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Research Article

Existence of Solutions of a Nonlocal Elliptic System via Galerkin Method

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By means of the Galerkin method and by using a suitable version of the Brouwer fixed-point theorem, we establish the existence of at least one positive solution of a nonlocal elliptic N-dimensional system coupled with Dirichlet boundary conditions.

1. Introduction

This paper is devoted to the study of the nonlocal elliptic system

$$\begin{gathered}
-\Delta u(x) + a(x)\int_\Omega b(y)v^p(y)dy = g(u(x), v(x)), \quad x \in \Omega, \\
-\Delta v(x) + c(x)\int_\Omega d(y)u^q(y)dy = h(u(x), v(x)), \quad x \in \Omega, \\
u(x), v(x) > 0, \quad x \in \Omega, \\
u(x) = v(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded smooth domain, $p,\ q$ are positive numbers, $a, b, c, d \in C(\overline{\Omega})$, and the nonlinearities g and h will be defined later.

The one-dimensional counterpart of this problem has been considered by Cabada et al. in [1]. There the authors, by using dual variational methods and Leray-Schauder degree, with $p = q = 1,\ b \equiv d \equiv 1$, and $a = c$ a real parameter, and under suitable assumptions on the nonlinear functions $g(u,v) \equiv g(v)$ and $h(u,v) \equiv h(u)$, show the existence of solution (not necessarily positive) depending upon the parameter a.

In this case, we present a different point of view from the one used in [1]. We note that, among other things, we assume $N \geq 2$.

Motivated by its many applications and the richness of the methods employed to solve it, this kind of problems has been studied by different authors when only one equation is considered, see, among others, [2–9]. Indeed, there is a lot of phenomena that may be modeled by equations of the form

$$u_t - \Delta u = f(x, u, B(u)), \tag{1.2}$$

where B is a nonlocal operator which, in some applications, is written in the form

$$B(u) = \int_\Omega b(x)[u(x)]^\beta dx. \tag{1.3}$$

See [10–13] for some surveys on these equations.

In particular, steady-state solutions deliver us to elliptic equations such as

$$\begin{gathered} -\Delta u(x) = f(x, u(x), B(u)), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.4}$$

which, in several cases, have a behaviour quite different from the local one

$$\begin{gathered} -\Delta u(x) = f(x, u(x)), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.5}$$

One of the most significant differences between these two types of problems is the nonexistence, in some particular cases, of maximum principles. For instance, Allegretto and Barabanova [4] consider the one-dimensional problem

$$\begin{gathered} -u''(x) + \eta \int_0^1 u(y)dy = \sin(\pi x), \quad 0 < x < 1, \\ u(0) = u(1) = 0, \quad \eta > 0. \end{gathered} \tag{1.6}$$

It is not difficult to verify that the explicit solution of this problem is given by the expression

$$u(x) = \frac{12\eta(x^2 - x)}{(\pi^3(12 + \eta))} + \frac{\sin(\pi x)}{\pi^2}. \tag{1.7}$$

So, when the values of the positive parameter η are small, the solution is positive. However, if η is large enough, function u becomes negative near the end points $x = 0$ and

$x = 1$. That is, $\sin(\pi x) > 0$ in $(0, 1)$ but, for η large enough, the corresponding solution is not positive.

This contrasts with the local equation

$$\begin{aligned} -u''(x) + \eta u(x) &= f(x), \quad 0 < x < 1, \\ u(0) = u(1) &= 0, \quad \eta > 0, \end{aligned} \tag{1.8}$$

for which it is very well known, see [14], that, for all $\eta > 0$, function $u > 0$ in $(0, 1)$ whenever $f > 0$ in $(0, 1)$.

Remark 1.1. We have to point out that the lack of a general maximum principle seems to be characteristic of integrodifferential operator. Indeed, in [15], the authors consider a noncooperative system, arisen in the classical FitzHugh-Nagumo systems, which serves as a model for nerve conduction. More precisely, it is studied the system

$$\begin{gathered} -\Delta u(x) = f(x, u) - v, \quad x \in \Omega, \\ -\Delta v(x) = \delta u - \gamma v, \quad x \in \Omega, \\ u(x), v(x) > 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.9}$$

where $\delta, \gamma > 0$ are constants and $f(x, u)$ is a given function. Taking $B \equiv \delta(-\Delta + \gamma)^{-1}$, under Dirichlet boundary condition, problem (1.9) is equivalent to the integrodifferential problem

$$\begin{gathered} -\Delta u(x) + Bu = f(x, u), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.10}$$

Consider now the problem

$$\begin{gathered} -\Delta u(x) + Bu - \lambda u = f(x), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.11}$$

with $f \in L^2(\Omega)$ and $f \geq 0$ in Ω.

Let λ_1 be the first eigenvalue of operator $-\Delta$ in the space $H_0^1(\Omega)$, and assume that $\lambda_1 > \sqrt{\delta} - \gamma$. Then, for all $\lambda \in (2\sqrt{\delta} - \gamma, \lambda_1 + (\delta/(\gamma + \lambda_1\ )))$, problem (1.11) satisfies a maximum principle, that is, under the above assumptions, the solution u of (1.11) satisfies $u \geq 0$ a.e in Ω. See [15] for the proof of this result.

After that, it is proved in [16], by using semigroup theory, that this maximum principle does not hold for all $\lambda < 2\sqrt{\delta} - \gamma$. Indeed, the approach used in [16] may be used to prove that a general maximum principle for the problem (1.6) is not valid. In view of this, the method of sub- and supersolution should be used carefully by considering a relation between the growth of the nonlinearity and the parameters of the problem.

Remark 1.2. It is worthy to remark that problem (1.1) has no variational structure even in the scalar case. So the most usual variational techniques cannot be used to study it.

To attack problem (1.1), we will use the Galerkin method through the following version of the Brouwer fixed-Point Theorem whose proof may be found in Lions [17, Lemma 4.3].

Proposition 1.3. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function such that*

$$\langle F(\xi), \xi \rangle > 0 \quad \text{if } |\xi| = r, \tag{1.12}$$

for some $r > 0$, where $\langle \cdot, \cdot \rangle$ is the Euclidian Scalar product and $|\cdot| = \langle \cdot, \cdot \rangle^{1/2}$ is the corresponding Euclidian norm in $\mathbb{R}^m$. Then, there exists $\xi_0 \in \mathbb{R}^m$, $|\xi_0| \leq r$ such that $F(\xi_0) = 0$.

2. A Sublinear Problem

In this section, we consider the problem

$$\begin{gathered} -\Delta u(x) + \lambda \int_\Omega v^p(y)\,dy = u^\alpha(x) + v^\beta(x), \quad x \in \Omega, \\ -\Delta v(x) + \lambda \int_\Omega u^q(y)\,dy = u^\gamma(x) + v^\delta(x), \quad x \in \Omega, \\ u(x), v(x) > 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{2.1}$$

Here, λ is a real parameter and $\alpha,\ \beta,\ \gamma,\ \delta$ are positive constants whose properties will be precised later.

In order to use Proposition 1.3, we have to introduce a suitable setup. First of all, we consider an orthonormal Hilbertian basis $\mathbb{B} = \{\varphi_1, \varphi_2, \ldots\}$ in $H_0^1(\Omega)$ whose norm is the usual one

$$\|u\|^2 = \int_\Omega |\nabla u(y)|^2\, dy, \quad \forall u \in H_0^1(\Omega). \tag{2.2}$$

Next, let $\mathbb{V}_m$ be the finite dimensional vector space

$$\mathbb{V}_m = [\varphi_1, \ldots, \varphi_m] \subset \mathbb{B}, \tag{2.3}$$

equipped with the norm induced by the one in $H_0^1(\Omega)$.

Thus, if $u \in \mathbb{V}_m$, there is a unique $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ such that

$$u = \sum_{j=1}^m \xi_j \varphi_j, \tag{2.4}$$

and, as a consequence,

$$\|u\|^2 = |\xi|^2. \tag{2.5}$$

So, the spaces $\mathbb{V}_m$ and $\mathbb{R}^m$ are isomorphic and isometric by

$$\begin{gathered}\mathbb{V}_m \longleftrightarrow \mathbb{R}^m,\\ u = \sum_{j=1}^{m} \xi_j \varphi_j \longleftrightarrow \xi = (\xi_1, \dots, \xi_m).\end{gathered} \tag{2.6}$$

From now on, we identify, with no additional comments, $u \leftrightarrow \xi$ via this isometry.

In order to obtain a nontrivial solution of problem (2.1), let $\epsilon > 0$ be a constant and consider the auxiliary problem

$$\begin{gathered}-\Delta u(x) + \lambda \int_\Omega (v^+)^p(y)\,dy = (u^+)^\alpha(x) + (v^+)^\beta(x) + \epsilon, \quad x \in \Omega,\\ -\Delta v(x) + \lambda \int_\Omega (u^+)^q(y)\,dy = (u^+)^\gamma(x) + (v^+)^\delta(x), \quad x \in \Omega,\\ u(x), v(x) > 0, \quad x \in \Omega,\\ u(x) = v(x) = 0, \quad x \in \partial\Omega.\end{gathered} \tag{2.7}$$

Theorem 2.1. *Assume that Ω is a bounded C^2 domain of $\mathbb{R}^N$ and the constants $p, q, \alpha, \beta, \gamma, \delta \in (0,1)$. Then, for all $\lambda < 0$, problem* (2.1) *has at least one solution in* $(C^2(\Omega) \cap C(\overline{\Omega})) \times (C^2(\Omega) \cap C(\overline{\Omega}))$.

Proof. First of all, we consider a map $(F, G) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$, $(F, G) = (F_1, \dots, F_m, G_1, \dots, G_m)$, defined, for all $i = 1, \dots, m$, as

$$\begin{gathered}F_i(\xi, \eta) = \int_\Omega \nabla u \nabla \varphi_i + \lambda \int_\Omega (v^+)^p \int_\Omega \varphi_i - \int_\Omega (u^+)^\alpha \varphi_i - \int_\Omega (v^+)^\beta \varphi_i - \epsilon \int_\Omega \varphi_i,\\ G_i(\xi, \eta) = \int_\Omega \nabla v \nabla \varphi_i + \lambda \int_\Omega (u^+)^q \int_\Omega \varphi_i - \int_\Omega (u^+)^\gamma \varphi_i - \int_\Omega (v^+)^\delta \varphi_i,\end{gathered} \tag{2.8}$$

where we are identifying $(u, v) \in \mathbb{V}_m \times \mathbb{V}_m$, $u = \sum_{j=1}^m \xi_j \varphi_j$, $v = \sum_{j=1}^m \eta_j \varphi_j$, with $(\xi, \eta) \in \mathbb{R}^m \times \mathbb{R}^m$, $\xi = (\xi_1, \dots, \xi_m)$, $\eta = (\eta_1, \dots, \eta_m)$.

Now, we have that, for all $i = 1, \dots, m$, the following equations hold:

$$\begin{aligned}F_i(\xi, \eta) \cdot \xi_i &= \int_\Omega \nabla u \cdot \nabla(\xi_i \varphi_i) + \lambda \int_\Omega (v^+)^p \int_\Omega \xi_i \varphi_i - \int_\Omega (u^+)^\alpha (\xi_i \varphi_i)\\ &\quad - \int_\Omega (v^+)^\beta (\xi_i \varphi_i) - \epsilon \int_\Omega \xi_i \varphi_i,\\ G_i(\xi, \eta) \cdot \eta_i &= \int_\Omega \nabla v \cdot \nabla(\eta_i \varphi_i) + \lambda \int_\Omega (u^+)^q \int_\Omega \eta_i \varphi_i - \int_\Omega (u^+)^\gamma (\eta_i \varphi_i) - \int_\Omega (v^+)^\delta (\eta_i \varphi_i).\end{aligned} \tag{2.9}$$

Therefore,

$$\begin{aligned}\langle (F,G)(\xi,\eta),(\xi,\eta)\rangle = &\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2 + \lambda \int_\Omega (v^+)^p \int_\Omega u - \int_\Omega (u^+)^\alpha u \\ &- \int_\Omega (v^+)^\beta u - \epsilon \int_\Omega u + \lambda \int_\Omega (u^+)^q \int_\Omega v - \int_\Omega (u^+)^\gamma v - \int_\Omega (v^+)^\delta v.\end{aligned} \tag{2.10}$$

Denoting as $\|(x,y)\|^2 = |x|^2 + |y|^2$ for all $x,\ y \in \mathbb{R}^m$, using the isometry between $\mathbb{V}_m$ and $\mathbb{R}^m$, and the inequalities of Hölder, Poincaré, and Sobolev, we arrive at the following estimations:

$$\left(\int_\Omega (v^+)^p\right)\int_\Omega u \le C\left(\int_\Omega (|v|^p)^{2/p}\right)^{p/2} \cdot \|u\| \le C\|(u,v)\|^{p+1}, \tag{I_1}$$

$$\int_\Omega (v^+)^\beta u \le \left(\int_\Omega |v|^{2\beta}\right)^{1/2} |u|_2 \le C\|(u,v)\|^{\beta+1}, \tag{I_2}$$

and, in a similar way,

$$\int_\Omega (u^+)^\alpha u \le C\|(u,v)\|^{\alpha+1}, \tag{I_3}$$

$$\int_\Omega (u^+)^q \int_\Omega v \le C\|(u,v)\|^{q+1}, \tag{I_4}$$

$$\int_\Omega (u^+)^\gamma v \le \|(u,v)\|^{\gamma+1}, \tag{I_5}$$

$$\int_\Omega (v^+)^\delta v \le C\|(u,v)\|^{\delta+1}. \tag{I_6}$$

We note that the positive constant C depends on Ω, but it does not depend on the rest of the parameters involved in problem (2.7).

Using the previous estimations (I_1)–(I_6) together with the fact that $\lambda < 0$, we deduce that

$$\begin{aligned}\langle (F,G)(\xi,\eta),(\xi,\eta)\rangle \geq{}& \|(u,v)\|^2 + \lambda C\|(u,v)\|^{p+1} - C\|(u,v)\|^{\beta+1}\\ &- \epsilon\, C\|(u,v)\| + \lambda C\|(u,v)\|^{q+1} - C\|(u,v)\|^{\gamma+1}\\ &- C\|(u,v)\|^{\delta+1} - C\|(u,v)\|^{\gamma+1} - C\|(u,v)\|^{\alpha+1}.\end{aligned} \tag{2.11}$$

Now, since $0 < p,q,\alpha,\beta,\gamma,\delta < 1$, there is $r > 0$ such that

$$\langle (F,G)(\xi,\eta),(\xi,\eta)\rangle > 0 \quad \text{if } \|(\xi,\eta)\| > r. \tag{2.12}$$

In view of Proposition 1.3, there exists $r > 0$ that does not depend on m, and a pair $(u_m,v_m) \in \mathbb{V}_m \times \mathbb{V}_m$, such that $\|u_m\|,\|v_m\| \leq r$, and satisfies the following equalities for all $i = 1,\ldots,m$:

$$\begin{gathered}\int_\Omega \nabla u_m \nabla\varphi_i + \lambda\int_\Omega (v_m^+)^p \int_\Omega \varphi_i - \int_\Omega (u_m^+)^\alpha\varphi_i - \int_\Omega (v_m^+)^\alpha\varphi_i - \epsilon\int_\Omega \varphi_i = 0,\\ \int_\Omega \nabla v_m \nabla\varphi_i + \lambda\int_\Omega (u_m^+)^q \int_\Omega \varphi_i - \int_\Omega (u_m^+)^\gamma\varphi_i - \int_\Omega (v_m^+)^\delta\varphi_i = 0.\end{gathered} \tag{2.13}$$

So, for all $\varphi \in \mathbb{V}_m$, it is satisfied that

$$\begin{gathered}\int_\Omega \nabla u_m \nabla\varphi + \lambda\int_\Omega (v_m^+)^p \int_\Omega \varphi - \int_\Omega (u_m^+)^\alpha\varphi - \int_\Omega (v_m^+)^\alpha\varphi - \epsilon\int_\Omega \varphi = 0,\\ \int_\Omega \nabla v_m \nabla\varphi + \lambda\int_\Omega (u_m^+)^q \int_\Omega \varphi - \int_\Omega (u_m^+)^\gamma\varphi - \int_\Omega (v_m^+)^\delta\varphi = 0.\end{gathered} \tag{2.14}$$

In view of the boundedness of the approximate solutions u_m, v_m in $H_0^1(\Omega)$, we have, perhaps for subsequences, that $u_m \rightharpoonup u_\epsilon$ and $v_m \rightharpoonup v_\epsilon$ in $H_0^1(\Omega)$.

Fixing $k < m$ and making $m \to +\infty$ in the last two equalities, we obtain, after using Sobolev immersions, that the following equalities hold for all $\varphi \in \mathbb{V}_k$:

$$\begin{gathered}\int_\Omega \nabla u_\epsilon \nabla\varphi + \lambda\int_\Omega (v_\epsilon^+)^p \int_\Omega \varphi - \int_\Omega (u_\epsilon^+)^\alpha\varphi - \int_\Omega (v_\epsilon^+)^\alpha\varphi - \epsilon\int_\Omega \varphi = 0,\\ \int_\Omega \nabla v_\epsilon \nabla\varphi + \lambda\int_\Omega (u_\epsilon^+)^q \int_\Omega \varphi - \int_\Omega (u_\epsilon^+)^\gamma\varphi - \int_\Omega (v_\epsilon^+)^\delta\varphi = 0.\end{gathered} \tag{2.15}$$

Since k is arbitrary, the last two identities are valid for all $\varphi \in H_0^1(\Omega)$. Consequently, $(u_\epsilon, v_\epsilon) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a weak solution of the auxiliary problem (2.7).

Now, since $\lambda < 0$ and $\epsilon > 0$, we have, from (2.7), that $-\Delta u_\epsilon \geq \epsilon$ and $-\Delta v_\epsilon \geq 0$ on Ω. This fact, together with the Dirichlet boundary conditions, says that $u_\epsilon, v_\epsilon > 0$ on Ω. In consequence, the following equalities hold for all $\varphi \in H_0^1(\Omega)$:

$$\begin{gathered}\int_\Omega \nabla u_\epsilon \nabla \varphi + \lambda \int_\Omega v_\epsilon^p \int_\Omega \varphi - \int_\Omega u^\alpha \varphi - \int_\Omega v^\alpha \varphi - \epsilon \int_\Omega \varphi = 0, \\ \int_\Omega \nabla v_\epsilon \nabla \varphi + \lambda \int_\Omega u_\epsilon^q \int_\Omega \varphi - \int_\Omega u_\epsilon^\gamma \varphi - \int_\Omega v_\epsilon^\delta \varphi = 0.\end{gathered} \tag{2.16}$$

That is to say, (u_ϵ, v_ϵ) is a weak solution of problem

$$\begin{gathered}-\Delta u(x) + \lambda \int_\Omega v^p(y)dy = u^\alpha(x) + v^\beta(x) + \epsilon, \quad x \in \Omega, \\ -\Delta v(x) + \lambda \int_\Omega u^q(y)dy = u^\gamma(x) + v^\delta(x), \quad x \in \Omega, \\ u(x), v(x) > 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega.\end{gathered} \tag{2.17}$$

In particular, it satisfies the following inequalities:

$$\begin{gathered}-\Delta u_\epsilon(x) \geq u_\epsilon^\alpha(x), \quad x \in \Omega, \\ -\Delta v_\epsilon(x) \geq v_\epsilon^\delta(x), \quad x \in \Omega, \\ u_\epsilon(x), v_\epsilon(x) > 0, \quad x \in \Omega, \\ u_\epsilon(x) = v_\epsilon(x) = 0, \quad x \in \partial\Omega.\end{gathered} \tag{2.18}$$

Let $u_\alpha, v_\delta > 0$ be the unique solution of the problem

$$\begin{gathered}-\Delta u(x) = u^\alpha(x), \quad x \in \Omega, \\ -\Delta v(x) = v^\delta(x), \quad x \in \Omega, \\ u(x), v(x) > 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega.\end{gathered} \tag{2.19}$$

We point out that the existence and uniqueness of the solutions of each of the above equations follow from [18, 19] because $0 < \alpha,\ \delta < 1$.

In the sequel, we use the following comparison result, due to Ambrosetti, Brezis, and Cerami.

Lemma 2.2. *[20, Lemma 3.3]. Assume that $f(t)$ is a continuous function such that $t^{-1}f(t)$ is decreasing for $t > 0$. Let v and w satisfy*

$$\begin{gathered} -\Delta v(x) \leq f(v(x)), \quad x \in \Omega, \\ v(x) > 0, \quad x \in \Omega, \\ v(x) = 0, \quad x \in \partial\Omega, \\ -\Delta w(x) \geq f(w(x)), \quad x \in \Omega, \\ w(x) > 0, \quad x \in \Omega, \\ w(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{2.20}$$

Then, $w(x) \geq v(x)$ for all $x \in \Omega$. □

So we conclude that $u_\epsilon \geq u_\alpha > 0$ and $v_\epsilon \geq v_\delta > 0$ in Ω.

Taking limits on both members of (2.16) and (2.6) as $\epsilon \to 0^+$, we deduce that $u_\epsilon \rightharpoonup u$ and $v_\epsilon \rightharpoonup v$, for some $u, v \in H_0^1(\Omega)$ such that $u, v > 0$ in Ω.

Proceeding as before, by using Sobolev embeddings and elliptic regularity, we conclude that (u, v) is a classical solution of the system (2.1).

Remark 2.3. We note that, from the fact that problem (1.6) is a one-dimensional particular case of problem (2.1), when $\lambda > 0$, we cannot ensure that, in general, the problem (2.1) has a positive solution in Ω.

Remark 2.4. We should point out that we may consider a more general system than (2.1). To be more precise, we may consider a system like

$$\begin{gathered} -\Delta_m u(x) + \lambda \int_\Omega v^p(y)\,dy = u^\alpha(x) + v^\beta(x), \quad x \in \Omega, \\ -\Delta_n v(x) + \lambda \int_\Omega u^q(y)\,dy = u^\gamma(x) + v^\delta(x), \quad x \in \Omega, \\ u(x), v(x) > 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.21}$$

where $-\Delta_m u = -\operatorname{div}(|\nabla u|^{m-2}\nabla u)$ and $-\Delta_n u = -\operatorname{div}(|\nabla u|^{n-2}\nabla u)$ are, respectively, the *m-Laplacian* and *n-Laplacian*. Although the proof of the existence of solution follows similar ideas as those used in Theorem 2.1, the calculations are more complicated because we have to work with Schauder's basis in $W_0^{1,m}(\Omega)$ and $W_0^{1,n}(\Omega)$ and these spaces do not enjoy a Hilbert space structure.

If we would like to dare a little more, we may consider a system like

$$\begin{gathered}
-\Delta_{m(x)} u(x)+\lambda \int_{\Omega} v^{p(x)}(y) d y=u^{\alpha(x)}+\mathrm{v}^{\beta(x)}, \quad x \in \Omega, \\
-\Delta_{n(x)} v(x)+\lambda \int_{\Omega} u^{q(x)}(y) d y=u^{\gamma(x)}+v^{\delta(x)}, \quad x \in \Omega, \\
u(x),\ v(x)>0, \quad x \in \Omega, \\
u(x)=v(x)=0, \quad x \in \partial \Omega,
\end{gathered} \tag{2.22}$$

where

$$\begin{gathered}
-\Delta_{m(x)} u=-\operatorname{div}\left(|\nabla u|^{m(x)-2} \nabla u\right), \\
-\Delta_{n(x)} u=-\operatorname{div}\left(|\nabla u|^{n(x)-2} \nabla u\right)
\end{gathered} \tag{2.23}$$

are, respectively, the $m(x)$-*Laplacian* and $n(x)$-*Laplacian* and $p(x)$, $\alpha(x)$, $\beta(x)$, $n(x)$, $q(x)$, $\gamma(x)$, $\delta(x)$ are continuous functions on $\overline{\Omega}$ satisfying suitable conditions. In this case, we have to work in the generalized Lebesgue-Sobolev spaces $W_0^{1,m(x)}(\Omega)$ and $W_0^{1,n(x)}(\Omega)$. See, for example, [21, 22] and the references therein, for more detailed information on this subject.

3. A Singular Problem

The Galerkin method may also be used to attack a singular version of the problem (2.1). More precisely, let us consider a simple version of a singular problem as

$$\begin{gathered}
-\Delta u(x)+\lambda \int_{\Omega} v^{p}(y) d y=\frac{1}{u^{\alpha}(x)}, \quad x \in \Omega, \\
-\Delta v(x)+\lambda \int_{\Omega} u^{q}(y) d y=v^{\delta}(x), \quad x \in \Omega, \\
u(x), v(x)>0, \quad x \in \Omega, \\
u(x)=v(x)=0, \quad x \in \partial \Omega
\end{gathered} \tag{3.1}$$

with $\alpha, \delta>0$.

We should point out that other combinations of u and v may be considered, including convection terms like $|\nabla u|^{\gamma},\ \gamma > 0$. More precisely, we may consider problems like

$$\begin{gathered}
-\Delta u(x) + \lambda \int_{\Omega} v^{p}(y)dy = \frac{1}{u^{\alpha}(x)} + |\nabla u(x)|^{\gamma}, \quad x \in \Omega, \\
-\Delta v(x) + \lambda \int_{\Omega} u^{q}(y)dy = v^{\delta}(x), \quad x \in \Omega, \\
u(x), v(x) > 0, \quad x \in \Omega, \\
u(x) = v(x) = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{3.2}$$

or

$$\begin{gathered}
-\Delta u(x) + \lambda \int_{\Omega} v^{p}(y)dy = \frac{1}{u^{\alpha}(x)} + |\nabla u(x)|^{\gamma}, \quad x \in \Omega, \\
-\Delta v(x) + \lambda \int_{\Omega} u^{q}(y)dy = \frac{1}{v^{\delta}(x)}, \quad x \in \Omega, \\
u(x), v(x) > 0, \quad x \in \Omega, \\
u(x) = v(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.3}$$

However, for the sake of simplicity and to illustrate the method, we restrict our discussion to the problem (3.1).

To approach problem (3.1), we consider a nonsingular perturbation as

$$\begin{gathered}
-\Delta u(x) + \lambda \int_{\Omega} (v^{+})^{p}(y)dy = \frac{1}{((u^{+}(x)) + \epsilon)^{\alpha}}, \quad x \in \Omega, \\
-\Delta v(x) + \lambda \int_{\Omega} (u^{+})^{q}(y)dy = (v^{+}(x))^{\delta}, \quad x \in \Omega, \\
u(x) = v(x) = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{3.4}$$

with $\epsilon > 0$, to obtain approximate solutions $(u_{\epsilon}, v_{\epsilon})$. In this way, we avoid the singular term.

We have the following result.

Theorem 3.1. *Let $0 < \alpha,\ \delta < 1$ be real numbers and $\lambda < 0$ a real parameter. Then, problem* (3.1) *possesses a positive solution $(u, v) \in (C^{2}(\Omega) \cap C(\overline{\Omega})) \times (C^{2}(\Omega) \cap C(\overline{\Omega}))$.*

Proof. Reasoning as in the proof of Theorem 2.1 we obtain, for each $\epsilon > 0$, a solution $(u_{\epsilon}, v_{\epsilon})$ of problem (3.4). So, since $\lambda < 0$, we obtain

$$\begin{gathered}
-\Delta u_{\epsilon}(x) \geq \frac{1}{((u_{\epsilon}^{+}(x)) + \epsilon)^{\alpha}} > 0, \quad x \in \Omega, \\
-\Delta v_{\epsilon}(x) \geq (v_{\epsilon}^{+}(x))^{\delta} \geq 0, \quad x \in \Omega, \\
u_{\epsilon}(x) = v_{\epsilon}(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.5}$$

In view of the maximum principle, we obtain that $u_\epsilon > 0$ in Ω and $v_\epsilon \geq 0$ in Ω. Consequently,

$$\begin{gathered}
-\Delta u_\epsilon(x) + \lambda \int_\Omega v_\epsilon^p(y)dy = \frac{1}{(u_\epsilon(x)+\epsilon)^\alpha}, \quad x \in \Omega, \\
-\Delta v_\epsilon(x) + \lambda \int_\Omega u_\epsilon^q(y)dy = v_\epsilon^\delta(x), \quad x \in \Omega, \\
u_\epsilon(x) = v_\epsilon(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.6}$$

If $v_\epsilon \equiv 0$, we obtain, in view of (3.6), that $\int_\Omega u_\epsilon^q(y)dy = 0$, which contradicts the fact that $u_\epsilon > 0$ in Ω. Thus, $v_\epsilon \not\equiv 0$ and, because $v_\epsilon \geq 0$, the maximum principle gives $v_\epsilon > 0$ in Ω.

The solution of problem (3.1) will be obtained by studying the limit when $\epsilon \to 0$. Thus, we may suppose that $0 < \epsilon < 1$. In view of this, we obtain from (3.6) that

$$\begin{gathered}
-\Delta u_\epsilon(x) > \frac{1}{(u_\epsilon(x)+1)^\alpha}, \quad x \in \Omega, \\
u_\epsilon(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.7}$$

Let ω_ϵ be the unique positive solution of

$$\begin{gathered}
-\Delta \omega_\epsilon(x) = \frac{1}{(u_\epsilon(x)+1)^\alpha}, \quad x \in \Omega, \\
\omega_\epsilon(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.8}$$

From the maximum principle, we deduce that $u_\epsilon > \omega_\epsilon > 0$ in Ω. Since $1/(u_\epsilon + 1)^\alpha$ is bounded, by using interior elliptic regularity, we obtain that there is $\omega \in C^2(\Omega)$ such that $\omega_\epsilon \to \omega$ in $C^2(\Omega')$, for all $\Omega' \subset\subset \Omega$.

Using the Galerkin method and reasoning as in the proof of Theorem 2.1, we deduce that the approximate solutions u_ϵ, v_ϵ are uniformly bounded in $H_0^1(\Omega)$ with respect to $0 < \epsilon < 1$. In consequence, we conclude that $u_\epsilon \rightharpoonup u$ and $v_\epsilon \rightharpoonup v$ in $H_0^1(\Omega)$ in the weak sense. Hence, in view of (3.8), we obtain

$$\begin{gathered}
-\Delta \omega(x) = \frac{1}{(u(x)+1)^\alpha}, \quad x \in \Omega, \\
\omega(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{3.9}$$

Invoking again the maximum principle, we get $\omega > 0$ in Ω. Since $u_\epsilon > \omega_\epsilon$ in Ω, we conclude that $u \geq \omega > 0$ in Ω. Reasoning as before, $v > 0$ in Ω, and (u, v) is a classical solution of problem (3.1), which finishes the proof of this result. □

4. On a Superlinear Problem

At last we will make some remarks on a superlinear problem. In order to simplify the exposition, let us consider the one equation case

$$\begin{gathered}
-\Delta u(x) + \lambda \int_\Omega u^p(y)dy = u^\alpha(x) + f(x), \quad x \in \Omega, \\
u(x) > 0, \quad x \in \Omega, \\
u(x) = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{4.1}$$

where $1 < p,\ \alpha < (N+2)/(N-2),\ N \geq 3$ and $f \in L^2(\Omega), f \geq 0, f \not\equiv 0$. Such a function f is introduced in order to ensure that the solution is nonnegative and nontrivial. Note that $u \equiv 0$ is a solution of

$$\begin{gathered}
-\Delta u(x) + \lambda \int_\Omega u^p(y)dy = u^\alpha(x), \quad x \in \Omega, \\
u(x) = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{4.2}$$

and we do not have at our disposal the results in [19, 20].

Theorem 4.1. *Suppose that* $1 < p,\ \alpha < (N+2)/(N-2),\ N \geq 3$ *and* $\lambda < 0$. *Then, for each* $f \in L^2(\Omega)$, *with* $\|f\|_2$ *sufficiently small, problem* (4.1) *possesses a nonnegative and nontrivial solution.*

Proof. First of all, let us consider the auxiliary problem

$$\begin{gathered}
-\Delta u(x) + \lambda \int_\Omega (u^+)^p(y)dy = (u^+(x))^\alpha + f(x), \quad x \in \Omega, \\
u(x) = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{4.3}$$

As before, let us consider $F : \mathbb{R}^m \to \mathbb{R}^m,\ F = (F_1, \ldots, F_m)$, defined, for all $i = 1, \ldots, m$, as

$$F_i(\xi) = \int_\Omega \nabla u \nabla \varphi_i + \lambda \int_\Omega (u^+)^p \int_\Omega \varphi_i - \int_\Omega (u^+)^\alpha \varphi_i - \int_\Omega f\ \varphi_i, \tag{4.4}$$

where we are using the previous identifications.

For all $i = 1, \ldots, m$, the following equations hold

$$F_i(\xi) \cdot \xi_i = \int_\Omega \nabla u \cdot \nabla(\xi_i \varphi_i) + \lambda \int_\Omega (u^+)^p \int_\Omega \xi_i \varphi_i - \int_\Omega (u^+)^\alpha (\xi_i \varphi_i) - \int_\Omega f\ \xi_i \varphi_i, \tag{4.5}$$

and so

$$\langle F(\xi), \xi \rangle = \int_\Omega |\nabla u|^2 + \lambda \int_\Omega (u^+)^p \int_\Omega u - \int_\Omega (u^+)^\alpha u - \int_\Omega f\ u. \tag{4.6}$$

Using the classical immersions, we get

$$\langle F(\xi),\xi\rangle \geq \|u\|^2 + \lambda C\|u\|^{p+1} - C\|u\|^{\alpha} - C\|f\|_2\|u\| \tag{4.7}$$

for all $u \in \mathbb{V}_m$.

We now consider the function

$$g(t) = t^2 + \lambda C t^{p+1} - C t^{\alpha+1}, \qquad t \geq 0, \tag{4.8}$$

and note that

$$g(t) = t^2 \left(1 + \lambda C t^{p-1} - C t^{\alpha-1}\right). \tag{4.9}$$

So, we may find a $t_0 > 0$, sufficiently small, such that

$$g(t_0) = t_0^2 \left(1 + \lambda C t_0^{p-1} - C t_0^{\alpha-1}\right) > 0. \tag{4.10}$$

As consequence, we can choose $\|f\|_2$, small enough, such that

$$g(t_0) = t_0^2 \left(1 + \lambda C t_0^{p-1} - C t_0^{\alpha-1}\right) > C\|f\|_2 t_0, \tag{4.11}$$

and, taking $u \in \mathbb{V}_m$ such that $\|u\| = t_0$, we get

$$\langle F(\xi),\xi\rangle \ > \ 0, \qquad |\xi| = \|u\| = t_0, \qquad u \in \mathbb{V}_m, \tag{4.12}$$

and note that such a t_0 does not depend on m.

Proceeding as in the proof of Theorem 2.1, we obtain, for each $m \in \mathbb{N}$, an approximate solution $u_m \in \mathbb{V}_m$ satisfying $\|u_{\mathrm{m}}\| \leq t_0$ and, for all $i = 1, \ldots, m$,

$$\int_\Omega \nabla u_m \nabla \varphi_i + \lambda \int_\Omega (u_m^+)^p \int_\Omega \varphi_i - \int_\Omega (u_m^+)^\alpha \varphi_i - \int_\Omega f\ \varphi_i = 0. \tag{4.13}$$

Thus, for all $\varphi \in \mathbb{V}_m$, we have

$$\int_\Omega \nabla u_m \nabla \varphi + \lambda \int_\Omega (u_m^+)^p \int_\Omega \varphi - \int_\Omega (u_m^+)^\alpha \varphi - \int_\Omega f\ \varphi = 0. \tag{4.14}$$

Since we are working in the superlinear and subcritical case and using the previous arguments, we obtain that there is a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla \varphi + \lambda \int_\Omega (u^+)^p \int_\Omega \varphi - \int_\Omega (u^+)^\alpha \varphi - \int_\Omega f\ \varphi = 0 \tag{4.15}$$

for all $\varphi \in H_0^1(\Omega)$.

We now use the weak maximum principle to conclude that $u \geq 0$ in Ω. From the fact that $f \not\equiv 0$, the solution is not the trivial one. Then, u is a solution of our original problem (4.1). □

Remark 4.2. Note that, when we are working with a problem with variational structure the function f may be discarded because we may use the Mountain Pass Theorem. In this case the critical level given by this theorem is positive and so we may conclude that the solution obtained in this way is not null. However, in our case, we can not dispose of the variational techniques. In view of this and to show that we obtain a nontrivial solution, we had to introduce the function f.

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