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Research Article

Multiple Solutions for a Fractional Difference Boundary Value Problem via Variational Approach

Zuoshi Xie,[1] Yuanfeng Jin,[2] and Chengmin Hou[2]

[1] *School of Economics and International Trade, Zhejiang University of Finance and Economics, Hangzhou, Zhejiang 310018, China*
[2] *Department of Mathematics, Yanbian University, Yanji 133002, China*

Correspondence should be addressed to Chengmin Hou, houchengmin@yahoo.com.cn

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By establishing the corresponding variational framework and using the mountain pass theorem, linking theorem, and Clark theorem in critical point theory, we give the existence of multiple solutions for a fractional difference boundary value problem with parameter. Under some suitable assumptions, we obtain some results which ensure the existence of a well precise interval of parameter for which the problem admits multiple solutions. Some examples are presented to illustrate the main results.

1. Introduction

Variational methods for dealing with difference equations have appeared as early as 1985 in [1] in which the positive definiteness of quadratic forms (which are functionals) is related to the existence of "nodes" of solutions (or positive solutions satisfying "conjugate" boundary conditions) of linear self-adjoint second-order difference equations of the form

$$\Delta\big(p(k-1)\Delta y(k-1)\big) + q(k)y(k) = 0, \quad k = 1, 2, \ldots, n, \tag{1.1}$$

where $p(k)$ is real and positive for $k = 0, 1, \ldots, n$ and $q(k)$ is real for $k = 1, 2, \ldots, n$. Later there are interests in solutions of nonlinear difference equations under various types of boundary or subsidiary conditions, and more sophisticated methods such as the mountain pass theorems are needed to handle the existence problem (see, e.g., [2–11]).

Recently, fractional differential and difference "operators" are found themselves in concrete applications, and hence attention has to be paid to associated fractional difference and differential equations under various boundary or side conditions. For example, a recent paper by Atici and Eloe [12] explores some of the theories of a discrete conjugate fractional BVP. Similarly, in [13], a discrete right-focal fractional BVP is analyzed. Other recent advances in the theory of the discrete fractional calculus may be found in [14, 15]. In particular, an interesting recent paper by Atici and Şengül [16] addressed the use of fractional difference equations in tumor growth modeling. Thus, it seems that there exists some promise in using fractional difference equations as mathematical models for describing physical problems in more accurate manners.

In order to handle the existence problem for fractional BVPs, various methods (among which are some standard fixed-point theorems) can be used. In this paper, however, we show that variational methods can also be applied. A good reason for picking such an approach is that, in Atici and Şengül [16], some basic fractional calculuses are developed and a simple variational problem is demonstrated, and hence advantage can be taken in obvious manners. We remark, however, that fractional difference operators can be approached in different manners and one by means of operator convolution rings can be found in the book by Cheng [17, Chapter 3] published in 2003.

More specifically, in this paper, we are interested in the existence of multiple solutions for the following 2ν-order fractional difference boundary value problem

$$
{}_T\Delta_{t-1}^{\nu}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right) = \lambda f(t+\nu-1, x(t+\nu-1)), \quad t \in [0,T]_{\mathbb{N}_0}, \tag{1.2}
$$

$$
x(\nu-2) = \left[{}_t\Delta_{\nu-1}^{\nu}x(t)\right]_{t=T} = 0, \tag{1.3}
$$

where $\nu \in (0,1)$, ${}_t\Delta_{\nu-1}^{\nu}$ and ${}_T\Delta_t^{\nu}$ are, respectively, left fractional difference and the right fractional difference operators (which will be explained in more detail later), $t \in [0,T]_{\mathbb{N}_0} = \{0,1,2,\ldots,T\}$, $f(t+\nu-1,\cdot) : [\nu-1, T+\nu-1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous, and λ is a positive parameter.

By establishing the corresponding variational framework and using critical point theory, we will establish various existence results (which naturally depend on f, ν, and λ).

For convenience, throughout this paper, we arrange $\sum_{i=j}^{m} x(i) = 0$, for $m < j$.

2. Preliminaries

We first collect some basic lemmas for manipulating discrete fractional operators. These and other related results can be found in [14, 16].

First, for any integer β, we let $\mathbb{N}_\beta = \{\beta, \beta+1, \beta+2, \ldots\}$. We define $t^{(\nu)} := \Gamma(t+1)/\Gamma(t+1-\nu)$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that, if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{(\nu)} = 0$.

Definition 2.1. The νth fractional sum of f for $\nu > 0$ is defined by

$$
\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s), \tag{2.1}
$$

for $t \in \mathbb{N}_{a-\nu}$. We also define the νth fractional difference for $\nu > 0$ by $\Delta^{\nu} f(t) := \Delta^N \Delta^{\nu-N} f(t)$, where $t \in \mathbb{N}_{a+N-\nu}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \nu \le N$.

Definition 2.2. Let f be any real-valued function and $\nu \in (0,1)$. The left discrete fractional difference and the right discrete fractional difference operators are, respectively, defined as

$$
\begin{aligned}
{}_t\Delta_a^{\nu} f(t) &= \Delta_t \Delta_a^{-(1-\nu)} f(t) = \frac{1}{\Gamma(1-\nu)} \Delta \sum_{s=a}^{t+\nu-1} (t-s-1)^{(-\nu)} f(s), \quad t \equiv a-\nu+1 \ (\mathrm{mod}1), \\
{}_b\Delta_t^{\nu} f(t) &= -\Delta_b \Delta_t^{-(1-\nu)} f(t) = \frac{1}{\Gamma(1-\nu)} (-\Delta) \sum_{s=t+1-\nu}^{b} (s-t-1)^{(-\nu)} f(s), \quad t \equiv b+\nu-1 \ (\mathrm{mod}1).
\end{aligned}
\tag{2.2}
$$

Definition 2.3. For $I \in C^1(\mathbb{E}, \mathbb{R})$, we say I satisfies the Palais-Smale condition (henceforth denoted by (PS) condition) if any sequence $\{x_n\} \subset \mathbb{E}$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence.

Lemma 2.4 (see [18]). *A real symmetric matrix A is positive definite if there exists a real nonsingular matrix M such that $A = M^{\dagger} M$, where $M^{\dagger}$ is the transpose.*

Lemma 2.5 (see [9]: linking theorem). *Let $\mathbb{E}$ be a real Banach space, and $I \in C^1(\mathbb{E}, \mathbb{R})$ satisfies (PS) condition and is bounded from below. Suppose I has a local linking at the origin θ, namely, there is a decomposition $\mathbb{E} = \mathbb{Y} \oplus \mathbb{W}$ and a positive number ρ such that $k = \dim \mathbb{Y} < \infty$, $I(y) < I(\theta)$ for $y \in \mathbb{Y}$ with $0 < \|y\| \le \rho$; $I(y) \ge I(\theta)$ for $y \in \mathbb{W}$ with $\|y\| \le \rho$. Then I has at least three critical points.*

Lemma 2.6 (see [6]). *Let $\mathbb{E}$ be a real reflexive Banach space, and let the functional $I : \mathbb{E} \to \mathbb{R}$ be weakly lower (upper) semicontinuous and coercive, that is, $\lim_{\|x\| \to \infty} I(x) = \infty$ (resp., anticoercive, i.e., $\lim_{\|x\| \to \infty} I(x) = -\infty$). Then there exists $x_0 \in \mathbb{E}$ such that $I(x_0) = \inf_{\mathbb{E}} I(x)$ (resp., $I(x_0) = \sup_{\mathbb{E}} I(x)$). Moreover, if $I \in C^1(\mathbb{E}, \mathbb{R})$, then x_0 is a critical point of functional I.*

Recall that, in the finite dimensional setting, it is well known that a coercive functional satisfies the (PS) condition.

Let $\mathbb{B}_r$ denote the open ball in a real Banach space of radius r about 0, and let $\partial \mathbb{B}_r$ denote its boundary. Now some critical point theorems needed later can be stated.

Lemma 2.7 (mountain pass theorem [8]). *Let $\mathbb{E}$ be a real Banach space and $I \in C^1(\mathbb{E}, \mathbb{R})$, satisfying (PS) condition. Suppose $I(\theta) = 0$ and*

(I_1) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial \mathbb{B}_\rho} \ge \alpha$,*

(I_2) *there is $e \in \mathbb{E} \setminus \overline{\mathbb{B}}_\rho$ such that $I(e) \le 0$.*

Then I possesses a critical value $c \ge \alpha$. Moreover c can be characterized as

$$
c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u), \tag{2.3}
$$

where

$$
\Gamma = \{ g \in C([0,1], \mathbb{E}) \mid g(0) = \theta, g(1) = e \}. \tag{2.4}
$$

Lemma 2.8 (see [7]). *Let $\mathbb{E}$ be a reflexive Banach space and $I \in C^1(\mathbb{E}, \mathbb{R})$ with $I(\theta) = 0$. Suppose that I is an even functional satisfying (PS) condition and the following conditions:*

(I_3) *there are constants $\rho, \overline{\alpha} > 0$ and a closed linear subspace $\mathbb{X}_1$ of $\mathbb{E}$ such that codim $\mathbb{X}_1 = l < \infty$ and $I|_{\partial\mathbb{B}_\rho \cap \mathbb{X}_1} \geq \overline{\alpha}$,*

(I_4) *there is a finite dimensional subspace $\mathbb{X}_2$ of $\mathbb{E}$ with $\dim \mathbb{X}_2 = m$, $m > l$, such that $I(x) \to -\infty$ as $\|x\| \to \infty$, $x \in \mathbb{X}_2$. Then I possesses at least $m-l$ distinct pairs of nontrivial critical points.*

Lemma 2.9 (the Clark theorem [8]). *Let $\mathbb{E}$ be a real Banach space, $I \in C^1(\mathbb{E}, \mathbb{R})$ with I even, bounded from below, and satisfying (PS) condition. Suppose $I(\theta) = 0$, there is a set $\mathbb{K} \subset \mathbb{E}$ such that $\mathbb{K}$ is homeomorphic to $\mathbb{S}^{j-1}$ (the $j-1$ dimensional unit sphere) by an odd map and $\sup_{\mathbb{K}} I < 0$. Then I possesses at least j distinct pairs of critical points.*

3. Main Results

Firstly, we establish variational framework. Let

$$\Omega = \left\{ x = (x(\nu-1), x(\nu), \dots, x(\nu+T-1))^{\dagger} \mid x(\nu+i-1) \in \mathbb{R}, i = 0, 1, \dots, T \right\} \tag{3.1}$$

be the $T+1$-dimensional Hilbert space with the usual inner product and the usual norm

$$\langle x, z \rangle = \sum_{t=\nu-1}^{T+\nu-1} x(t) z(t), \quad \|x\| = \left(\sum_{t=\nu-1}^{T+\nu-1} |x(t)|^2 \right)^{1/2}, \quad x, z \in \Omega. \tag{3.2}$$

For $r > 1$, we recall the r-norm on Ω: $\|x\|_r = (\sum_{t=\nu-1}^{T+\nu-1} |x(t)|^r)^{1/r}$. We also recall the standard fact that there exist positive constants c_r and $\overline{c}_r$, such that

$$c_r \|x\| \leq \|x\|_r \leq \overline{c}_r \|x\|, \quad x \in \Omega. \tag{3.3}$$

Define a functional on Ω by

$$I(x) = \frac{1}{2} \sum_{t=-1}^{T} \left({}_t\Delta_{\nu-1}^{\nu} x(t) \right)^2 - \lambda \sum_{t=-1}^{T} F(t+\nu-1, x(t+\nu-1)) \tag{3.4}$$

for $x = (x(\nu-1), x(\nu), \dots, x(\nu+T-1))^{\dagger} \in \Omega$, where

$$\begin{gathered} F(t+\nu-1, x(t+\nu-1)) = \int_0^{x(t+\nu-1)} f(t+\nu-1, s) ds, \\ x(\nu-2) = 0, \qquad \left[{}_t\Delta_{\nu-1}^{\nu} x(t) \right]_{t=T} = \frac{-\nu}{\Gamma(1-\nu)} \sum_{s=\nu-1}^{T+\nu} (T-s-1)^{(-\nu-1)} x(s) = 0. \end{gathered} \tag{3.5}$$

Obviously, $I(\theta) = 0$. Let

$$\mathbb{E} = \left\{\chi = (x(\nu-2), x(\nu-1), \dots, x(\nu+T))^{\dagger} \in \mathbb{R}^{T+3} \mid x(\nu-2) = 0, \left[{}_t\Delta_{\nu-1}^{\nu} x(t)\right]_{t=T} = 0\right\}. \tag{3.6}$$

Then by (1.3) it is easy to see that $\mathbb{E}$ is isomorphic to Ω. In the following, when we say $x \in \Omega$, we always imply that x can be extended to $\chi \in \mathbb{E}$ if it is necessary. Now we claim that if $x = (x(\nu-1), x(\nu), \dots, x(\nu+T-1))^{\dagger} \in \Omega$ is a critical point of I, then $\chi = (x(\nu-2), x(\nu-1), \dots, x(\nu+T))^{\dagger} \in \mathbb{E}$ is precisely a solution of BVP (1.2) and (1.3). Indeed, since I can be viewed as a continuously differentiable functional defined on the finite dimensional Hilbert space Ω, the Frechet derivative $I'(x)$ is zero if and only if $\partial I(x)/\partial x(i) = 0$ for all $i = \nu-1, \nu, \dots, \nu+T-1$.

By computation,

$$\begin{aligned}
\frac{\partial I(x)}{\partial x(\nu-1)} &= \sum_{t=-1}^{T} \left({}_t\Delta_{\nu-1}^{\nu} x(t)\right) \frac{\partial_t \Delta_{\nu-1}^{\nu} x(t)}{\partial x(\nu-1)} - \lambda f(\nu-1, x(\nu-1)) \\
&= \frac{-\nu}{\Gamma(1-\nu)} \sum_{t=-1}^{T} \frac{\partial \sum_{s=\nu-1}^{t+\nu} (t-s-1)^{(-\nu-1)} x(s)}{\partial x(\nu-1)} \left({}_t\Delta_{\nu-1}^{\nu} x(t)\right) - \lambda f(\nu-1, x(\nu-1)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)} \sum_{t=-1}^{T} (t-\nu)^{(-\nu-1)} \sum_{s=\nu-1}^{t+\nu} (t-s-1)^{(-\nu-1)} x(s) - \lambda f(\nu-1, x(\nu-1)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)} \sum_{s=-1}^{T} (s-\nu)^{(-\nu-1)} \sum_{u=\nu-1}^{s+\nu} (s-u-1)^{(-\nu-1)} x(u) - \lambda f(\nu-1, x(\nu-1)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)} \left[\sum_{s=t-\nu}^{T} (s-t-1)^{(-\nu-1)} \sum_{u=\nu-1}^{s+\nu} (s-u-1)^{(-\nu-1)} x(u)\right]_{t=\nu-1} \\
&\quad - \lambda f(\nu-1, x(\nu-1)) \\
&= \frac{-\nu}{\Gamma(1-\nu)} \left[\sum_{s=t-\nu}^{T} (s-t-1)^{(-\nu-1)} \left({}_s\Delta_{\nu-1}^{\nu}\right) x(s)\right]_{t=\nu-1} - \lambda f(\nu-1, x(\nu-1)) \\
&= \frac{-1}{\Gamma(1-\nu)} \left[\sum_{s=t-\nu}^{T} \left((s-t-1)^{(-\nu)} - (s-t)^{(-\nu)}\right) \left({}_s\Delta_{\nu-1}^{\nu}\right) x(s)\right]_{t=\nu-1} \\
&\quad - \lambda f(\nu-1, x(\nu-1)) \\
&= \frac{1}{\Gamma(1-\nu)} \left[(-\Delta) \sum_{s=t-\nu}^{T} (s-t)^{(-\nu)} \left({}_s\Delta_{\nu-1}^{\nu}\right) x(s)\right]_{t=\nu-1} - \lambda f(\nu-1, x(\nu-1)) \\
&= \left[{}_T\Delta_{t-1}^{\nu} \left({}_t\Delta_{\nu-1}^{\nu}\right) x(t)\right]_{t=\nu-1} - \lambda f(\nu-1, x(\nu-1)),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial I(x)}{\partial x(\nu)} &= \sum_{t=-1}^{T}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right)\frac{\partial_t\Delta_{\nu-1}^{\nu}x(t)}{\partial x(\nu)} - \lambda f(\nu, x(\nu)) \\
&= \frac{-\nu}{\Gamma(1-\nu)}\sum_{t=-1}^{T}\frac{\partial\sum_{s=\nu-1}^{t+\nu}(t-s-1)^{(-\nu-1)}x(s)}{\partial x(\nu)}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right) - \lambda f(\nu, x(\nu)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)}\sum_{t=0}^{T}(t-\nu-1)^{(-\nu-1)}\sum_{s=\nu-1}^{t+\nu}(t-s-1)^{(-\nu-1)}x(s) - \lambda f(\nu, x(\nu)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)}\sum_{s=0}^{T}(s-\nu-1)^{(-\nu-1)}\sum_{u=\nu-1}^{s+\nu}(s-u-1)^{(-\nu-1)}x(u) - \lambda f(\nu, x(\nu)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)}\left[\sum_{s=t-\nu}^{T}(s-t-1)^{(-\nu-1)}\sum_{u=\nu-1}^{s+\nu}(s-u-1)^{(-\nu-1)}x(u)\right]_{t=\nu} - \lambda f(\nu, x(\nu)) \\
&= \frac{-\nu}{\Gamma(1-\nu)}\left[\sum_{s=t-\nu}^{T}(s-t-1)^{(-\nu-1)}\left({}_s\Delta_{\nu-1}^{\nu}\right)x(s)\right]_{t=\nu} - \lambda f(\nu, x(\nu)) \\
&= \frac{-1}{\Gamma(1-\nu)}\left[\sum_{s=t-\nu}^{T}\left((s-t-1)^{(-\nu)} - (s-t)^{(-\nu)}\right)\left({}_s\Delta_{\nu-1}^{\nu}\right)x(s)\right]_{t=\nu} - \lambda f(\nu, x(\nu)) \\
&= \frac{1}{\Gamma(1-\nu)}\left[(-\Delta)\sum_{s=t-\nu}^{T}(s-t)^{(-\nu)}\left({}_s\Delta_{\nu-1}^{\nu}\right)x(s)\right]_{t=\nu} - \lambda f(\nu, x(\nu)) \\
&= \left[{}_T\Delta_{t-1}^{\nu}\left({}_t\Delta_{\nu-1}^{\nu}\right)x(t)\right]_{t=\nu} - \lambda f(\nu, x(\nu)), \\
&\ \ \vdots \\
\frac{\partial I(x)}{\partial x(\nu+T-1)} &= \sum_{t=-1}^{T}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right)\frac{\partial_t\Delta_{\nu-1}^{\nu}x(t)}{\partial x(\nu+T-1)} - \lambda f(\nu+T-1, x(\nu+T-1)) \\
&= \frac{-\nu}{\Gamma(1-\nu)}\sum_{t=-1}^{T}\frac{\partial\sum_{s=\nu-1}^{t+\nu}(t-s-1)^{(-\nu-1)}x(s)}{\partial x(\nu+T-1)}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right) \\
&\quad - \lambda f(\nu+T-1, x(\nu+T-1)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)}\sum_{t=T-1}^{T}(t-\nu-T)^{(-\nu-1)}\sum_{s=\nu-1}^{t+\nu}(t-s-1)^{(-\nu-1)}x(s) \\
&\quad - \lambda f(\nu+T-1, x(\nu+T-1)) \\
&= \frac{\nu^2}{\Gamma^2(1-\nu)}\sum_{s=T-1}^{T}(s-\nu-T)^{(-\nu-1)}\sum_{u=\nu-1}^{s+\nu}(s-u-1)^{(-\nu-1)}x(u) \\
&\quad - \lambda f(\nu+T-1, x(\nu+T-1))
\end{aligned}$$

$$\begin{aligned}
&= \frac{\nu^2}{\Gamma^2(1-\nu)}\left[\sum_{s=t-\nu}^{T}(s-t-1)^{(-\nu-1)}\sum_{u=\nu-1}^{s+\nu}(s-u-1)^{(-\nu-1)}x(u)\right]_{t=\nu+T-1} \\
&\quad - \lambda f(\nu+T-1, x(\nu+T-1)) \\
&= \frac{-\nu}{\Gamma(1-\nu)}\left[\sum_{s=t-\nu}^{T}(s-t-1)^{(-\nu-1)}\left({}_s\Delta_{\nu-1}^{\nu}\right)x(s)\right]_{t=\nu+T-1} \\
&\quad - \lambda f(\nu+T-1, x(\nu+T-1)) \\
&= \frac{-1}{\Gamma(1-\nu)}\left[\sum_{s=t-\nu}^{T}\left((s-t-1)^{(-\nu)}-(s-t)^{(-\nu)}\right)\left({}_s\Delta_{\nu-1}^{\nu}\right)x(s)\right]_{t=\nu+T-1} \\
&\quad - \lambda f(\nu+T-1, x(\nu+T-1)) \\
&= \frac{1}{\Gamma(1-\nu)}\left[(-\Delta)\sum_{s=t-\nu}^{T}(s-t)^{(-\nu)}\left({}_s\Delta_{\nu-1}^{\nu}\right)x(s)\right]_{t=\nu+T-1} \\
&\quad - \lambda f(\nu+T-1, x(\nu+T-1)) \\
&= \left[{}_T\Delta_{t-1}^{\nu}\left({}_t\Delta_{\nu-1}^{\nu}\right)x(t)\right]_{t=\nu+T-1} - \lambda f(\nu+T-1, x(\nu+T-1)).
\end{aligned} \tag{3.7}$$

So to obtain the existence of solutions for problem (1.2) and (1.3), we just need to study the existence of critical points, that is, $x \in \Omega$ such that $I'(x) = 0$, of the functional I on Ω.

Next, observe by Definition 2.2 that, for $t \in [-1, T]_{\mathbb{N}_{-1}}$,

$$ {}_t\Delta_{\nu-1}^{\nu}x(t) = \Delta\frac{1}{\Gamma(1-\nu)}\sum_{t=\nu-1}^{t-(1-\nu)}(t-s-1)^{(-\nu)}x(s). \tag{3.8}$$

We let

$$z(t+\nu-1) = \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{t-(1-\nu)}(t-s-1)^{(-\nu)}x(s), \tag{3.9}$$

then

$$\begin{aligned}
&z(\nu-2) = 0, \\
&z(\nu-1) = \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{0-(1-\nu)}(-s-1)^{(-\nu)}x(s) = x(\nu-1), \\
&z(\nu) = \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{1-(1-\nu)}(1-s-1)^{(-\nu)}x(s) = (1-\nu)x(\nu-1) + x(\nu),
\end{aligned}$$

$$
\begin{aligned}
z(\nu+1) &= \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{2-(1-\nu)}(2-s-1)^{(-\nu)}x(s) \\
&= \frac{(2-\nu)(1-\nu)}{2!}x(\nu-1)+(1-\nu)x(\nu)+x(\nu+1), \\
&\vdots \\
z(\nu+T-1) &= \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{T-(1-\nu)}(T-s-1)^{(-\nu)}x(s) \\
&= \frac{(T-\nu)(T-1-\nu)\cdots(1-\nu)}{(T)!}x(\nu-1)+\frac{(T-\nu-1)(T-2-\nu)\cdots(1-\nu)}{(T-1)!}x(\nu) \\
&\quad +\cdots+(1-\nu)x(\nu+T-2)+x(\nu+T-1),
\end{aligned}
\tag{3.10}
$$

that is, $z = Bx$, where $z = (z(\nu-1), z(\nu), \ldots, z(\nu+T-1))^{\dagger}$, $x = (x(\nu-1), x(\nu), \ldots, x(\nu+T-1))^{\dagger}$:

$$
B = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1-\nu & 1 & 0 & \cdots & 0 \\
\dfrac{(2-\nu)(1-\nu)}{2!} & 1-\nu & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\dfrac{(T-\nu)(T-1-\nu)\cdots(1-\nu)}{T!} & \dfrac{(T-1-\nu)(T-2-\nu)\cdots(1-\nu)}{(T-1)!} & \cdots & \cdots & 1
\end{pmatrix}_{(T+1)\times(T+1)}.
\tag{3.11}
$$

By Lemma 2.4, $(B^{-1})^{\dagger}B^{-1}$ is a positive definite matrix. Let $\lambda_{\min}$ and $\lambda_{\max}$ denote, respectively, the minimum and the maximum eigenvalues of $(B^{-1})^{\dagger}B^{-1}$.

Since $x = B^{-1}z$, we may easily see that

$$
\lambda_{\min}\|z\|^2 \le \|x\|^2 = \left\langle z^{\dagger}\left(B^{-1}\right)^{\dagger}, B^{-1}z\right\rangle \le \lambda_{\max}\|z\|^2.
\tag{3.12}
$$

Then $\|x\| \to \infty$ if and only if $\|z\| \to \infty$. Next, let

$$
A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & \cdots & -1 & 1
\end{pmatrix}_{(T+1)\times(T+1)}.
\tag{3.13}
$$

By direct verifications, we may find that A is a positive definite matrix. Let $\eta_1, \eta_2, \ldots, \eta_{T+1}$ be the orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{T+1}$ of A, where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{T+1}$.

For convenience, we list the following assumptions.

(C_1) There exists $\mu \in (0,2)$ such that $\limsup_{|x|\to\infty}(F(t+\nu-1,x)/|x|^{\mu}) < a$ for $t \in [0,T]_{\mathbb{N}_0}$, where a is a constant.

(C_2) There is a constant $\mu > 2$ such that $\liminf_{|x|\to\infty}(F(t+\nu-1,x)/|x|^{\mu}) > 0$ for $t \in [0,T]_{\mathbb{N}_0}$.

(C_3) There exists a constant $d > 0$ such that $\limsup_{|x|\to 0}(F(t+\nu-1,x)/|x|^2) < d$ for $t \in [0,T]_{\mathbb{N}_0}$.

(C_4) $F(t+\nu-1,x)$ satisfies $\lim_{x\to 0}(F(t+\nu-1,x)/|x|^2) = q_1 > 0$ for $t \in [0,T]_{\mathbb{N}_0}$, where q_1 is a constant.

(C_5) $f(t+\nu-1,x)$ is odd with respect to x, that is, $f(t+\nu-1,-x) = -f(t+\nu-1,x)$, for $t \in [0,T]_{\mathbb{N}_0}$, and $x \in \mathbb{R}$.

(C_6) There is a positive constant p_2 such that $\liminf_{x\to 0}(F(t+\nu-1,x)/|x|^2) > p_2$ for $t \in [0,T]_{\mathbb{N}_0}$.

Theorem 3.1. *If (C_1) holds, then for all $\lambda > 0$, BVP* (1.2), (1.3) *has at least one solution.*

Proof. By (C_1), we obtain

$$F(t+\nu-1,x) \le a|x|^{\mu} + b, \quad t \in [0,T]_{\mathbb{N}_0},\ |x| \ge \varsigma, \tag{3.14}$$

where ς is some sufficiently large numbers and $b > 0$. Thus, by the continuity of $F(t+\nu-1,x) - a|x|^{\mu}$ on $[0,T]_{\mathbb{N}_0} \times [-\varsigma,\varsigma]$, there exists $a' > 0$ such that

$$F(t+\nu-1,x) \le a|x|^{\mu} + a', \quad (t,s) \in [0,T]_{\mathbb{N}_0} \times \mathbb{R}. \tag{3.15}$$

Combining with (3.3)–(3.15), we have

$$\begin{aligned} I(x) &= \frac{1}{2}\sum_{t=-1}^{T}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right)^2 - \lambda\sum_{t=-1}^{T}F(t+\nu-1,x(t+\nu-1)) \\ &= \frac{1}{2}\sum_{t=-1}^{T}(\Delta z(t+\nu-1))^2 - \lambda\sum_{t=-1}^{T}F(t+\nu-1,x(t+\nu-1)) \\ &= \frac{1}{2}\sum_{t=-1}^{T-1}(\Delta z(t+\nu-1))^2 - \lambda\sum_{t=0}^{T}F(t+\nu-1,x(t+\nu-1)) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\lambda_1\|z\|^2 - \lambda\sum_{t=0}^{T}F(t+\nu-1, x(t+\nu-1)) \\
&\geq \frac{1}{2}\lambda_1\|z\|^2 - \lambda|a|\sum_{t=0}^{T}|x(t+\nu-1)|^{\mu} - a'\lambda(T+1) \\
&\geq \frac{1}{2}\lambda_1\|z\|^2 - \lambda|a|(\overline{c}_{\mu})^{\mu}\|x\|^{\mu} - a'\lambda(T+1) \\
&\geq \frac{1}{2}\lambda_1\|z\|^2 - \lambda|a|(\overline{c}_{\mu})^{\mu}\lambda_{\max}^{(1/2)\mu}\|z\|^{\mu} - a'\lambda(T+1).
\end{aligned} \tag{3.16}$$

So, in view of our assumption $\mu \in (0,2)$, we see that, for $\lambda > 0$, $I(x) \to \infty$ as $\|x\| \to \infty$, that is, $I(x)$ is a coercive map. In view of Lemma 2.6, we know that there exists at least one $\overline{x} \in \Omega$ such that $I'(\overline{x}) = 0$; hence BVP (1.2), (1.3) has at least one solution. The proof is completed. □

Remark 3.2. If $\mu = 2$ and $a > 0$, from the proof of Theorem 3.1, we can get that, for $\lambda \in (0, \lambda_1/2|a|\lambda_{\max})$, our functional I is also coercive.

Theorem 3.3. *If (C$_2$) holds, then for all $\lambda > 0$, BVP* (1.2), (1.3) *has at least one solution.*

Proof. Similar to the proof Theorem 3.1, we have

$$I(x) \leq \frac{1}{2}\lambda_{T+1}\|z\|^2 - \lambda\sum_{t=0}^{T}F(t+\nu-1, x(t+\nu-1)). \tag{3.17}$$

By (C$_2$), there exists $\varsigma > 0$ and $\varsigma' > 0$ such that $F(t+\nu-1, x) \geq \varsigma|x|^{\mu}$ for $|x| > \varsigma'$ with $t \in [0,T]_{\mathbb{N}_0}$, so

$$F(t+\nu-1, x) \geq \varsigma|x|^{\mu} - c, \quad (t,x) \in [0,T]_{\mathbb{N}_0} \times \mathbb{R}, \tag{3.18}$$

where $c > 0$. Since $F(t+\nu-1,x) - \varsigma|x|^{\mu}$ is continuous on $[0,T]_{\mathbb{N}_0} \times [-\varsigma', \varsigma']$, through (3.17), we obtain

$$I(x) \leq \frac{1}{2}\lambda_{T+1}\|z\|^2 - \lambda\varsigma(c_{\mu})^{\mu}\lambda_{\min}^{\mu/2}\|z\|^{\mu} + c\lambda(T+1). \tag{3.19}$$

Thus $I(x) \to -\infty$ as $||x|| \to \infty$ for $\mu > 2$. That is, $I(x)$ is an anticoercive. In view of Lemma 2.6, we know that there exists at least one $\overline{x} \in \Omega$ such that $I'(\overline{x}) = 0$; hence BVP (1.2), (1.3) has at least one solution. The proof is completed. □

Theorem 3.4. *Assume (C$_2$) and (C$_3$) hold. Then, for $\lambda \in (0, \lambda_1/2d\lambda_{\max})$, the BVP* (1.2), (1.3) *possesses at least two nontrivial solutions.*

Proof. First, we know from Theorem 3.3 that $I(x) \to -\infty$ as $||x|| \to \infty$. Clearly, Ω is a real reflexive finite dimensional Banach space and $I \in C^1(\Omega, \mathbb{R})$, so functional I is weakly upper semicontinuous. By Lemma 2.6, there exists $x_0 \in \Omega$ such that $I(x_0) = \sup_{\Omega} I$ and $I'(x_0) = 0$. Set $c_0 = \sup_{\Omega} I$. Let $\{x_n\} \subset \Omega$, such that there exists $M > 0$ and $|I(x_n)| \le M$ for $n \in \mathbb{N}$. By (C_2) and (3.19), we may see that

$$-M \le I(x_n) \le \frac{1}{2}\lambda_{T+1}\|z_n\|^2 - \lambda_{\min}^{\mu/2}\lambda\varsigma(c_\mu)^\mu\|z_n\|^\mu + c\lambda(T+1), \tag{3.20}$$

that is,

$$-M - c\lambda(T+1) \le \frac{1}{2}\lambda_{T+1}\|z_n\|^2 - \lambda_{\min}^{\mu/2}\lambda\varsigma(c_\mu)^\mu\|z_n\|^\mu. \tag{3.21}$$

In view of $\mu > 2$, we see that $\{z_n\} \subset \Omega$ is bounded, and hence $\{x_n\} \subset \Omega$ is bounded. Since Ω is finite dimensional, there is a subsequence of $\{x_n\}$, which is convergent in Ω. Therefore, the (PS) condition is verified.

By (C_3), there exists $\delta > 0, F(t+\nu-1, x) \le dx^2$ for $|x| \le \delta$, $t \in [0,T]_{\mathbb{N}_0}$. Thus, for $x \in \Omega$ with $\|x\| \le \delta$, we have

$$\begin{aligned} I(x) &= \frac{1}{2}\sum_{t=-1}^{T-1}(\Delta z(t+\nu-1))^2 - \lambda\sum_{t=0}^{T}F(t+\nu-1, x(t+\nu-1)) \\ &\ge \frac{1}{2}\lambda_1\|z\|^2 - \lambda\sum_{t=0}^{T}F(t+\nu-1, x(t+\nu-1)) \\ &\ge \left(\frac{1}{2}\lambda_1 - \lambda_{\max}\lambda d\right)\|z\|^2. \end{aligned} \tag{3.22}$$

For $\lambda \in (0, \lambda_1/2d\lambda_{\max})$, we choose $\rho = \delta$ and $\gamma = ((1/2)\lambda_1 - \lambda\lambda_{\max})\rho^2$. Then we have $I|_{\partial B_\rho} \ge \gamma > 0$, so that the condition (I_1) in Lemma 2.7 holds.

Since $I(x) \to -\infty$ as $\|x\| \to \infty$, we can find $e \in \Omega$ with sufficiently large norm $\|e\|$ such that $I(e) < 0$. Hence (I_2) in Lemma 2.7 is satisfied. Thus, functional I has one critical value

$$c = \inf_{g\in\Gamma}\max_{u\in g([0,1])} I(u), \tag{3.23}$$

where $\Gamma = \{g \in C([0,1], \Omega) \mid g(0) = \theta, g(1) = e\}$. If $c_0 > c$, the proof is completed. It suffices to consider the case $c_0 = c$. Then

$$c_0 = c = \inf_{g\in\Gamma}\max_{u\in g([0,1])} I(u), \tag{3.24}$$

that is, $c_0 = \max_{u\in g([0,1])} I(u)$ for each $g \in \Gamma$.

Similarly, we can also choose $-e \in \Omega$ such that $I(-e) < 0$. Applying Lemma 2.7 again, we obtain another critical value of the functional I,

$$\overline{c} = \inf_{\overline{g}\in\overline{\Gamma}}\max_{u\in\overline{g}([0,1])} I(u), \tag{3.25}$$

where $\overline{\Gamma} = \{\overline{g} \in C([0,1],\Omega) \mid \overline{g}(0) = \theta, \overline{g}(1) = -e\}$. If $c_0 > \overline{c}$, then the proof is completed. It suffices to consider the case where $c_0 = \overline{c}$. Then $c_0 = \max_{u \in g([0,1])} I(u)$ for each $g \in \overline{\Gamma}$. By the definitions of Γ and $\overline{\Gamma}$, we may choose $g_0 \in \Gamma$ and $\overline{g}_0 \in \overline{\Gamma}$ such that $g_0([0,1]) \cap \overline{g}_0([0,1]) = \{\theta\}$. Therefore, we get the maximum of the functional I on $g_0([0,1]) \setminus \{\theta\}$ and $\overline{g}_0([0,1]) \setminus \{\theta\}$, respectively, that is, we find two distinct nontrivial critical points of the functional I. Therefore, our BVP (1.2), (1.3) possesses at least two nontrivial solutions. □

Theorem 3.5. *Assume that* (C_1) *and* (C_4) *hold and that there exists* $N \in [1, T+1]_{\mathbb{N}_1}$ *such that* $\lambda_N < \lambda_{N+1}$. *Then, for* $\lambda \in (\lambda_N / 2q_1\lambda_{\min}, \lambda_{N+1}/2q_1\lambda_{\max})$, *BVP* (1.2), (1.3) *has at least three solutions.*

Proof. By (C_1) and Theorem 3.1, we obtain $\lim_{\|x\| \to \infty} I(x) = \infty$, thus functional I is bounded from below. Similar to the proof of (PS) condition in Theorem 3.4, we can verify that functional I satisfies (PS) condition in our hypothesis. In order to apply linking theorem, we prove functional I is local linking at origin θ as follows. Clearly, $\Omega = \operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_{T+1}\}$. Let $\mathbb{X} = \operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_N\}$, $\mathbb{W} = \operatorname{span}\{\eta_{N+1}, \eta_{N+2}, \ldots, \eta_{T+1}\}$, then $\Omega = \mathbb{X} \oplus \mathbb{W}$.

By (C_4), for $\varepsilon \in (0, q_1)$, there exists $\rho > 0$, such that

$$(q_1 - \varepsilon)x^2 \le F(t+\nu-1, x) \le (q_1 + \varepsilon)x^2, \quad |x| \le \rho,\ t \in [0,T]_{\mathbb{N}_0}. \tag{3.26}$$

So, for $x \in \mathbb{X}$ with $0 < \|x\| \le \rho$, such that

$$\begin{gathered}\sum_{t=-1}^{T-1} (\Delta z(t+\nu-1))^2 = z' A z \le \lambda_N \|z\|^2 \le \frac{\lambda_N \|x\|^2}{\lambda_{\min}}, \\ \sum_{t=0}^{T} F(t+\nu-1, x(t+\nu-1)) \ge (q_1 - \varepsilon) \sum_{t=0}^{T} (x(t+\nu-1))^2 = (q_1 - \varepsilon)\|x\|^2.\end{gathered} \tag{3.27}$$

Since $z = Bx \in \mathbb{X}$, we have

$$\begin{aligned} I(x) &= \frac{1}{2}\sum_{t=-1}^{T-1} (\Delta z(t+\nu-1))^2 - \lambda \sum_{t=0}^{T} F(t+\nu-1, x(t+\nu-1)) \\ &\le \left(\frac{\lambda_N}{2\lambda_{\min}} - \lambda(q_1 - \varepsilon) \right) \|x\|^2. \end{aligned} \tag{3.28}$$

Thus, for $\lambda > \lambda_N / 2(q_1 - \varepsilon)\lambda_{\min}$, we have $I(x) < 0$ for $x \in \mathbb{X}$ with $0 < \|x\| \le \rho$.

Similarly, for $x \in \mathbb{W}$ with $0 < \|x\| \le \rho$,

$$I(x) = \frac{1}{2}\sum_{t=-1}^{T-1} (\Delta z(t+\nu-1))^2 - \lambda \sum_{t=0}^{T} F(t+\nu-1, x(t+\nu-1)) \ge \left(\frac{\lambda_{N+1}}{2\lambda_{\max}} - \lambda(q_1 + \varepsilon) \right) \|x\|^2, \tag{3.29}$$

then for $\lambda < \lambda_{N+1}/2(q_1+\varepsilon)\lambda_{\max}$, we have $I(x) > 0$ for $x \in \mathbb{W}$ with $0 < \|x\| \le \rho$. So, by Lemma 2.5, for $\varepsilon \in (0, q_1)$, if $\lambda \in (\lambda_N/2(q_1-\varepsilon)\lambda_{\min}, \lambda_{N+1}/2(q_1+\varepsilon)\lambda_{\max})$, functional I possesses at least three critical points. By the arbitrariness of ε, we get for $\lambda \in (\lambda_N/2q_1\lambda_{\min}, \lambda_{N+1}/2q_1\lambda_{\max})$, the problem (1.2), (1.3) possesses at least three solutions. □

Theorem 3.6. *Assume (C_2), (C_3), and (C_5) hold. Then, for each $N \in [0,T]_{\mathbb{N}_0}$, if $\lambda \in (0, \lambda_{N+1}/2d\lambda_{\max})$, then BVP* (1.2),(1.3) *possesses at least $T+1-N$ pairs of solutions.*

Proof. By (C_5), functional I is even, and based on the proof of Theorem 3.4, we know that I satisfies (PS) condition. In order to obtain our result, we need to verify (I_3) and (I_4) of Lemma 2.8.

First, in view of (C_3), there exists $\rho > 0$ such that

$$F(t+\nu-1, x) \leq dx^2 \quad \text{for } |x| \leq \rho,\ t \in [0,T]_{\mathbb{N}_0}. \tag{3.30}$$

For $N \in [1, T+1]_{\mathbb{N}_1}$, if we choose $\mathbb{X}_1 = \operatorname{span}\{\eta_{N+1}, \eta_{N+2}, \ldots, \eta_{T+1}\}$, then codim $\mathbb{X}_1 = N$. So for $x \in \mathbb{X}_1$ with $\|x\| \leq \rho$, since $z = Bx$, we have

$$I(x) = \frac{1}{2}\sum_{t=-1}^{T-1}(\Delta z(t+\nu-1))^2 - \lambda\sum_{t=0}^{T}F(t+\nu-1, x(t+\nu-1)) \geq \left(\frac{\lambda_{N+1}}{2\lambda_{\max}} - \lambda d\right)\|x\|^2. \tag{3.31}$$

Thus, for $\lambda \in (0, \lambda_{N+1}/2d\lambda_{\max})$, $I|_{\mathbb{X}_1 \cap \partial\mathbb{B}_\rho} \geq \beta > 0$, where $\beta = (\lambda_{N+1}/2\lambda_{\max} - \lambda d)\rho^2$, ($I_3$) of Lemma 2.8 holds.

Next if we choose $\mathbb{X}_2 = \operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_{T+1}\}$, then for $x \in \mathbb{X}_2$, in view of (C_2) and Theorem 3.3, we get $I(x) \to -\infty$ as $\|x\| \to \infty$. (I_4) of Lemma 2.8 is satisfied.

Therefore, for $\lambda \in (0, \lambda_{N+1}/2d\lambda_{\max})$, functional I possesses at least $T+1-N$ pair of critical points in Ω, and problem (1.2),(1.3) has at least $T+1-N$ pairs of solutions. □

Remark 3.7. In Theorem 3.6, if we choose $N = 0$, then for $\lambda \in (0, \lambda_1/2d\lambda_{\max})$, the BVP (1.2), (1.3) possesses at least $T+1$ pairs of solutions.

Obviously, compared with Theorem 3.4, the even condition (C_5) ensures that the problem (1.2), (1.3) possesses more solutions.

Theorem 3.8. *Suppose (C_1), (C_5), and (C_6) hold. Then for every $N \in [1, T+1]_{\mathbb{N}_1}$, when $\lambda \in (\lambda_N/2p_2\lambda_{\min}, \infty)$, problem* (1.2), (1.3) *possesses at least N pairs of nontrivial solutions.*

Proof. $I(x)$ is an even functional on Ω by (C_5). From (C_1), we obtain $I(x) \to \infty$ as $\|x\| \to \infty$, so it is clear that I is bounded from below on Ω and satisfies the (PS) condition. For $N \in [1, T+1]_{\mathbb{N}_1}$, if we choose $\mathbb{X}_1 = \operatorname{span}\{\eta_1, \ldots, \eta_N\}$ and set $\mathbb{K} = \mathbb{X}_1 \cap \partial\mathbb{B}_\rho$, then $\mathbb{K}$ is homeomorphic to $\mathbb{S}^{N-1}$ by an odd map. By (C_6), there exists $\rho_1 > 0$ such that $F(t+\nu-1, x) \geq p_2 x^2$ for $|x| \leq \rho_1$, $t \in [0,T]_{\mathbb{N}_0}$. So for $x \in \mathbb{K}_1 = \mathbb{X}_1 \cap \partial\mathbb{B}_{\rho_1}$,

$$\begin{aligned} I(x) &= \frac{1}{2}\sum_{t=-1}^{T}(\Delta z(t+\nu-1))^2 - \lambda\sum_{t=-1}^{T}F(t+\nu-1, x(t+\nu-1)) \leq \frac{1}{2}\lambda_N\|z\|^2 - \lambda p_2\|x\|^2 \\ &= \left(\frac{\lambda_N}{2\lambda_{\min}} - \lambda p_2\right)\rho_1^2. \end{aligned} \tag{3.32}$$

For $\lambda \in (\lambda_N/2p_2\lambda_{\min x}, +\infty)$, we have $\sup_{\mathbb{K}_1} I(x) < 0$. Therefore, by Lemma 2.9, functional I has at least N pairs of nontrivial solutions. □

Remark 3.9. From Theorem 3.5, it is easy to see that, when f is odd about the second variable, we can obtain more solutions of the problem (1.2), (1.3), and the number of solutions depends on where λ lies.

4. Applications

In the final section, we apply the results developed in Section 3 to some examples.

Example 4.1. Consider the following problem

$$\begin{gathered}{}_{T}\Delta_{t-1}^{\nu}\left({}_{t}\Delta_{\nu-1}^{\nu}x(t)\right)=\lambda 4(t+\nu-1)x^{3}(t+\nu-1)(\sin x(t+\nu-1)+2)\\+(t+\nu-1)x^{4}(t+\nu-1)\cos(x(t+\nu-1)),\quad t\in[0,T]_{\mathbb{N}0},\\x(\nu-2)=\left[{}_{t}\Delta_{\nu-1}^{\nu}x(t)\right]_{t=T}=0,\end{gathered}\tag{4.1}$$

where $f(t+\nu-1,x)=4(t+\nu-1)x^3(\sin x+2)+(t+\nu-1)x^4\cos x$. Choose $\mu=4$ and $d=1$ in (C_2) and (C_3). Since

$$\begin{gathered}\liminf_{|x|\to\infty}\frac{F(t+\nu-1,x)}{|x|^{4}}=\liminf_{|x|\to\infty}(t+\nu-1)(\sin x+2)=t+\nu-1>0,\quad t\in[0,T]_{\mathbb{N}_0},\\\limsup_{|x|\to 0}\frac{F(t+\nu-1,x)}{x^{2}}=\limsup_{|x|\to\infty}(t+\nu-1)x^{2}(\sin x+2)=0,\quad t\in[0,T]_{\mathbb{N}_0},\end{gathered}\tag{4.2}$$

we see that (C_2) and (C_3) hold. Thus, by Theorem 3.4, when $\lambda\in(0,\lambda_1/2\lambda_{\max})$, problem (4.1) has at least two nontrivial solutions.

Example 4.2. Consider the problem

$$\begin{gathered}{}_{T}\Delta_{t-1}^{\nu}\left({}_{t}\Delta_{\nu-1}^{\nu}x(t)\right)=\lambda 4\sin(x(t+\nu-1))\cos(x(t+\nu-1))\\-e^{-(t+\nu-1)x(t+\nu-1)}\\\times\Big(2x(t+\nu-1)-(t+\nu-1)x^{2}(t+\nu-1)\Big)\quad t\in[0,T]_{\mathbb{N}_0},\\x(\nu-2)=\left[{}_{t}\Delta_{\nu-1}^{\nu}x(t)\right]_{t=T}=0.\end{gathered}\tag{4.3}$$

Suppose there exists $N_0\in[0,T]_{\mathbb{N}_0}$ such that $\lambda_{\max}\lambda_{N_0}<\lambda_{\min}\lambda_{N_0+1}$. If we choose $\mu=q_1=a=1$ in (C_1) and (C_4), then for $t\in[0,T]_{\mathbb{N}_0}$, we have

$$\begin{gathered}\limsup_{|x|\to\infty}\frac{F(t+\nu-1,x)}{|x|^{\mu}}=\limsup_{|x|\to\infty}\frac{2\sin^{2}x-x^{2}e^{-(t+\nu-1)x}}{|x|^{\mu}}<1=a,\\\lim_{x\to 0}\frac{F(t+\nu-1,x)}{x^{2}}=1=q_1>0,\end{gathered}\tag{4.4}$$

and hence (C_1) and (C_4) are satisfied. So, in view of Theorem 3.5, for $\lambda \in (\lambda_{N_0}/2\lambda_{\min}, \lambda_{N_0+1}/2\lambda_{\max})$, problem (4.3) has at least three solutions.

Example 4.3. Consider the problem

$$\begin{aligned} {}_T\Delta_{t-1}^{\nu}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right) &= \lambda 4(t+\nu-1)x^3(t+\nu-1)(\cos x(t+\nu-1)+2) \\ &\quad -(t+\nu-1)x^4(t+\nu-1)\sin(x(t+\nu-1)), \quad t \in [0,T]_{\mathbb{N}_0}, \\ x(\nu-2) &= \left[{}_t\Delta_{\nu-1}^{\nu}x(t)\right]_{t=T} = 0. \end{aligned} \tag{4.5}$$

Condition (C_5) is satisfied. If we choose $\mu = 4$ and $d = 1$ in (C_2) and (C_3), then by some simple calculation, we may show that the hypotheses (C_2) and (C_3) are fulfilled. Therefore, by Theorem 3.6, for any $N \in [0,T]_{\mathbb{N}_0}$ and $\lambda \in (0, \lambda_{N+1}/2\lambda_{\max})$, problem (4.5) has at least $T+1-N$ pairs of solutions.

Example 4.4. Finally, consider the problem

$$\begin{aligned} {}_T\Delta_{t-1}^{\nu}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right) &= \lambda \frac{1}{t+\nu-1}\sin((t+\nu-1)x(t+\nu-1)) \\ &\quad + x(t+\nu-1)\cos((t+\nu-1)x(t+\nu-1)), \quad t \in [0,T]_{\mathbb{N}_0}, \\ x(\nu-2) &= \left[{}_t\Delta_{\nu-1}^{\nu}x(t)\right]_{t=T} = 0, \end{aligned} \tag{4.6}$$

where $f(t+\nu-1,x) = (1/(t+\nu-1))\sin((t+\nu-1)x) + x\cos((t+\nu-1)x)$. Let $a = \mu = 1$ and $p_2 = 1/2$. Then it is easy to verify that (C_1), (C_5), and (C_6) hold. Thus, by Theorem 3.8, for each $N \in [0,T]_{\mathbb{N}_0}$ and $\lambda \in (\lambda_N/\lambda_{\min}, +\infty)$, problem (4.6) has at least N pairs of solutions.

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References

[1] S. S. Cheng, “Sturmian comparison theorems for three-term recurrence equations,” *Journal of Mathematical Analysis and Applications*, vol. 111, no. 2, pp. 465–474, 1985.
[2] O. P. Agrawal, J. Gregory, and K. Pericak-Spector, “A Bliss-type multiplier rule for constrained variational problems with time delay,” *Journal of Mathematical Analysis and Applications*, vol. 210, no. 2, pp. 702–711, 1997.
[3] J. N. Reddy, *Energy and Variational Methods in Applied Mechanics: With an Introduction to the Finite Element Method*, John Wiley and Sons, New York, NY, USA, 1984.
[4] O. P. Agrawal, “Formulation of Euler-Lagrange equations for fractional variational problems,” *Journal of Mathematical Analysis and Applications*, vol. 272, no. 1, pp. 368–379, 2002.
[5] B. Gompertz, “On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies,” *Philosophical Transactions of the Royal Society of London*, vol. 115, pp. 513–585, 1825.
[6] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5, Academic Press, Orlando, Fla, USA, 1988.
[7] H. Brezis and L. Nirenberg, “Remarks on finding critical points,” *Communications on Pure and Applied Mathematics*, vol. 44, no. 8-9, pp. 939–963, 1991.

[8] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.
[9] D. Goeleven, D. Motreanu, and P. D. Panagiotopoulos, "Multiple solutions for a class of eigenvalue problems in hemivariational inequalities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 29, no. 1, pp. 9–26, 1997.
[10] G. Q. Wang and S. S. Cheng, "Steady state solutions of a toroidal neural network via a mountain pass theorem," *Annals of Differential Equations*, vol. 22, no. 3, pp. 360–363, 2006.
[11] H. H. Liang and P. X. Weng, "Existence and multiple solutions for a second-order difference boundary value problem via critical point theory," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 511–520, 2007.
[12] F. M. Atici and P. W. Eloe, "A transform method in discrete fractional calculus," *International Journal of Difference Equations*, vol. 2, no. 2, pp. 165–176, 2007.
[13] C. S. Goodrich, "Solutions to a discrete right-focal fractional boundary value problem," *International Journal of Difference Equations*, vol. 5, no. 2, pp. 195–216, 2010.
[14] F. M. Atici and P. W. Eloe, "Two-point boundary value problems for finite fractional difference equations," *Journal of Difference Equations and Applications*, vol. 17, no. 4, pp. 445–456, 2011.
[15] F. M. Atici and P. W. Eloe, "Discrete fractional calculus with the nabla operator," *Electronic Journal of Qualitative Theory of Differential Equations Edition I*, no. 3, pp. 1–12, 2009.
[16] F. M. Atici and S. Şengül, "Modeling with fractional difference equations," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 1, pp. 1–9, 2010.
[17] S. S. Cheng, *Partial Difference Equations*, vol. 3 of *Advances in Discrete Mathematics and Applications*, Taylor & Francis, London, UK, 2003.
[18] F. Ayres Jr, *Schaum's Outline of Theory and problems of Matrices*, Schaum, New York, NY, USA, 1962.