z)\prec g(z)$ $(z\in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0)=g(0)$, $f(U)\subset g(U)$.

Lemma 5.2 (see [13]). *If the functions f and g are analytic in U with $g \prec f$, then*

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\eta} d\theta, \quad \eta > 0,\ z = re^{i\theta},\ 0 < r < 1. \tag{5.1}$$

Applying Theorem 2.1 with the extremal function and Lemma 5.2, we prove the following theorem.

Theorem 5.3. *Let $\eta > 0$. If $f \in \mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$ and $\{\Phi(\alpha, \beta, k, \upsilon, n)\}_{n=2}^{\infty}$ are nondecreasing sequences, then, for $z = re^{i\theta}$ and $0 < r < 1$, one has*

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)^{\beta}\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f_2\left(re^{i\theta}\right)^{\beta}\right|^{\eta} d\theta, \tag{5.2}$$

where

$$\begin{gathered} f_2(z)^{\beta} = z^{\beta} - \beta \frac{(k-\alpha) + (1-k)\left(\beta + |\upsilon|\right)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z^{\beta+1}, \\ \Phi(\alpha, \beta, k, \upsilon, n) = (\beta + n - 1)\left[(\beta + n - 1 + |\upsilon|)(1-k) + (k-\alpha)\right] Q(n, \beta, \ell). \end{gathered} \tag{5.3}$$

Proof. Let f of the form of (1.2) and

$$f_2(z)^{\beta} = z^{\beta} - \beta \frac{(k-\alpha) + (1-k)\left(\beta + |\upsilon|\right)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z^{\beta+1}, \tag{5.4}$$

then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} \beta a_n z^{n-1}\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|1 - \beta \frac{(k-\alpha) + (1-k)\left(\beta + |\upsilon|\right)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z\right|^{\eta} d\theta. \tag{5.5}$$

By Lemma 5.2, it suffices to show that

$$1 - \sum_{n=2}^{\infty} \beta a_n z^{n-1} \prec 1 - \beta \frac{(k-\alpha) + (1-k)\left(\beta + |\upsilon|\right)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z. \tag{5.6}$$

Setting

$$1 - \sum_{n=2}^{\infty} \beta a_n z^{n-1} = 1 - \beta \frac{(k-\alpha) + (1-k)\left(\beta + |\upsilon|\right)}{\Phi(\alpha, \beta, k, \upsilon, 2)} w(z), \tag{5.7}$$

from (5.7) and (2.1) we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty}\frac{\Phi(\alpha,\beta,k,\upsilon,2)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}a_n z^{n-1}\right| \\ &\le |z|\sum_{n=2}^{\infty}\frac{\Phi(\alpha,\beta,k,\upsilon,n)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}a_n \\ &\le |z| < 1.\end{aligned} \tag{5.8}$$

This completes the proof of Theorem 5.3. □

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