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Research Article

On a Class of Abstract Time-Fractional Equations on Locally Convex Spaces

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This paper is devoted to the study of abstract time-fractional equations of the following form: $\mathbf{D}_t^{\alpha_n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha_i}u(t)=A\mathbf{D}_t^{\alpha}u(t)+f(t)$, $t>0$, $u^{(k)}(0)=u_k$, $k=0,\ldots,\lceil\alpha_n\rceil-1$, where $n\in\mathbb{N}\setminus\{1\}$, A and $A_1,\ldots,A_{n-1}$ are closed linear operators on a sequentially complete locally convex space E, $0\leq\alpha_1<\cdots<\alpha_n$, $0\leq\alpha<\alpha_n$, $f(t)$ is an E-valued function, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α (Bazhlekova (2001)). We introduce and systematically analyze various classes of k-regularized (C_1,C_2)-existence and uniqueness (propagation) families, continuing in such a way the researches raised in (de Laubenfels (1999, 1991), Kostić (Preprint), and Xiao and Liang (2003, 2002). The obtained results are illustrated with several examples.

1. Introduction and Preliminaries

A great number of abstract time-fractional equations appearing in engineering, mathematical physics, and chemistry can be modeled through the abstract Cauchy problem

$$\begin{gathered}\mathbf{D}_t^{\alpha n}u(t)+\sum_{i=1}^{n-1}A_i\mathbf{D}_t^{\alpha i}u(t)=A\mathbf{D}_t^{\alpha}u(t)+f(t),\quad t>0,\\ u^{(k)}(0)=u_k,\quad k=0,\ldots,\lceil\alpha_n\rceil-1.\end{gathered}\tag{1.1}$$

For further information about the applications of fractional calculus, the interested reader may consult the monographs by Baleanu et al. [1], Klafter et al. (Eds.) [2], Kilbas et al. [3], Mainardi [4], Podlubny [5], and Samko et al. [6]; we also refer to the references [7–19].

The aim of this paper is to develop some operator theoretical methods for solving the abstract time-fractional equations of the form (1.1). We start by quoting some special cases. The study of qualitative properties of the abstract Basset-Boussinesq-Oseen equation:

$$u'(t) - A\mathbf{D}_t^{\alpha}u(t) + u(t) = f(t), \quad t \geq 0, \ u(0) = 0 \ \ (\alpha \in (0,1)), \tag{1.2}$$

describing the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, has been initiated by Lizama and Prado in [17]. For further results concerning the C-wellposedness of (1.2), [20, 21] are of importance. In [12], Karczewska and Lizama have recently analyzed the following stochastic fractional oscillation equation:

$$u(t) + \int_0^t (t-s)[A\mathbf{D}_s^{\alpha}u(s) + u(s)]ds = W(t), \quad t > 0, \tag{1.3}$$

where $1 < \alpha < 2$, A is the generator of a bounded analytic C_0-semigroup on a Hilbert space H and $W(t)$ denotes an H-valued Wiener process defined on a stochastic basis $(\Omega, \mathcal{F}, P)$. The theory of (a,k)-regularized resolvent families (cf. [12, Theorems 3.1 and 3.2]) can be applied in the study of deterministic counterpart of (1.3) in integrated form:

$$u(t) + \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}Au(s)ds + \int_0^t (t-s)u(s)ds = \int_0^t (t-s)f(s)ds, \quad t > 0, \tag{1.4}$$

where $\Gamma(\cdot)$ denotes the Gamma function and $f \in L^1_{\text{loc}}([0,\infty) : E)$. Equation (1.4) generalizes the so-called Bagley-Torvik equation, which can be obtained by plugging $\alpha = 3/2$ in (1.4), and models an oscillation process with fractional damping term (cf. [21] for the analysis of C-wellposedness and perturbation properties of (1.4)). After differentiation, (1.4) becomes, in some sense,

$$u''(t) + A\mathbf{D}_t^{\alpha}u(t) + u(t) = f(t), \quad t \geq 0; \ u(0) = u'(0) = 0. \tag{1.5}$$

Notice also that the periodic solutions for the equation

$$D^{\alpha}u(t) + BD^{\beta}u(t) + Au(t) = f(t), \quad t \in [0, 2\pi], \tag{1.6}$$

where A and B are closed linear operators defined on a complex Banach space X, $0 \leq \beta < \alpha \leq 2$, $f \in C([0,2\pi] : X)$ and D^{α} denotes the Liouville-Grünwald fractional derivative of order α, have been studied by Keyantuo and Lizama in [13]. Observe also that Diethelm analyzed in [22, Chapter 8] scalar-valued multiterm Caputo fractional differential equations. Consider, for illustration purposes, the following abstract time-fractional equation:

$$\mathbf{D}_t^{\alpha}u(t) + \mathbf{D}_t^{\beta}u(t) = au(t), \quad t > 0; \ u(0) = u_0, \ u'(0) = 0, \tag{1.7}$$

where $1 < \alpha < 2$, $0 < \beta < \alpha$ and $A = a$ is a certain complex constant. Applying the Laplace transform (see, e.g., [10, (1.23)]), we get:

$$\left(\lambda^{\alpha} + \lambda^{\beta}\right)\tilde{u}(\lambda) - \left(\lambda^{\alpha-1} + \lambda^{\beta-1}\right)u_0 = a\tilde{u}(\lambda). \tag{1.8}$$

Therefore,

$$\tilde{u}(\lambda) = \frac{\lambda^{\alpha-1} + \lambda^{\beta-1}}{\lambda^{\alpha} + \lambda^{\beta} - a}u_0. \tag{1.9}$$

By (24) and (26) in [19], it readily follows that:

$$u(t) = \sum_{n=0}^{\infty}(-1)^n t^{(\alpha-\beta)n}\left[E^{n+1}_{\alpha,(\alpha-\beta)n+1}(at^{\alpha}) + t^{\alpha-\beta}E^{n+1}_{\alpha,(\alpha-\beta)(n+1)+1}(at^{\alpha})\right]u_0, \tag{1.10}$$

where

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(n\alpha+\beta)n!} \tag{1.11}$$

is the generalized Mittag-Leffler function. Here $(\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+n-1)$ ($n \in \mathbb{N}$) and $(\gamma)_0 = 1$. The formula (1.10) shows that it is quite complicated to apply Fourier multiplier theorems to the abstract time-fractional equations of the form (1.1); for some basic references in this direction, the reader may consult [16, 23]. Before going any further, we would also like to observe that Atanacković et al. considered in [8], among many other authors, the following fractional generalization of the telegraph equation:

$$\tau\mathbf{D}_t^{\alpha}u(t) + \mathbf{D}_t^{\beta}u(t) = Du_{xx}, \quad x \in (0,l),\ t > 0, \tag{1.12}$$

where $0 < \beta \leq \alpha \leq 2$, $\tau > 0$ and $D > 0$. In that paper, solutions to signalling and Cauchy problems in terms of a series and integral representation are given.

In the second section, we continue the analysis from our recent paper [15], where it has been assumed that $A_j = c_j I$ for some complex constants $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$; here, and in the sequel of the second section, I denotes the identity operator on E. We introduce and clarify the basic structural properties of various types of k-regularized (C_1, C_2)-existence and uniqueness propagation families. This is probably the best concept for the investigation of integral solutions of the abstract time-fractional equation (1.1) with $A_j \in L(E)$, $1 \leq j \leq n-1$. If there exists an index $j \in \mathbb{N}_{n-1}$ such that $A_j \notin L(E)$, then the vector-valued Laplace transform cannot be so easily applied (cf. Theorems 2.10–2.11), which implies, however, that there exist some limitations to the introduced classes of propagation families. The notion of a strong solution of (1.1) is introduced in Definition 2.1, and the notions of strong and mild solutions of inhomogeneous equations of the form (2.15) below are introduced in Definition 2.7. The generalized variation of parameters formula is proved in Theorem 2.8.

On the other hand, the notions of C_1-existence families and C_2-uniqueness families for the higher order abstract Cauchy problem (ACP_n) were introduced by Xiao and Liang in

[24, Definition 2.1]. In the third section, we will introduce more general classes of (local) k-regularized C_1-existence families for (1.1), k-regularized C_2-uniqueness families for (1.1), and k-regularized C-resolvent families for (1.1). Our intention in this section is to transfer results of [24] to abstract time-fractional equations. In addition, various adjoint type theorems for k-regularized C-resolvent families are considered in Theorem 3.6.

Throughout this paper, we will always assume that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short, and that the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E; in this place, we would like to mention in passing that the locally convex spaces are very important to describe a set of mixed states in quantum theory [2]. The completeness of E, if needed, will be explicitly emphasized. By $L(E)$ is denoted the space of all continuous linear mappings from E into E. Let $\mathcal{B}$ be the family of bounded subsets of E and let $p_B(T) := \sup_{x\in B} p(Tx)$, $p \in \circledast$, $B \in \mathcal{B}$, $T \in L(E)$. Then $p_B(\cdot)$ is a seminorm on $L(E)$ and the system $(p_B)_{(p,B)\in\circledast\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E)$. Recall that $L(E)$ is sequentially complete provided that E is barreled. Henceforth A is a closed linear operator acting on E, $L(E) \ni C$ is an injective operator, and the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)ds$. The domain, resolvent set and range of A are denoted by $D(A)$, $\rho(A)$ and $R(A)$, respectively. Since it makes no misunderstanding, we will identify A with its graph. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \left\{\lambda \in \mathbb{C};\ \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\right\}. \tag{1.13}$$

Suppose F is a linear subspace of E. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax,\ x \in D(A_{|F})$.

Define $E_p := E/p^{-1}(0)$ $(p \in \circledast)$. Then the norm of a class $x + p^{-1}(0)$ is defined by $\|x + p^{-1}(0)\|_{E_p} := p(x)$ $(x \in E)$. The canonical mapping $\Psi_p : E \to E_p$ is continuous and the completion of E_p under the norm $\|\cdot\|_{E_p}$ is denoted by $\overline{E_p}$. Since no confusion seems likely, we will also denote the norms on E_p and $L(E_p)$ ($\overline{E_p}$ and $L(\overline{E_p})$) by $\|\cdot\|$; $L_\circledast(E)$ denotes the subspace of $L(E)$ which consists of those bounded linear operators T on E such that, for every $p \in \circledast$, there exists $c_p > 0$ satisfying $p(Tx) \leq c_p p(x)$, $x \in E$. If $T \in L_\circledast(E)$ and $p \in \circledast$, then the operator $T_p : E_p \to E_p$, defined by $T_p(\Psi_p(x)) := \Psi_p(Tx)$, $x \in E$, belongs to $L(E_p)$. This operator is uniquely extensible to a bounded linear operator $\overline{T_p}$ on $\overline{E_p}$, and the following holds: $\|T_p\| = \|\overline{T_p}\|$. The function $\pi_{qp} : E_p \to E_q$, defined by $\pi_{qp}(\Psi_p(x)) := \Psi_q(x)$, $x \in E$, is a continuous homomorphism of E_p onto E_q, and extends therefore, to a continuous linear homomorphism π_{qp} of $\overline{E_p}$ onto $\overline{E_q}$. The reader may consult [25] for the basic facts about projective limits of Banach spaces (closed linear operators acting on Banach spaces) and their projective limits. Recall, a closed linear operator A acting on E is said to be compartmentalized (w.r.t. $\circledast$) if, for every $p \in \circledast$, $A_p := \{(\Psi_p(x), \Psi_p(Ax)) : x \in D(A)\}$ is a function. Therefore, $T \in L_\circledast(E)$ is a compartmentalized operator.

Given $s \in \mathbb{R}$ in advance, set $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The principal branch is always used to take the powers. Set $\mathbb{N}_l := \{1, \ldots, l\}$, $\mathbb{N}_l^0 := \{0, 1, \ldots, l\}$, $0^\zeta := 0$, $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ $(\zeta > 0, t > 0)$ and $g_0 :=$ the Dirac δ-distribution. If $\gamma \in (0, \pi]$, then we define $\Sigma_\gamma := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \gamma\}$. We refer the reader to [26] and references cited there for the basic material concerning integration in sequentially complete locally convex spaces and vector-valued analytic functions.

Let $\alpha > 0$, let $\beta \in \mathbb{R}$, and let the Mittag-Leffler function $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} z^n/\Gamma(\alpha n+\beta)$, $z \in \mathbb{C}$. In this place, we assume that $1/\Gamma(\alpha n+\beta) = 0$ if $\alpha n+\beta \in -\mathbb{N}_0$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. The Wright function $\Phi_\gamma(t)$ is defined by $\Phi_\gamma(t) := \mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t \geq 0$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. For further information concerning Mittag-Leffler and Wright functions, we refer the reader to [10, Section 1.3].

The following definition has been recently introduced in [27].

Definition 1.1. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$ and A is a closed linear operator on E.

(i) Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) (a,k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\in[0,\tau)} \subseteq L(E) \times L(E)$ if and only if the mapping $t \mapsto (R_1(t)x, R_2(t)x)$, $t \in [0,\tau)$ is continuous for every fixed $x \in E$ and if the following conditions hold:

(a) $R_i(0) = k(0)C_i$, $i = 1, 2$,

(b) C_2 is injective,

(c)

$$A\int_0^t a(t-s)R_1(s)x\,ds = R_1(t)x - k(t)C_1x, \quad t \in [0,\tau),\ x \in E, \tag{1.14}$$

$$\int_0^t a(t-s)R_2(s)Ax\,ds = R_2(t)x - k(t)C_2x, \quad t \in [0,\tau),\ x \in D(A). \tag{1.15}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(E)$ be strongly continuous. Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ if and only if $R_1(0) = k(0)C_1$ and (1.14) holds.

(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(E)$ be strongly continuous. Then it is said that A is a subgenerator of a (local, if $\tau < \infty$) (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ if and only if $R_2(0) = k(0)C_2$, C_2 is injective and (1.15) holds.

It will be convenient to remind us of the following definitions from [14, 20, 26].

Definition 1.2. (i) Let $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$ and let $a \in L^1_{\mathrm{loc}}([0,\tau))$, $a \neq 0$. A strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) (a,k)-regularized C-resolvent family having A as a subgenerator if and only if the following holds:

(a) $R(t)A \subseteq AR(t)$, $t \in [0,\tau)$, $R(0) = k(0)C$ and $CA \subseteq AC$,

(b) $R(t)C = CR(t)$, $t \in [0,\tau)$,

(c) $R(t)x = k(t)Cx + \int_0^t a(t-s)AR(s)x\,ds$, $t \in [0,\tau)$, $x \in D(A)$,

$(R(t))_{t\in[0,\tau)}$ is said to be nondegenerate if the condition $R(t)x = 0$, $t \in [0,\tau)$ implies $x = 0$, and $(R(t))_{t\in[0,\tau)}$ is said to be locally equicontinuous if, for every $t \in (0,\tau)$, the family $\{R(s) : s \in [0,t]\}$ is equicontinuous. In the case $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially equicontinuous (equicontinuous) if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous.

(ii) Let $\beta \in (0,\pi]$ and let $(R(t))_{t\geq 0}$ be an (a,k)-regularized C-resolvent family. Then it is said that $(R(t))_{t\geq 0}$ is an analytic (a,k)-regularized C-resolvent family of angle β, if there exists a function $\mathbf{R} : \Sigma_\beta \to L(E)$ satisfying that, for every $x \in E$, the mapping $z \mapsto \mathbf{R}(z)x$, $z \in \Sigma_\beta$ is analytic as well as that

(a) $\mathbf{R}(t) = R(t)$, $t > 0$ and

(b) $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{R}(z)x = k(0)Cx$ for all $\gamma \in (0,\beta)$ and $x \in E$,

$(R(t))_{t\geq 0}$ is said to be an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family, respectively, equicontinuous analytic (a,k)-regularized C-resolvent family of angle β, if for every $\gamma \in (0,\beta)$, there exists $\omega_\gamma \geq 0$, respectively, $\omega_\gamma = 0$, such that the set $\{e^{-\omega_\gamma |z|}\mathbf{R}(z) : z \in \Sigma_\gamma\}$ is equicontinuous. Since there is no risk for confusion, we will identify in the sequel $R(\cdot)$ and $\mathbf{R}(\cdot)$.

Definition 1.3. (i) Let $k \in C([0,\infty))$ and $a \in L^1_{\text{loc}}([0,\infty))$. Suppose that $(R(t))_{t\geq 0}$ is a global (a,k)-regularized C-resolvent family having A as a subgenerator. Then it is said that $(R(t))_{t\geq 0}$ is a quasi-exponentially equicontinuous (q-exponentially equicontinuous, for short) (a,k)-regularized C-resolvent family having A as subgenerator if and only if, for every $p \in \circledast$, there exist $M_p \geq 1$, $\omega_p \geq 0$ and $q_p \in \circledast$ such that:

$$p(R(t)x) \leq M_p e^{\omega_p t} q_p(x), \quad t \geq 0,\ x \in E. \tag{1.16}$$

(ii) Let $\beta \in (0,\pi]$, and let A be a subgenerator of an analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ of angle β. Then it is said that $(R(t))_{t\geq 0}$ is a q-exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family of angle β, if for every $p \in \circledast$ and $\varepsilon \in (0,\beta)$, there exist $M_{p,\varepsilon} \geq 1$, $\omega_{p,\varepsilon} \geq 0$ and $q_{p,\varepsilon} \in \circledast$ such that

$$p(R(z)x) \leq M_{p,\varepsilon} e^{\omega_{p,\varepsilon}|z|} q_{p,\varepsilon}(x), \quad z \in \Sigma_{\beta-\varepsilon},\ x \in E. \tag{1.17}$$

For a global (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t\geq 0}$ having A as subgenerator, it is said that is locally equicontinuous (exponentially equicontinuous, (q-)exponentially equicontinuous, analytic, (q-)exponentially analytic,...) if and only if both $(R_1(t))_{t\geq 0}$ and $(R_2(t))_{t\geq 0}$ are.

The reader may consult [26, Theorems 2.7 and 2.8] for the basic Hille-Yosida type theorems for exponentially equicontinuous (a,k)-regularized C-resolvent families. The characterizations of exponentially equicontinuous, analytic (a,k)-regularized C-resolvent families in terms of spectral properties of their subgenerators are given in [26, Theorems 3.6 and 3.7]. For further information concerning q-exponentially equicontinuous (a,k)-regularized C-resolvent families, we refer the reader to [20, 25].

Henceforth, we assume that $k, k_1, k_2, \ldots$ are scalar-valued kernels and that $a \neq 0$ in $L^1_{\text{loc}}([0,\tau))$. All considered operator families will be nondegenerate.

The following conditions will be used in the sequel:

(H1) A is densely defined and $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous.

(H2) $\rho(A) \neq \emptyset$.

(H3) $\rho_C(A) \neq \emptyset$, $\overline{R(C)} = E$ and $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous.

(H3)′ $\rho_C(A) \neq \emptyset$ and $C^{-1}AC = A$.

(H4) A is densely defined and $(R(t))_{t\in[0,\tau)}$ is locally equicontinuous, or $\rho_C(A) \neq \emptyset$.

(H5) (H1) $\vee$ (H2) $\vee$ (H3) $\vee$ (H3)$'$.

(P1) $k(t)$ is Laplace transformable, that is, it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{k}(\lambda) = \mathcal{L}(k)(\lambda) := \lim_{b\to\infty}\int_0^b e^{-\lambda t}k(t)dt := \int_0^\infty e^{-\lambda t}k(t)dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $\operatorname{abs}(k) := \inf\{\Re\lambda : \tilde{k}(\lambda) \text{ exists}\}$.

2. The Main Structural Properties of k-Regularized (C_1, C_2)-Existence and Uniqueness Propagation Families

In this section, we will always assume that E is a SCLCS, A and $A_1,\ldots,A_{n-1}$ are closed linear operators acting on E, $n \in \mathbb{N}\setminus\{1\}$, $0 \le \alpha_1 < \cdots < \alpha_n$ and $0 \le \alpha < \alpha_n$. Our intention is to clarify the most important results concerning the C-wellposedness of (1.1). Set $m_j := \lceil \alpha_j \rceil$, $1 \le j \le n$, $m := m_0 := \lceil \alpha \rceil$, $A_0 := A$ and $\alpha_0 := \alpha$.

Definition 2.1. A function $u \in C^{m_n-1}([0,\infty) : E)$ is called a (strong) solution of (1.1) if and only if $A_i \mathbf{D}_t^{\alpha_i} u \in C([0,\infty) : E)$ for $0 \le i \le n-1$, $g_{m_n-\alpha_n} * (u - \sum_{k=0}^{m_n-1} u_k g_{k+1}) \in C^{m_n}([0,\infty) : E)$ and (1.1) holds. The abstract Cauchy problem (1.1) is said to be (strongly) C-wellposed if:

(i) for every $u_0,\ldots,u_{m_n-1} \in \bigcap_{0\le j\le n-1} C(D(A_j))$, there exists a unique solution $u(t; u_0,\ldots,u_{m_n-1})$ of (1.1);

(ii) for every $T > 0$ and $q \in \circledast$, there exist $c > 0$ and $r \in \circledast$ such that, for every $u_0,\ldots,u_{m_n-1} \in \bigcap_{0\le j\le n-1} C(D(A_j))$, the following holds:

$$q(u(t; u_0,\ldots,u_{m_n-1})) \le c\sum_{k=0}^{m_n-1} r\left(C^{-1}u_k\right), \quad t \in [0,T]. \tag{2.1}$$

In the case of abstract Cauchy problem (ACP$_n$), the definition of C-wellposedness introduced above is slightly different from the corresponding definition introduced by Xiao and Liang [28, Definition 5.2, page 116] in the Banach space setting (cf. also [28, Definition 1.2, page 46] for the case $C = I$). Recall that the notion of a strong C-propagation family is important in the study of existence and uniqueness of strong solutions of the abstract Cauchy problem (ACP$_n$); compare [28, Section 3.5, pages 115–130] for further information in this direction. Suppose now that $u(t) \equiv u(t; u_0,\ldots,u_{m_n-1})$, $t \ge 0$ is a strong solution of (1.1), with $f(t) \equiv 0$ and initial values $u_0,\ldots,u_{m_n-1} \in R(C)$. Convoluting both sides of (1.1) with $g_{\alpha_n}(t)$, and making use of the equality [10, (1.21)], it readily follows that $u(t)$, $t \ge 0$ satisfies the following:

$$\begin{aligned} u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j\left[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot)\right] \\ = g_{\alpha_n-\alpha} * A\left[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot)\right]. \end{aligned} \tag{2.2}$$

In the sequel of this section, we will primarily consider various types of solutions of the integral equation (2.2).

Given $i \in \mathbb{N}^0_{m_n-1}$ in advance, set $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$. Then it is clear that $D_{m_n-1} \subseteq \cdots \subseteq D_0$. Plugging $u_j = 0$, $0 \leq j \leq m_n - 1$, $j \neq i$, in (2.2), one gets:

$$\begin{aligned}
&\left[u(\cdot;\ 0,\ldots,u_i,\ldots,0) - u_i g_{i+1}(\cdot)\right] \\
&\quad + \sum_{j \in D_i} g_{\alpha_n-\alpha_j} * A_j \left[u(\cdot;\ 0,\ldots,u_i,\ldots,0) - u_i g_{i+1}(\cdot)\right] \\
&\quad + \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left[g_{\alpha_n-\alpha_j} * A_j u(\cdot;\ 0,\ldots,u_i,\ldots,0)\right] \\
&= \begin{cases} g_{\alpha_n-\alpha} * Au(\cdot;\ 0,\ldots,u_i,\ldots,0), & m-1 < i, \\ g_{\alpha_n-\alpha} * A\left[u(\cdot;\ 0,\ldots,u_i,\ldots,0) - u_i g_{i+1}(\cdot)\right], & m-1 \geq i, \end{cases}
\end{aligned} \tag{2.3}$$

where u_i appears in the ith place $(0 \leq i \leq m_n - 1)$ starting from 0. Suppose now $0 < \tau \leq \infty$, $0 \neq K \in L^1_{\text{loc}}([0,\tau))$ and $k(t) = \int_0^t K(s)ds$, $t \in [0,\tau)$. Denote $R_i(t)C^{-1}u_i = (K * u(\cdot;\ 0,\ldots, u_i,\ldots,0))(t)$, $t \in [0,\tau)$, $0 \leq i \leq m_n - 1$. Convoluting formally both sides of (2.3) with $K(t)$, $t \in [0,\tau)$, one obtains that, for $0 \leq i \leq m_n - 1$:

$$\begin{aligned}
&\left[R_i(\cdot)C^{-1}u_i - (k * g_i)(\cdot)u_i\right] + \sum_{j \in D_i} g_{\alpha_n-\alpha_j} * A_j\left[R_i(\cdot)C^{-1}u_i - (k * g_i)(\cdot)u_i\right] \\
&\quad + \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left[g_{\alpha_n-\alpha_j} * A_j R_i(\cdot)C^{-1}u_i\right] \\
&= \begin{cases} \left(g_{\alpha_n-\alpha} * AR_i\right)(\cdot)C^{-1}u_i, & m-1 < i, \\ g_{\alpha_n-\alpha} * A\left[R_i(\cdot)C^{-1}u_i - (k * g_i)(\cdot)u_i\right], & m-1 \geq i. \end{cases}
\end{aligned} \tag{2.4}$$

Motivated by the above analysis, we introduce the following definition.

Definition 2.2. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, C, C_1, $C_2 \in L(E)$, C and C_2 are injective. A sequence $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(E)$ is called a (local, if $\tau < \infty$):

(i) k-regularized C_1-existence propagation family for (1.1) if and only if $R_i(0) = (k * g_i)(0)C_1$ and the following holds:

$$\begin{aligned}
&\left[R_i(\cdot)x - (k * g_i)(\cdot)C_1x\right] + \sum_{j \in D_i} A_j\left[g_{\alpha_n-\alpha_j} * \left(R_i(\cdot)x - (k * g_i)(\cdot)C_1x\right)\right] \\
&\quad + \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} A_j\left(g_{\alpha_n-\alpha_j} * R_i\right)(\cdot)x \\
&= \begin{cases} A\left(g_{\alpha_n-\alpha} * R_i\right)(\cdot)x, & m-1 < i,\ x \in E, \\ A\left[g_{\alpha_n-\alpha} * \left(R_i(\cdot)x - (k * g_i)(\cdot)C_1x\right)\right](\cdot), & m-1 \geq i,\ x \in E, \end{cases}
\end{aligned} \tag{2.5}$$

for any $i = 0,\ldots,m_n - 1$.

(ii) k-regularized C_2-uniqueness propagation family for (1.1) if and only if $R_i(0) = (k * g_i)(0)C_2$ and

$$\begin{aligned}
&\left[R_i(\cdot)x - (k * g_i)(\cdot)C_2x\right] + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \left[R_i(\cdot)A_jx - (k * g_i)(\cdot)C_2A_jx\right] \\
&\quad + \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} \left(g_{\alpha_n-\alpha_j} * R_i(\cdot)A_jx\right)(\cdot) \\
&= \begin{cases} \left(g_{\alpha_n-\alpha} * R_i(\cdot)Ax\right)(\cdot), & m-1 < i, \\ g_{\alpha_n-\alpha} * \left[R_i(\cdot)Ax - (k * g_i)(\cdot)C_2Ax\right](\cdot), & m-1 \geq i, \end{cases}
\end{aligned} \tag{2.6}$$

for any $x \in \bigcap_{0\leq j\leq n-1} D(A_j)$ and $i \in \mathbb{N}^0_{m_n-1}$.

(iii) k-regularized C-resolvent propagation family for (1.1), in short k-regularized C-propagation family for (1.1), if $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C-uniqueness propagation family for (1.1), and if for every $t \in [0,\tau)$, $i \in \mathbb{N}^0_{m_n-1}$ and $j \in \mathbb{N}^0_{n-1}$, one has $R_i(t)A_j \subseteq A_jR_i(t)$, $R_i(t)C = CR_i(t)$ and $CA_j \subseteq A_jC$.

The above classes of propagation families can be defined by purely algebraic equations (cf. [11, 15, 27]). We will not go into further details about this topic here.

As indicated before, we will consider only nondegenerate k-regularized C-resolvent propagation families for (1.1). In case $k(t) = g_{\zeta+1}(t)$, where $\zeta \geq 0$, it is also said that $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a ζ-times integrated C-resolvent propagation family for (1.1); 0-times integrated C-resolvent propagation family for (1.1) is simply called C-resolvent propagation family for (1.1). For a k-regularized (C_1, C_2)-existence and uniqueness family $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$, it is said that is locally equicontinuous (exponentially equicontinuous, (q-)exponentially equicontinuous, analytic, (q-)exponentially analytic,...) if and only if all single operator families $(R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)}$ are. The above terminological agreements and abbreviations can be simply understood for the classes of k-regularized C_1-existence propagation families and k-regularized C_2-uniqueness propagation families. The class of k-regularized (C_1, C_2)-existence and uniqueness propagation families for (1.1) can be also introduced (cf. Definitions 1.1 and 3.1 below).

In case that $A_j = c_jI$, where $c_j \in \mathbb{C}$ for $1 \leq j \leq n-1$, it is also said that the operator A is a subgenerator of $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$. Now we would like to notice the following: if A is a subgenerator of a k-regularized C-resolvent propagation family $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ for (1.1), then, in general, there do not exist $a_i \in L^1_{\text{loc}}([0,\tau))$, $i \in \mathbb{N}^0_{m_n-1}$ and $k_i \in C([0,\tau))$ such that $(R_i(t))_{t\in[0,\tau)}$ is an (a_i, k_i)-regularized C-resolvent family with subgenerator A; the same observation holds for the classes of k-regularized C_1-existence propagation families and k-regularized C_2-uniqueness propagation families. Despite this fact, the structural results for k-regularized C-resolvent propagation families can be derived by using appropriate modifications of the proofs of corresponding results for (a,k)-regularized C-resolvent families. Furthermore, these results can be clarified for any single operator family $(R_i(t))_{t\in[0,\tau)}$ of the tuple $((R_0(t))_{t\in[0,\tau)}, \dots, (R_{m_n-1}(t))_{t\in[0,\tau)})$.

Let $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ be a k-regularized C-resolvent propagation family with subgenerator A. Then one can simply prove that the validity of condition (H5) implies the following functional equation:

$$\begin{aligned}
&\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]+\sum_{j=1}^{n-1}c_j g_{\alpha_n-\alpha_j}*\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]\\
&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}c_j\left[g_{\alpha_n-\alpha_j+i}*k\right](\cdot)Cx\\
&\quad=\begin{cases}A\left[g_{\alpha_n-\alpha}*R_i\right](\cdot)x, & m-1<i,\quad x\in E,\\ A\left[g_{\alpha_n-\alpha}*\left(R_i(\cdot)x-(k*g_i)(\cdot)Cx\right)\right], & m-1\geq i,\quad x\in E,\end{cases}
\end{aligned}\tag{2.7}$$

for any $i=0,\ldots,m_n-1$. The set consisted of all subgenerators of $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$, denoted by $\chi(R)$, need not to be finite. Notice that the supposition $A\in\chi(R)$ obviously implies $C^{-1}AC\in\chi(R)$. The integral generator $\widehat{A}$ of $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ is defined as the set of all pairs $(x,y)\in E\times E$ such that, for every $i=0,\ldots,m_n-1$ and $t\in[0,\tau)$, the following holds:

$$\begin{aligned}
&\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]+\sum_{j=1}^{n-1}c_j g_{\alpha_n-\alpha_j}*\left[R_i(\cdot)x-(k*g_i)(\cdot)Cx\right]\\
&\quad+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}c_j\left[g_{\alpha_n-\alpha_j+i}*k\right](\cdot)Cx\\
&\quad=\begin{cases}\left[g_{\alpha_n-\alpha}*R_i\right](\cdot)y, & m-1<i,\\ g_{\alpha_n-\alpha}*\left[R_i(\cdot)y-(k*g_i)(\cdot)Cy\right], & m-1\geq i.\end{cases}
\end{aligned}\tag{2.8}$$

It is a linear operator on E which extends any subgenerator $A\in\chi(R)$ and satisfies $\widehat{A}=C^{-1}\widehat{A}C$. We have the following.

(i) $R_i(t)(\lambda-A)^{-1}C=(\lambda-A)^{-1}CR_i(t)$, $t\in[0,\tau)$, provided $A\in\chi(R)$, $\lambda\in\rho_C(A)$ and $0\leq i\leq m_n-1$.

(ii) Let $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ be locally equicontinuous. Then:

  (a) $\widehat{A}$ is a closed linear operator.

  (b) $\widehat{A}\in\chi(R)$, if $R_i(t)R_i(s)=R_i(s)R_i(t)$, $0\leq t,s<\tau$, $i\in\mathbb{N}^0_{m_n-1}$.

  (c) $\widehat{A}=C^{-1}AC$, if $A\in\chi(R)$ and (H5) holds. Furthermore, the condition (H5) can be replaced by (2.7).

(iii) Let $\{A,B\}\subseteq\chi(R)$. Then $Ax=Bx$, $x\in D(A)\cap D(B)$, and $A\subseteq B\Leftrightarrow D(A)\subseteq D(B)$. Assume that (2.7) holds for A, and that (2.7) holds for A replaced by B. Then we have the following:

  (a) $C^{-1}AC=C^{-1}BC$ and $C(D(A))\subseteq D(B)$.

  (b) A and B have the same eigenvalues.

  (c) $A\subseteq B\Rightarrow\rho_C(A)\subseteq\rho_C(B)$.

Albeit the similar assertions can be considered in general case, we will omit the corresponding discussion even in the case that $A_j \in L(E)$ for $1 \leq j \leq n-1$.

Proposition 2.3. *Let $i \in \mathbb{N}^0_{m_n-1}$, and let $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ be a locally equicontinuous k-regularized C-resolvent propagation family for* (1.1). *If* (2.5) *holds with $C_1 = C$, then the following holds:*

(i) *the equality*

$$R_i(t)R_i(s) = R_i(s)R_i(t), \quad 0 \leq t, \ s < \tau \tag{2.9}$$

holds provided $m-1 < i$ and the following condition:

($\diamond$) *any of the assumptions $f(t) + \sum_{j\in D_i} A_j(g_{\alpha_n-\alpha_j} * f)(t) = 0$, $t \in [0,\tau)$, or $A(g_{\alpha_n-\alpha} * f)(t) = 0$, for some $f \in C([0,\tau) : E)$, implies $f(t) = 0$, $t \in [0,\tau)$;*

(ii) *the equality* (2.9) *holds provided $m-1 \geq i$, $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$, and the following condition:*

($\diamond\diamond$) *if $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} A_j(g_{\alpha_n-\alpha_j} * f)(t) = 0$, $t \in [0,\tau)$, for some $f \in C([0,\tau) : E)$, then $f(t) = 0$, $t \in [0,\tau)$.*

Proof. Let $x \in E$ and $s \in [0,\tau)$ be fixed. Define $u_i(t) := R_i(t)R_i(s)x - R_i(s)R_i(t)x$, $t \in [0,\tau)$. Using (2.5), it is not difficult to prove that

$$A\int_0^t g_{\alpha_n-\alpha}(t-r)u(r)dr = u(t) + \sum_{j=1}^{n-1}\int_0^t A_j\Big(g_{\alpha_n-\alpha_j} * u\Big)(r)dr = 0, \quad t \in [0,\tau). \tag{2.10}$$

Let $m-1 < i$. Convoluting both sides of (2.10) with $R_i(\cdot)$, we easily infer that $u(t) + \sum_{j=1}^{n-1} A_j(g_{\alpha_n-\alpha_j} * u)(t) = 0$, $t \in [0,\tau)$ and $A(g_{\alpha_n-\alpha} * u)(t) = 0$, $t \in [0,\tau)$. Now the equality (2.9) follows from ($\diamond$). The proof is quite similar in the case $m-1 \geq i$. □

Remark 2.4. The equations (1.1) with $\alpha = 0$ are much easier to deal with, since in this case, $m = 0$ and $m-1 < i$ for all $i \in \mathbb{N}^0_{m_n-1}$. In general, (1.1) with $\alpha > 0$ cannot be reduced to an equivalent equation of the previously considered form.

Proposition 2.5. *Suppose $((R_{j,0}(t))_{t\in[0,\tau)}, \ldots, (R_{j,m_n-1}(t))_{t\in[0,\tau)})$ is a locally equicontinuous k_j-regularized C-resolvent propagation family for* (1.1), *$j = 1, 2$, and $0 \leq i \leq m_n - 1$. Then we have the following.*

(i) *If $m-1 < i$ and ($\diamond$) holds, then*

$$(k_1 * R_{2,i})(t)x = (k_2 * R_{1,i})(t)x, \quad x \in \bigcap_{j=0}^{n-1} D(A_j), \ t \in [0,\tau). \tag{2.11}$$

If, additionally,

$$\bigcap_{j=0}^{n-1} D(A_j) \text{ is dense in } E, \tag{2.12}$$

then (2.11) *holds for all* $x \in E$.

(ii) *The equality* (2.11) *holds provided* $m-1 \geq i$, $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$ *and* ($\diamond\diamond$)*; assuming additionally* (2.12)*, we have the validity of* (2.11) *for all* $x \in E$.

Proof. We will only prove the second part of proposition. Let $x \in \bigcap_{j=0}^{n-1} D(A_j)$. Then the functional equation of $(R_{j,i}(t))_{t\in[0,\tau)}$ $(j = 1, 2)$ implies:

$$\begin{aligned}
&\left[(k_2 * g_i) * (R_{1,i}(\cdot)x - (k_1 * g_i)(\cdot)Cx)\right](\cdot) \\
&= \left\{ R_{2,i}(\cdot) + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \left[R_{2,i}(\cdot)A_j - (k * g_i)(\cdot)CA_j\right] \right. \\
&\quad \left. + \sum_{j\notin D_i} g_{\alpha_n-\alpha_j} * R_{2,i}(\cdot)A_j - g_{\alpha_n-\alpha} * \left[R_{2,i}(\cdot)A - (k * g_i)(\cdot)CA\right] \right\} \\
&* \left[R_{1,i}(\cdot)x - (k * g_i)(\cdot)Cx\right](\cdot) \\
&= \left\{ R_{2,i}(\cdot) + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \left[R_{2,i}(\cdot)A_j - (k * g_i)(\cdot)CA_j\right] + \sum_{j\notin D_i} g_{\alpha_n-\alpha_j} * R_{2,i}(\cdot)A_j \right\} \\
&* \left[R_{1,i}(\cdot) - (k_1 * g_i)(\cdot)Cx\right](\cdot) \\
&\quad - \left[R_{2,i}(\cdot)x - (k_2 * g_i)(\cdot)C\right] * A\left(g_{\alpha_n-\alpha} * \left[R_{1,i}(\cdot)x - (k_1 * g_i)(\cdot)Cx\right]\right)(\cdot),
\end{aligned} \tag{2.13}$$

which yields after a tedious computation:

$$\sum_{j\notin D_i} g_{\alpha_n-\alpha_j} * A_j\left[(k_2 * R_{1,i})(\cdot) - (k_1 * R_{2,i})(\cdot)\right] \equiv 0. \tag{2.14}$$

In view of ($\diamond\diamond$), the above equality shows that $(k_2 * R_{1,i})(t)x = (k_1 * R_{2,i})(t)x$, $t \in [0, \tau)$. It can be simply verified that the condition (2.12) implies that (2.9) holds for all $x \in E$. □

Proposition 2.6. *Let* $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ *be a locally equicontinuous* k*-regularized* C_1*-existence propagation family (*k*-regularized* C_2*-unique-ness propagation family,* k*-regularized* C*-resolvent propagation family) for* (1.1)*, and let* $b \in L^1_{\mathrm{loc}}([0, \tau))$ *be a kernel. Then the tuple*$(((b * R_0)(t))_{t\in[0,\tau)}, \ldots, ((b * R_{m_n-1})(t))_{t\in[0,\tau)})$ *is a locally equicontinuous* $(k * b)$*-regularized* C_1*-existence propagation family (*$(k * b)$*-regularized* C_2*-uniqueness propagation family,* $(k * b)$*-regularized* C*-resolvent propagation family) for* (1.1).

Suppose now E is complete, (1.1) is C-wellposed, $\bigcap_{j=0}^{n-1} D(A_j)$ is dense in E and $0 \leq i \leq m_n - 1$. Set $R_i(t)x := u(t;\ 0, \ldots, Cx, \ldots, 0)(t)$, $t \geq 0$, $x \in \bigcap_{j=0}^{n-1} D(A_j)$, where $0 \leq i \leq m_n - 1$ and Cx appears in the ith place in the preceding expression. Since we have assumed that E is complete, the operator $R_i(t)$ $(t \geq 0)$ can be uniquely extended (cf. also (ii) of Definition 2.1) to a bounded linear operator on E. It can be easily proved that $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a locally equicontinuous C-uniqueness propagation family for (1.1), and that the assumption $CA_j \subseteq A_jC$, $j \in \mathbb{N}_{n-1}^0$ implies $R_i(t)C = CR_i(t)$, $t \geq 0$. In case that $A_j = c_jI$, where $c_j \in \mathbb{C}$ for $1 \leq j \leq n-1$, one can apply the arguments given in the proof of [29, Proposition 1.1, page 32] in order to see that $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a locally equicontinuous C-resolvent propagation family for (1.1). Regrettably, it is not clear how one can prove in general case that $R_i(t)A_j \subseteq A_jR_i(t)$, $j \in \mathbb{N}_{n-1}^0$, $t \geq 0$.

The following definition also appears in [15].

Definition 2.7. Let $T > 0$ and $f \in C([0,T] : E)$. Consider the following inhomogeneous equation:

$$u(t) + \sum_{j=1}^{n-1} \Big(g_{\alpha_n-\alpha_j} * A_ju\Big)(t) = f(t) + \big(g_{\alpha_n-\alpha} * Au\big)(t), \quad t \in [0,T]. \tag{2.15}$$

A function $u \in C([0,T] : E)$ is said to be

(i) a strong solution of (2.15) if and only if $A_ju \in C([0,T] : E)$, $j \in \mathbb{N}_{n-1}^0$ and (2.15) holds for every $t \in [0,T]$;

(ii) a mild solution of (2.15) if and only if $(g_{\alpha_n-\alpha_j} * u)(t) \in D(A_j)$, $t \in [0,T]$, $j \in \mathbb{N}_{n-1}^0$ and

$$u(t) + \sum_{j=1}^{n-1} A_j\Big(g_{\alpha_n-\alpha_j} * u\Big)(t) = f(t) + A\big(g_{\alpha_n-\alpha} * u\big)(t), \quad t \in [0,T]. \tag{2.16}$$

It is clear that every strong solution of (2.15) is also a mild solution of the same problem. The converse statement is not true, in general. One can similarly define the notion of a strong (mild) solution of the problem (2.2).

Let $0 < \tau \leq \infty$, and let $T \in (0,\tau)$. Then the following holds:

(a) if $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a C_1-existence propagation family for (1.1), then the function $u(t) = \sum_{i=0}^{m_n-1} R_i(t)x_i$, $t \in [0,T]$, is a mild solution of (2.2) with $u_i = C_1x_i$ for $0 \leq i \leq m_n - 1$;

(b) if $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a C_2-uniqueness propagation family for (1.1), and $A_jR_i(t)x = R_i(t)A_jx$, $t \in [0,T]$, $x \in \bigcap_{j=0}^{n-1} D(A_j)$, $i \in \mathbb{N}_{m_n-1}^0$, $j \in \mathbb{N}_{n-1}^0$, then the function $u(t) = \sum_{i=0}^{m_n-1} R_i(t)C_2^{-1}u_i$, $t \in [0,T]$, is a strong solution of (2.2), provided $u_i \in C_2(\bigcap_{j=0}^{n-1} D(A_j))$ for $0 \leq i \leq m_n - 1$.

Theorem 2.8. *Suppose* $((R_0(t))_{t\in[0,\tau)},\ldots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ *is a locally equicontinuous* k*-regularized* C_2*-uniqueness propagation family for* (1.1), (2.5) *holds,* $T\in(0,\tau)$ *and* $f\in C([0,T]:E)$*. Then the following holds:*

(i) *if* $m-1<i$*, then any strong solution* $u(t)$ *of* (2.15) *satisfies the equality:*

$$(R_i * f)(t) = (k * g_i * C_2 u)(t) + \sum_{j\in D_i}\Big(g_{\alpha_n-\alpha_j+i} * k * C_2 A_j u\Big)(t), \tag{2.17}$$

for any $t\in[0,T]$*. Therefore, there is at most one strong (mild) solution for* (2.15)*, provided that* $(\diamond)$ *holds,*

(ii) *if* $m-1\geq i$*, then any strong solution* $u(t)$ *of* (2.15) *satisfies the equality:*

$$(R_i * f)(t) = -\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\Big(g_{\alpha_n-\alpha_j+i} * k * C_2 A_j u\Big)(t), \quad t\in[0,T]. \tag{2.18}$$

Therefore, there is at most one strong (mild) solution for (2.15)*, provided that* $\mathbb{N}_{n-1}\setminus D_i\neq\emptyset$ *and that* $(\diamond\diamond)$ *holds.*

Proof. We will only prove the second part of theorem. Let $m-1\geq i$. Taking into account (2.6), we get:

$$\begin{aligned}
\left[R_i - (k * g_i C)\right] * f &= \left[R_i - (k * g_i C)\right] * \left\{u + \sum_{j=1}^{n-1}\Big(g_{\alpha_n-\alpha_j} * A_j u\Big) - (g_{\alpha_n-\alpha} * Au)\right\}\\
&= \left[R_i - (k * g_i C)\right] * \left(u + \sum_{j=1}^{n-1}\Big(g_{\alpha_n-\alpha_j} * A_j u\Big)\right)\\
&\quad - \Bigg\{\left[R_i - (k * g_i C)\right] + \sum_{j\in D_i}\Big[g_{\alpha_n-\alpha_j} * \big(R_i(\cdot)A_j x - (k * g_i)(\cdot)C_2 A_j x\big)\Big]\\
&\quad + \sum_{j\notin D_i}\Big(g_{\alpha_n-\alpha_j} * R_i(\cdot)A_j x\Big)\Bigg\} * u\\
&= -\sum_{\mathbb{N}_{n-1}\setminus D_i}\Big(g_{\alpha_n-\alpha_j+i} * k * C_2 A_j u\Big)(t), \quad t\in[0,T].
\end{aligned} \tag{2.19}$$

This implies the uniqueness of strong solutions to (2.15), provided that $\mathbb{N}_{n-1}\setminus D_i\neq\emptyset$ and that $(\diamond\diamond)$ holds. The uniqueness of mild solutions in the above case follows from the fact that, for every such a solution $u(t)$, there exists a sufficiently large $\zeta>0$ such that the function $(g_\zeta * u)(\cdot)$ is a strong solution of (2.15), with $f(\cdot)$ replaced by $(g_\zeta * f)(\cdot)$ therein. □

If $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$ is a (local) k-regularized C-resolvent propagation family for (1.1), then Theorem 2.8 shows that there exist certain relations between single operator families $(R_0(t))_{t\geq 0}, \ldots,$ and $(R_{m_n-1}(t))_{t\geq 0}$ (cf. also [15] and [28, page 116]). It would take too long to analyze such relations in detail.

The subsequent theorems can be shown by modifying the arguments given in the proof of [30, Theorem 2.2.1].

Theorem 2.9. *Suppose $k(t)$ satisfies (P1), $\omega \geq \max(0, \mathrm{abs}(k))$, $(R_i(t))_{t\geq 0}$ is strongly continuous, and the family $\{e^{-\omega t}R_i(t) : t \geq 0\}$ is equicontinuous, provided $0 \leq i \leq m_n - 1$. Let A be a closed linear operator on E, let $C_1, C_2 \in L(E)$, and let C_2 be injective. Set $P_\lambda := \lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j - A$, $\lambda \in \mathbb{C} \setminus \{0\}$.*

(i) *Suppose $A_j \in L(E)$, $j \in \mathbb{N}_{n-1}$. Then $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C_1-existence propagation family for* (1.1) *if and only if the following conditions hold.*

(a) *The equality*

$$P_\lambda \int_0^\infty e^{-\lambda t}R_i(t)x\,dt = \lambda^{\alpha_n-\alpha-i}\widetilde{k}(\lambda)C_1x + \sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}\widetilde{k}(\lambda)A_jC_1x, \tag{2.20}$$

holds provided $x \in E$, $i \in \mathbb{N}^0_{m_n-1}$, $m - 1 < i$ and $\Re\lambda > \omega$.

(b) *The equality*

$$P_\lambda \int_0^\infty e^{-\lambda t}\left[R_i(t)x - (k * g_i)(t)C_1x\right]dt = -\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}\widetilde{k}(\lambda)A_jC_1x, \tag{2.21}$$

holds provided $x \in E$, $i \in \mathbb{N}^0_{m_n-1}$, $m - 1 \geq i$ and $\Re\lambda > \omega$.

(ii) *Suppose $R_i(0) = (k*g_i)(0)C_2x$, $x \in E\setminus\overline{\bigcap_{0\leq j\leq n-1}D(A_j)}$, $i \in \mathbb{N}^0_{m_n-1}$. Then $((R_0(t))_{t\geq 0}, \ldots,$ $(R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C_2-uniqueness propagation family for* (1.1) *if and only if, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, and for every $x \in \bigcap_{0\leq j\leq n-1}D(A_j)$, the following equality holds:*

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\left[R_i(t)x - (k * g_i)(t)C_2x\right]dt \\
&\quad + \sum_{j\in D_i}\lambda^{\alpha_j-\alpha_n}\int_0^\infty e^{-\lambda t}\left[R_i(t)x - (k * g_i)(t)C_2A_jx\right]dt \\
&\quad + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha_n}\int_0^\infty e^{-\lambda t}R_i(t)A_jx\;dt \\
&= \begin{cases} \lambda^{\alpha-\alpha_n}\int_0^\infty e^{-\lambda t}R_i(t)Ax\;dt, & m-1 < i, \\ \lambda^{\alpha-\alpha_n}\int_0^\infty e^{-\lambda t}\left[R_i(t)Ax - (k * g_i)(t)C_2Ax\right]dt, & m-1 \geq i. \end{cases}
\end{aligned} \tag{2.22}$$

Theorem 2.10. *Suppose $k(t)$ satisfies (P1), $\omega \geq \max(0, \mathrm{abs}(k))$, $(R_i(t))_{t\geq 0}$ is strongly continuous, and the family $\{e^{-\omega t}R_i(t) : t \geq 0\}$ is equicontinuous, provided $0 \leq i \leq m_n - 1$. Let $CA_j \subseteq A_jC$,*

$j \in \mathbb{N}_{n-1}^0$, $A_j \in L(E)$, $j \in \mathbb{N}_{n-1}$, $A_iA_j = A_jA_i$, i, $j \in \mathbb{N}_{n-1}$ and $A_jA \subseteq AA_j$, $j \in \mathbb{N}_{n-1}$. Assume, additionally, that the operator $\lambda^{\alpha_n - i} + \sum_{j\in D_i} \lambda^{\alpha_j - i}A_j$ is injective for every $i \in \mathbb{N}_{m_n-1}^0$ with $m-1 < i$ and for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$, and that the operator $\sum_{j\in \mathbb{N}_{n-1}\setminus D_i} \lambda^{\alpha_j - i}A_j$ is injective for every $i \in \mathbb{N}_{m_n-1}^0$ with $m-1 \geq i$ and for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$. Then $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C-resolvent propagation family for (1.1)*, and* (2.5) *holds, if and only if the equalities* (2.20)-(2.21) *are fulfilled.*

Keeping in mind Theorem 2.10, one can simply clarify the most important Hille-Yosida type theorems for exponentially equicontinuous k-regularized C-resolvent propagation families (cf. also [15] and [26, Theorem 2.8] for further information in this direction). Notice also that the preceding theorem can be slightly reformulated for k-regularized (C_1, C_2)-existence and uniqueness resolvent propagation families.

The analytical properties of k-regularized C-resolvent propagation families are stated in the following two theorems whose proofs are omitted (cf. [14, Theorems 2.16-2.17] and [26, Lemma 3.3, Theorems 3.4, 3.6, and 3.7]).

Theorem 2.11. *Suppose $\beta \in (0, \pi/2]$, $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is an analytic k-regularized C-resolvent propagation family for* (1.1)*, $k(t)$ satisfies (P1),* (2.5) *holds, and $\widetilde{k}(\lambda)$ can be analytically continued to a function $\widehat{k} : \omega + \Sigma_{(\pi/2)+\beta} \to \mathbb{C}$, where $\omega \geq \max(0, \mathrm{abs}(k))$. Suppose $CA_j \subseteq A_jC$, $j \in \mathbb{N}_{n-1}^0$, $A_j \in L(E)$, $j \in \mathbb{N}_{n-1}$, $A_iA_j = A_jA_i$, $i, j \in \mathbb{N}_{n-1}$ and $A_jA \subseteq AA_j$, $j \in \mathbb{N}_{n-1}$. Let the family*

$$\left\{e^{-\omega z}R_i(z) : z \in \Sigma_\gamma\right\} \text{ be equicontinuous, provided } i \in \mathbb{N}_{m_n-1}^0 \text{ and } \gamma \in (0, \beta), \tag{2.23}$$

and let the set

$$\left\{(\lambda - \omega)\widehat{k}(\lambda)\lambda^{-i} : \lambda \in \omega + \Sigma_{(\pi/2)+\gamma}\right\} \tag{2.24}$$

be bounded provided $\gamma \in (0, \beta)$ and $m - 1 \geq i$. Set

$$N_i := \left\{\lambda \in \omega + \Sigma_{(\pi/2)+\beta} : \widehat{k}(\lambda)\left(\lambda^{\alpha_n} + \sum_{j\in D_i} \lambda^{\alpha_j}A_j\right) \text{ is injective}\right\}, \tag{2.25}$$

provided $m - 1 < i$, and

$$N_i := \left\{\lambda \in \omega + \Sigma_{(\pi/2)+\beta} : \widehat{k}(\lambda)\left(\lambda^{\alpha_n} + \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} \lambda^{\alpha_j}A_j\right) \text{ is injective}\right\}, \tag{2.26}$$

provided $m - 1 \geq i$. Suppose N_i is an open connected subset of $\mathbb{C}$, and the set $N_i \cap \{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$ has a limit point in $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$, for any $i \in \mathbb{N}_{m_n-1}^0$. Then the operator P_λ is injective for every $\lambda \in N_i$ and $i \in \mathbb{N}_{m_n-1}^0$,

$$\lim_{\lambda\to+\infty, \lambda\in N_i} \lambda\widetilde{k}(\lambda)P_\lambda^{-1}\left(\lambda^{\alpha_n-\alpha-i} + \sum_{j\in D_i} \lambda^{\alpha_j-\alpha-i}A_j\right)Cx = (k * g_i)(0)Cx, \tag{2.27}$$

provided $m-1<i$ and $x \in E$, and

$$\lim_{\lambda \to +\infty, \lambda \in N_i} \lambda \widetilde{k}(\lambda) P_\lambda^{-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \lambda^{\alpha_j - \alpha - i} A_j C x = 0, \tag{2.28}$$

provided $m-1 \geq i$ and $x \in E$. Suppose, additionally, that there exists $\mu \in \mathbb{C}$ such that $P_\mu^{-1} C \in L(E)$. Then the family

$$\begin{gathered}\Bigg\{ (\lambda-\omega)\widehat{k}(\lambda)\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j - C^{-1}AC\right)^{-1} \\ \times\left(\lambda^{\alpha_n-\alpha-i}C+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}A_jC\right) : \lambda \in N_i \cap \left(\omega+\Sigma_{(\pi/2)+\gamma}\right)\Bigg\} \text{ is equicontinuous,}\end{gathered} \tag{2.29}$$

provided $m-1<i$ and $\gamma \in (0,\beta)$, respectively, the family

$$\begin{gathered}\Bigg\{ (\lambda-\omega)\widehat{k}(\lambda)\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j - C^{-1}AC\right)^{-1} \sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}A_jC \\ : \lambda \in N_i \cap \left(\omega+\Sigma_{(\pi/2)+\gamma}\right)\Bigg\} \text{ is equicontinuous,}\end{gathered} \tag{2.30}$$

provided $m-1 \geq i$ and $\gamma \in (0,\beta)$, the mapping

$$\lambda \longmapsto \left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j - C^{-1}AC\right)^{-1}\left(\lambda^{\alpha_n-\alpha-i}C+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}A_jC\right)x, \tag{2.31}$$

defined for $\lambda \in N_i$, is analytic, provided $m-1<i$ and $x \in E$, and the mapping

$$\lambda \longmapsto \left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}A_j - C^{-1}AC\right)^{-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}A_jCx, \quad \lambda \in N_i, \tag{2.32}$$

is analytic, provided $m-1 \geq i$ and $x \in E$.

Theorem 2.12. *Assume $k(t)$ satisfies (P1), $\omega \geq \max(0, \operatorname{abs}(k))$, $\beta \in (0,\pi/2]$ and, for every $i \in \mathbb{N}^0_{m_n-1}$ with $m-1 \geq i$, the function $(k * g_i)(t)$ can be analytically extended to a function $k_i : \Sigma_\beta \to \mathbb{C}$ satisfying that, for every $\gamma \in (0,\beta)$, the set $\{e^{-\omega z}k_i(z) : z \in \Sigma_\gamma\}$ is bounded. Let $CA_j \subseteq A_jC$, $j \in \mathbb{N}^0_{n-1}$, $A_j \in L(E)$, $j \in \mathbb{N}_{n-1}$, $A_iA_j = A_jA_i$, $i,j \in \mathbb{N}_{n-1}$ and $A_jA \subseteq AA_j$, $j \in \mathbb{N}_{n-1}$. Assume, additionally, that for each $i \in \mathbb{N}^0_{m_n-1}$ the set $V_i := N_i \cap \{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$ contains the set $\{\lambda \in \mathbb{C} : \Re\lambda > \omega, \widetilde{k}(\lambda) \neq 0\}$, and that $R(\lambda^{\alpha_n}C + \sum_{j\in D_i}\lambda^{\alpha_j}A_jC) \subseteq R(P_\lambda)$, provided $m-1<i$ and $\lambda \in V_i$,*

respectively, $R(\lambda^{\alpha_n}C+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j}A_jC)\subseteq R(P_\lambda)$, *provided* $m-1\geq i$ *and* $\lambda\in V_i$ *(cf. the formulation of preceding theorem). Suppose also that the operator* $\lambda^{\alpha_n}I+\sum_{j\in D_i}\lambda^{\alpha_j}A_j$ *is injective, provided* $m-1<i$ *and* $\lambda\in V_i$, *and that the operator* $\lambda^{\alpha_n}I+\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j}A_j$ *is injective, provided* $m-1\geq i$ *and* $\lambda\in V_i$. *Let* $q_i:\omega+\Sigma_{(\pi/2)+\beta}\to L(E)$ $(0\leq i\leq m_n-1)$ *satisfy that, for every* $x\in E$, *the mapping* $\lambda\mapsto q_i(\lambda)x$, $\lambda\in\omega+\Sigma_{(\pi/2)+\beta}$ *is analytic as well as that:*

$$q_i(\lambda)x=\widetilde{k}(\lambda)P_\lambda^{-1}\left(\lambda^{\alpha_n-\alpha-i}C+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}A_jC\right)x,\quad x\in E,\ \lambda\in V_i,\tag{2.33}$$

provided $m-1<i$,

$$q_i(\lambda)x=-\widetilde{k}(\lambda)P_\lambda^{-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha-i}A_jCx,\quad x\in E,\ \lambda\in V_i,\tag{2.34}$$

provided $m-1\geq i$,

$$\text{the family }\left\{(\lambda-\omega)q_i(\lambda):\lambda\in\omega+\Sigma_{(\pi/2)+\gamma}\right\}\text{ is equicontinuous}\quad\forall\gamma\in(0,\beta),\tag{2.35}$$

and, in the case $\overline{D(A)}\neq E$,

$$\lim_{\lambda\to+\infty}\lambda q_i(\lambda)x=\begin{cases}(k*g_i)(0)Cx, & x\notin\overline{D(A)},\ m-1<i,\\ 0, & x\notin\overline{D(A)},\ m-1\geq i.\end{cases}\tag{2.36}$$

Then there exists an exponentially equicontinuous, analytic k*-regularized* C*-resolvent propagation family* $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$ *for* (1.1). *Furthermore, the family* $\{e^{-\omega z}R_i(z):z\in\Sigma_\gamma\}$ *is equicontinuous for all* $i\in\mathbb{N}^0_{m_n-1}$ *and* $\gamma\in(0,\beta)$, (2.5) *holds, and* $R_i(z)A_j\subseteq A_jR_i(z)$, $z\in\Sigma_\beta$, $j\in\mathbb{N}^0_{n-1}$.

In this paper, we will not consider differential properties of k-regularized C-resolvent (propagation) families. For more details, the interested reader may consult [30], and especially, [26, Theorems 3.18–3.20]. Notice also that the assertion of [26, Proposition 3.12] can be reformulated for k-regularized C-resolvent (propagation) families.

In the following theorem, which possesses several obvious consequences, we consider q-exponentially equicontinuous k-regularized I-resolvent propagation families in complete locally convex spaces.

Theorem 2.13. *(i) Suppose* $k(0)\neq0$, $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$ *is a* q*-exponentially equicontinuous* k*-regularized* I*-resolvent propagation family for* (1.1), $A_j\in L_{\circledast}(E)$, $j\in\mathbb{N}_{n-1}$, *and for every* $p\in\circledast$, *there exist* $M_p\geq1$ *and* $\omega_p\geq0$ *such that*

$$p(R_i(t)x)\leq M_pe^{\omega_pt}p(x),\quad t\geq0,\ x\in E,\ 0\leq i\leq m_n-1.\tag{2.37}$$

Then A *is a compartmentalized operator and, for every seminorm* $p\in\circledast$, $((\overline{R_{0,p}(t)})_{t\geq0},\ldots,(\overline{R_{m_n-1,p}(t)})_{t\geq0})$ *is an exponentially bounded* k*-regularized* $\overline{I_p}$*-resolvent propagation family for* (1.1), *in $\overline{E_p}$, with A_j replaced by $\overline{A_{j,p}}$ $(0 \leq j \leq n-1)$. Furthermore,*

$$\left\|\overline{R_{i,p}(t)}\right\| \leq M_p e^{\omega_p t}, \quad t \geq 0,\ 0 \leq i \leq m_n - 1, \tag{2.38}$$

and $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ is a q-exponentially equicontinuous, analytic k-regularized $\overline{I_p}$-resolvent propagation family of angle $\beta \in (0, \pi]$, provided that $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is. Assume additionally that (2.5) holds. Then, for every $p \in \circledast$, (2.5) holds with A_j and $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ replaced by $\overline{A_{j,p}}$ and $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$.

(ii) Suppose $k(t)$ satisfies (P1), E is complete, A is a compartmentalized operator in E, $A_j = c_j I$ for some $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$ and, for every $p \in \circledast$, $\overline{A_p}$ is a subgenerator (the integral generator, in fact) of an exponentially bounded k-regularized $\overline{I_p}$-resolvent propaga-tion family $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ in $\overline{E_p}$ satisfying (2.38), and (2.5) with A and $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ replaced, respectively, by $\overline{A_p}$ and $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$. Suppose, additionally, that $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$ and $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} |c_j|^2 > 0$, provided $m-1 \geq i$. Then, for every $p \in \circledast$, (2.37) holds $(0 \leq i \leq m_n - 1)$ and A is a subgenerator (the integral generator, in fact) of a q-exponentially equicontinuous k-regularized I-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ satisfying (2.5). Further-more, $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a q-exponentially equicontinuous, analytic k-regularized I-resolvent propagation family of angle $\beta \in (0, \pi]$ provided that, for every $p \in \circledast$, $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ is a q-exponentially bounded, analytic k-regularized $\overline{I_p}$-resolvent propagation family of angle β.

Proof. The proof is almost completely similar to that of [20, Theorem 3.1], and we will only outline a few relevant facts needed for the proof of (i). Suppose $x, y \in D(A)$ and $p(x) = p(y)$ for some $p \in \circledast$. Then (2.6) in combination with (2.37) implies that $\Psi_p(R_i(t)A(x-y)) = 0$, $t \geq 0$, provided $m-1 < i$, and $\Psi_p(R_i(t)A(x-y) - (k * g_i)(t)(x-y)) = 0$, $t \geq 0$, provided $m-1 \geq i$. In any case, $\Psi_p(R_i(t)A(x-y)) = 0, t \geq 0$, which implies $p(R_i(t)A(x-y)) = 0$, $t \geq 0$, and in particular $p(k(0)A(x-y)) = 0$. Since $k(0) \neq 0$, we obtain $p(Ax - Ay) = 0$ and $p(Ax) = p(Ay)$. Therefore, A is a compartmentalized operator. It is clear that (2.38) holds and that the mapping $t \mapsto \overline{R_{i,p}(t)}x_p$, $t \geq 0$ is continuous for any $x_p \in E_p$. This implies by the standard limit procedure that the mapping $t \mapsto \overline{R_{i,p}(t)}\overline{x_p}$, $t \geq 0$ is continuous for any $\overline{x_p} \in \overline{E_p}$. Now we will prove that, for every $p \in \circledast$, the operator A_p is closable for the topology of $\overline{E_p}$. In order to do that, suppose (x_n) is a sequence in $D(A)$ with $\lim_{n\to\infty}\Psi_p(x_n) = 0$ and $\lim_{n\to\infty}\Psi_p(Ax_n) = y$, in $\overline{E_p}$. Using the dominated convergence theorem, (2.6) and (2.37), we get that $\int_0^t g_{\alpha_n-\alpha}(t-s)\overline{R_{i,p}(s)}y\,ds = \lim_{n\to\infty}\int_0^t g_{\alpha_n-\alpha}(t-s)\overline{R_{i,p}(s)}\Psi_p(Ax_n)ds = 0$, for any $t \geq 0$. Taking the Laplace transform, one obtains $\overline{R_{i,p}(t)}y = 0$, $t \geq 0$. Since $\overline{R_{i,p}(0)} = k(0)\overline{I_p}$, we get that $y = 0$ and that A_p is closable, as claimed. Suppose $0 \leq i \leq m_n - 1$. It is checked at once that $\overline{R_{i,p}(t)}\,\overline{A_{j,p}} \subseteq \overline{A_{j,p}}\,\overline{R_{i,p}(t)}$, $t \geq 0$, $i \in \mathbb{N}^0_{m_n-1}$, $j \in \mathbb{N}_{n-1}$. The functional equation (2.6) for the operators $\overline{A_{j,p}}$, $0 \leq j \leq n-1$ and $((\overline{R_{0,p}(t)})_{t\geq 0}, \ldots, (\overline{R_{m_n-1,p}(t)})_{t\geq 0})$ can be trivially verified, which also holds for the functional equation (2.6) in case of its validity for the operators A_j, $0 \leq j \leq n-1$, and $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$. The remaining part of the proof can be obtained by copying the final part of the proof of [20, Theorem 3.1(i)]. □

Remark 2.14. In the second part of Theorem 2.13, we must restrict ourselves to the case in which $A_j = c_j I$ for some $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$. As a matter of fact, it is not clear how one can prove that the operator $\lambda^{\alpha_n}\overline{I_p} + \sum_{j\in D_i}\lambda^{\alpha_j}\overline{A_{j,p}}$ is injective, provided $m-1 < i$, $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$, as well as that the operator $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j}\overline{A_{j,p}}$ is injective, provided $m-1 \geq i$, $\Re\lambda > \omega$ and $\widetilde{k}(\lambda) \neq 0$. Then Theorem 2.10 is inapplicable, which implies that the argumentation used in the proof of [20, Theorem 3.1(ii)] does not work for the proof of fact that, for every $i \in \mathbb{N}^0_{m_n-1}$ and $t > 0$, $\{\overline{R_{i,p}(t)} : p \in \circledast\}$ is a projective family of operators.

3. k-Regularized (C_1, C_2)-Existence and Uniqueness Families for (1.1)

Throughout this section, we will always assume that X and Y are sequentially complete locally convex spaces. By $L(Y, X)$ is denoted the space which consists of all bounded linear operators from Y into X. The fundamental system of seminorms which defines the topology on X, respectively, Y, is denoted by $\circledast_X$, respectively, $\circledast_Y$. The symbol I designates the identity operator on X.

Let $0 < \tau \leq \infty$. A strongly continuous operator family $(W(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be locally equicontinuous if and only if, for every $T \in (0, \tau)$ and for every $p \in \circledast_X$, there exist $q_p \in \circledast_Y$ and $c_p > 0$ such that $p(W(t)y) \leq c_p q_p(y)$, $y \in Y$, $t \in [0, T]$; the notion of equicontinuity of $(W(t))_{t\in[0,\tau)}$ is defined similarly. Notice that $(W(t))_{t\in[0,\tau)}$ is automatically locally equicontinuous in case that the space Y is barreled.

Following Xiao and Liang [24], we introduce the following definition.

Definition 3.1. Suppose $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (1.1) if and only if, for every $y \in Y$, the following holds: $E(\cdot)y \in C^{m_n-1}([0, \tau) : X)$, $E^{(i)}(0)y = 0$ for every $i \in \mathbb{N}_0$ with $i < m_n - 1$, $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0, \tau) : X)$ for $0 \leq j \leq n-1$, and

$$E^{(m_n-1)}(t)y + \sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} * E^{(m_n-1)}\right)(t)y - A\left(g_{\alpha_n-\alpha} * E^{(m_n-1)}\right)(t)y = k(t)C_1 y, \tag{3.1}$$

for any $t \in [0, \tau)$.

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (1.1) if and only if, for every $\tau \in [0, \tau)$ and $x \in \bigcap_{0\leq j\leq n-1} D(A_j)$, the following holds:

$$U(t)x + \sum_{j=1}^{n-1}\left(g_{\alpha_n-\alpha_j} * U(\cdot)A_j x\right)(t) - \left(g_{\alpha_n-\alpha} * U(\cdot)Ax\right)(t)y = \left(k * g_{m_n-1}\right)(t)C_2 x. \tag{3.2}$$

(iii) A strongly continuous family $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)}) \subseteq L(Y, X) \times L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized (C_1, C_2)-existence and uniqueness family for (1.1) if and only if $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (1.1), and $(U(t))_{t\in[0,\tau)}$ is a k-regularized C_2-uniqueness family for (1.1).

(iv) Suppose $Y = X$ and $C = C_1 = C_2$. Then a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C-resolvent family for (1.1) if and only if $(R(t))_{t\in[0,\tau)}$ is a k-regularized C-uniqueness family for (1.1), $R(t)A_j \subseteq A_jR(t)$, for $0 \le j \le n-1$ and $t \in [0,\tau)$, as well as $R(t)C = CR(t)$, $t \in [0,\tau)$, and $CA_j \subseteq A_jC$, for $0 \le j \le n-1$.

In case $k(t) = g_{\zeta+1}(t)$, where $\zeta \ge 0$, it is also said that $(E(t))_{t\in[0,\tau)}$ is a ζ-times integrated C_1-existence family for (1.1); 0-times integrated C_1-existence family for (1.1) is also said to be a C_1-existence family for (1.1). The notion of (exponential) analyticity of C_1-existence families for (1.1) is taken in the sense of Definition 1.2(ii); the above terminological agreement can be simply understood for all other classes of uniqueness and resolvent families introduced in Definition 3.1.

Integrating both sides of (3.1) sufficiently many times, we easily infer that (cf. [24, Definition 2.1, page 151; and (2.8), page 153]):

$$E^{(l)}(t)y + \sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} * E^{(l)}\right)(t)y - A\left(g_{\alpha_n-\alpha} * E^{(l)}\right)(t)y = \left(k * g_{m_n-1-l}\right)(t)C_1y, \tag{3.3}$$

for any $t \in [0,\tau)$, $y \in Y$ and $l \in \mathbb{N}^0_{m_n-1}$. In this place, it is worth noting that the identity (3.3), with $k(t) = 1$, $l = 0$, $\tau = \infty$ and $\alpha_j = j$ $(0 \le j \le n-1)$, has been used in [24] for the definition of a C_1-existence family for (ACP$_n$). It can be simply proved that this definition is equivalent with the corresponding one given by Definition 3.1.

Proposition 3.2. *Let $((E(t))_{t\in[0,\tau)}, (U(t))_{t\in[0,\tau)})$ be a k-regularized (C_1, C_2)-existence and uniqueness family for* (1.1), *and let $(U(t))_{t\in[0,\tau)}$ be locally equicontinuous. If $A_j \in L(X)$, $j \in \mathbb{N}_{n-1}$ or $\alpha \le \min(\alpha_1,\ldots,\alpha_{n-1})$, then $C_2E(t)y = U(t)C_1y$, $t \in [0,\tau)$, $y \in Y$.*

Proof. Let $y \in Y$ be fixed. Using the local equicontinuity of $(U(t))_{t\in[0,\tau)}$, we easily infer that the mappings $t \mapsto ((g_{\alpha_n-\alpha} * U) * E(\cdot)y)(t)$, $t \in [0,\tau)$ and $t \mapsto (U * (g_{\alpha_n-\alpha} * E(\cdot)y))(t)$, $t \in [0,\tau)$ are continuous and coincide. The prescribed assumptions also imply that, for every $j \in \mathbb{N}_{n-1}$, $t \in [0,\tau)$ and $y \in Y$,

$$\left(g_{\alpha_n-\alpha} * U * A_j\left(g_{\alpha_n-\alpha_j} * E(\cdot)y\right)\right)(t)y = \left(g_{\alpha_n-\alpha} * UA_j * g_{\alpha_n-\alpha} * E(\cdot)y\right)(t)y. \tag{3.4}$$

Keeping in mind (3.2)-(3.3) and the foregoing arguments, we get that

$$\begin{aligned} &g_{\alpha_n-\alpha} * U * \left[E(\cdot)y + \sum_{j=1}^{n-1} A_j\left(g_{\alpha_n-\alpha_j} * E\right)(\cdot)y - k(\cdot)C_1y\right] \\ &\quad = g_{\alpha_n-\alpha} * UA * \left[g_{\alpha_n-\alpha} * E\right](\cdot)y \\ &\quad = \left[U(\cdot) + \sum_{j=1}^{n-1}\left(g_{\alpha_n-\alpha_j} * U(\cdot)A_j\right) - k(\cdot)C_2\right] * g_{\alpha_n-\alpha} * E(\cdot)y. \end{aligned} \tag{3.5}$$

This, in turn, implies the required equality $C_2E(t)y = U(t)C_1y$, $t \in [0,\tau)$. □

Definition 3.3. Suppose $0 \leq i \leq m_n - 1$. Then we define $D_i' := \{j \in \mathbb{N}_{n-1}^0 : m_j - 1 \geq i\}$, $D_i'' := \mathbb{N}_{n-1}^0 \setminus D_i'$ and

$$\mathbf{D}_i := \left\{ x \in \bigcap_{j \in D_i''} D(A_j) : A_j u_i \in R(C_1), j \in D_i'' \right\}. \tag{3.6}$$

In the first part of subsequent theorem (cf. also [24, Remark 2.2, Example 2.5, Remark 2.6]), we will consider the most important case $k(t) = 1$. The analysis is similar if $k(t) = g_{n+1}(t)$ for some $n \in \mathbb{N}$.

Theorem 3.4. *(i) Suppose $(E(t))_{t\in[0,\tau)}$ is a C_1-existence family for (1.1), $T \in (0,\tau)$, and $u_i \in \mathbf{D}_i$ for $0 \leq i \leq m_n - 1$. Then the function*

$$\begin{aligned} u(t) = \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) &- \sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n - \alpha_j} * E^{(m_n-1-i)} \right)(t) v_{i,j} \\ &+ \sum_{i=m}^{m_n-1} \left(g_{\alpha_n - \alpha} * E^{(m_n-1-i)} \right)(t) v_{i,0}, \quad 0 \leq t \leq T, \end{aligned} \tag{3.7}$$

is a strong solution of the problem (2.2) *on* $[0,T]$*, where $v_{i,j} \in Y$ satisfy $A_j u_i = C_1 v_{i,j}$ for $0 \leq j \leq n-1$.*

(ii) Suppose $(U(t))_{t\in[0,\tau)}$ is a locally equicontinuous k-regularized C_2-uniqueness family for (1.1)*, and $T \in (0,\tau)$. Then there exists at most one strong (mild) solution of* (2.2) *on* $[0,T]$*, with $u_i = 0$, $i \in \mathbb{N}_{m_n-1}^0$.*

Proof. A straightforward computation involving (3.3) shows that

$$\begin{aligned} &u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot) + \sum_{j=1}^{n-1} A_j \left(g_{\alpha_n-\alpha_j} * \left[u(\cdot) - \sum_{i=0}^{m_j-1} u_i g_{i+1}(\cdot) \right] \right) \\ &= - \sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n-\alpha_j} * R^{(m_n-1-i)} \right)(\cdot) v_{i,j} + \sum_{i=m}^{m_n-1} \left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)} \right)(\cdot) v_{i,0} \\ &\quad + \sum_{j=1}^{n-1} A_j \left(g_{\alpha_n-\alpha_j} * \left\{ \sum_{i=m_j}^{m_n-1} g_{i+1}(\cdot) u_i - \sum_{i=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n-\alpha_l} * R^{(m_n-1-i)} \right)(\cdot) v_{i,l} \right.\right. \\ &\qquad\qquad \left.\left. + \sum_{i=m}^{m_n-1} \left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)} \right)(\cdot) v_{i,0} \right\} \right) \\ &= - \sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \left(g_{\alpha_n-\alpha_j} * R^{(m_n-1-i)} \right)(\cdot) v_{i,j} + \sum_{i=m}^{m_n-1} \left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)} \right)(\cdot) v_{i,0} \\ &\quad + \sum_{j=1}^{n-1} \sum_{i=m_j}^{m_n-1} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot) - \sum_{i=0}^{m_n-1} \sum_{l \in \mathbb{N}_{n-1} \setminus D_i} g_{\alpha_n-\alpha_l} \end{aligned}$$

$$
\begin{aligned}
&* \left[-R^{(m_n-1-i)}(\cdot)v_{i,l} + A\left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)}\right)(\cdot)v_{i,l} + g_{i+1}(\cdot)C_1 v_{i,l}\right] \\
&+ \sum_{i=m}^{m_n-1} g_{\alpha_n-\alpha} * \left[-R^{(m_n-1-i)}(\cdot)v_{i,0} + A\left(g_{\alpha_n-\alpha} * R^{(m_n-1-i)}\right)(\cdot)v_{i,0} + g_{i+1}(\cdot)C_1 v_{i,0}\right] \\
&= g_{\alpha_n-\alpha} * A\left[u(\cdot) - \sum_{i=0}^{m-1} u_i g_{i+1}(\cdot)\right],
\end{aligned}
\tag{3.8}
$$

since

$$
\sum_{j=1}^{n-1}\sum_{i=m_j}^{m_n-1} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot) = \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} C_1 v_{i,j} g_{\alpha_n-\alpha_j+i+1}(\cdot). \tag{3.9}
$$

This implies that $u(t)$ is a mild solution of (2.2) on $[0,T]$. In order to complete the proof of (i), it suffices to show that $\mathbf{D}_t^{\alpha_n}u(t) \in C([0,T] : X)$ and $A_i\mathbf{D}_t^{\alpha_i}u \in C([0,T] : X)$ for all $i \in \mathbb{N}_{n-1}^0$. Towards this end, notice that the partial integration implies that, for every $t \in [0,T]$,

$$
\begin{aligned}
g_{m_n-\alpha_n} * \left[u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\right](t) &= \sum_{i=m}^{m_n-1}\left(g_{m_n-\alpha+i} * E^{(m_n-1)}\right)(t)v_{i,0} \\
&\quad - \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\left(g_{m_n-\alpha_j+i} * E^{(m_n-1)}\right)(t)v_{i,j}.
\end{aligned}
\tag{3.10}
$$

Therefore, $\mathbf{D}_t^{\alpha_n}u \in C([0,T] : X)$ and

$$
\begin{aligned}
\mathbf{D}_t^{\alpha_n}u(t) &= \frac{d^{m_n}}{dt^{m_n}}\left\{g_{m_n-\alpha_n} * \left[u(\cdot) - \sum_{i=0}^{m_n-1} u_i g_{i+1}(\cdot)\right](t)\right\} \\
&= \sum_{i=m}^{m_n-1}\left(g_{i-\alpha} * E^{(m_n-1)}\right)(t)v_{i,0} - \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\left(g_{i-\alpha_j} * E^{(m_n-1)}\right)(t)v_{i,j}.
\end{aligned}
\tag{3.11}
$$

Suppose, for the time being, $i \in \mathbb{N}_{n-1}^0$. Then $A_i u_j \in R(C_1)$ for $j \geq m_i$. Moreover, the inequality $l \geq \alpha_j$ holds provided $0 \leq l \leq m_n-1$ and $j \in \mathbb{N}_{n-1}\setminus D_l$, and $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(\cdot)y \in C([0,T] : X)$ for $0 \leq j \leq n-1$ and $y \in Y$. Now it is not difficult to prove that

$$
\begin{aligned}
A_i\mathbf{D}_t^{\alpha_i}u(\cdot) &= \sum_{j=m_i}^{m_n-1} g_{j+1-\alpha_i}(\cdot)A_i u_j - \sum_{l=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_l}\left[g_{l-\alpha_j} * A_i\left(g_{\alpha_n-\alpha_i} * E^{(m_n-1)}\right)\right](\cdot)v_{l,j} \\
&\quad + \sum_{l=m}^{m_n-1}\left[g_{l-\alpha} * A_i\left(g_{\alpha_n-\alpha_i} * E^{(m_n-1)}\right)\right](\cdot)v_{l,0} \in C([0,T] : X),
\end{aligned}
\tag{3.12}
$$

finishing the proof of (i). The second part of theorem can be proved as follows. Suppose $u(t)$ is a strong solution of (2.2) on $[0,T]$, with $u_i=0$, $i\in\mathbb{N}^0_{m_n-1}$. Using this fact and the equality

$$\int_0^t\int_0^{t-s} g_{\alpha_n-\alpha_j}(r)U(t-s-r)A_ju(s)\,dr\,ds=\int_0^t\int_0^{s} g_{\alpha_n-\alpha_j}(r)U(t-s)A_ju(s-r)\,dr\,ds, \tag{3.13}$$

for any $t\in[0,T]$ and $j\in\mathbb{N}^0_{n-1}$, we easily infer that (for more general results, see [31, Proposition 2.4(i)], and [29, page 155]):

$$\begin{aligned}(U*u)(t)&=\left(k*g_{m_n-1}C_2*u\right)(t)\\&\quad+\int_0^t\int_0^{t-s}\left[g_{\alpha_n-\alpha_j}(r)U(t-s-r)A_ju(s)-g_{\alpha_n-\alpha}(r)U(t-s-r)Au(s)\right]dr\,ds\\&=\left(k*g_{m_n-1}C_2*u\right)(t)+(U*u)(t),\quad t\in[0,T].\end{aligned} \tag{3.14}$$

Therefore, $(k*g_{m_n-1}C_2*u)(t)=0$, $t\in[0,T]$ and $u(t)=0$, $t\in[0,T]$. □

Before proceeding further, we would like to notice that the solution $u(t)$, given by (3.7), need not to be of class $C^1([0,T]:X)$, in general. Using integration by parts, it is checked at once that (3.7) is an extension of the formula [24, (2.5); Theorem 2.4, page 152]. Notice, finally, that the proof of Theorem 3.4(ii) is much simpler than that of [24, Theorem 2.4(ii)].

The standard proof of following theorem is omitted (cf. also [24, Theorem 2.7, Remark 2.8, Theorem 2.9] and [28, Chapter 1]).

Theorem 3.5. *Suppose $k(t)$ satisfies (P1), $(E(t))_{t\geq0}\subseteq L(Y,X)$, $(U(t))_{t\geq0}\subseteq L(X)$, $\omega\geq\max(0,\operatorname{abs}(k))$, $C_1\in L(Y,X)$ and $C_2\in L(X)$ is injective. Set $\mathbf{P}_\lambda:=I+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha_n}A_j-\lambda^{\alpha-\alpha_n}A$, $\Re\lambda>0$.*

(i) *(a) Let $(E(t))_{t\geq0}$ be a k-regularized C_1-existence family for (1.1), let the family $\{e^{-\omega t}E(t):t\geq0\}$ be equicontinuous, and let the family $\{e^{-\omega t}A_j(g_{\alpha_n-\alpha_j}*E)(t):t\geq0\}$ be equicontinuous ($0\leq j\leq n-1$). Then the following holds:*

$$\mathbf{P}_\lambda\int_0^\infty e^{-\lambda t}E(t)y\,dt=\tilde{k}(\lambda)\lambda^{1-m_n}C_1y,\quad y\in Y,\ \Re\lambda>\omega. \tag{3.15}$$

(b) Let the operator $\mathbf{P}_\lambda$ be injective for every $\lambda>\omega$ with $\tilde{k}(\lambda)\neq0$. Suppose, additionally, that there exist strongly continuous operator families $(W(t))_{t\geq0}\subseteq L(Y,X)$ and $(W_j(t))_{t\geq0}\subseteq L(Y,X)$ such that $\{e^{-\omega t}W(t):t\geq0\}$ and $\{e^{-\omega t}W_j(t):t\geq0\}$ are equicontinuous ($0\leq j\leq n-1$) as well as that

$$\begin{aligned}\int_0^\infty e^{-\lambda t}W(t)y\,dt&=\tilde{k}(\lambda)\mathbf{P}_\lambda^{-1}C_1y,\\ \int_0^\infty e^{-\lambda t}W_j(t)y\,dt&=\tilde{k}(\lambda)\lambda^{\alpha_j-\alpha_n}A_j\mathbf{P}_\lambda^{-1}C_1y,\end{aligned} \tag{3.16}$$

for every $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$ *and* $\widetilde{k}(\lambda) \neq 0$, $y \in Y$ *and* $j \in \mathbb{N}_{n-1}^0$. *Then there exists a* k-*regularized* C_1-*existence family for* (1.1), *denoted by* $(E(t))_{t\geq 0}$. *Furthermore,* $E^{(m_n-1)}(t)y = W(t)y$, $t \geq 0$, $y \in Y$ *and* $A_j(g_{\alpha_n-\alpha_j} * E^{(m_n-1)})(t)y = W_j(t)y$, $t \geq 0$, $y \in Y$, $j \in \mathbb{N}_{n-1}^0$.

(ii) *Let the assumptions of* (i) *hold with* $k(t) = 1$. *If* $m_n > 1$, *then one suppose additionally that, for every* $j \in \mathbb{N}_{n-1}^0$, *there exists a strongly continuous operator family* $(V_j(t))_{t\geq 0} \subseteq L(Y, X)$ *such that* $\{e^{-\omega t}V_j(t) : t \geq 0\}$ *is equicontinuous as well as that*

$$\int_0^\infty e^{-\lambda t}V_j(t)y\,dt = \lambda^{\alpha_j-\alpha_n-1}\mathbf{P}_\lambda^{-1}A_jC_1y, \tag{3.17}$$

for every $\lambda \in \mathbb{C}$ *with* $\Re\lambda > \omega$, *and* $y \in D(A_jC_1)$. *Let* $u_i \in \mathbf{D}_i$, *and let* $C_1v_i = u_i$ *for some* $v_i \in Y$ $(0 \leq i \leq m_n - 1)$. *Then, for every* $p \in \circledast_X$, *there exist* $c_p > 0$ *and* $q_p \in \circledast_Y$ *such that the corresponding solution* $u(t)$ *satisfies the following estimate:*

$$p(u(t)) \leq c_pe^{\omega t}\sum_{i=0}^{m_n-1} q_p(v_i), \quad t \geq 0, \text{ if } \omega > 0, \tag{3.18}$$

$$p(u(t)) \leq c_pg_{m_n}(t)\sum_{i=0}^{m_n-1} q_p(v_i), \quad t \geq 0, \text{ if } \omega = 0. \tag{3.19}$$

(iii) *Suppose* $(U(t))_{t\geq 0}$ *is strongly continuous and the operator family* $\{e^{-\omega t}U(t) : t \geq 0\}$ *is equicontinuous. Then* $(U(t))_{t\geq 0}$ *is a* k-*regularized* C_2-*uniqueness family for* (1.1) *if and only if, for every* $x \in \bigcap_{j=0}^{n-1} D(A_j)$, *the following holds:*

$$\int_0^\infty e^{-\lambda t}U(t)\mathbf{P}_\lambda x\,dt = \widetilde{k}(\lambda)\lambda^{1-m_n}C_2x, \quad \Re\lambda > \omega. \tag{3.20}$$

The Hausdorff locally convex topology on E^* defines the system $(|\cdot|_B)_{B\in\mathcal{B}}$ of seminorms on E^*, where $|x^*|_B := \sup_{x\in B}|\langle x^*, x\rangle|$, $x^* \in E^*$, $B \in \mathcal{B}$. Let us recall that E^* is sequentially complete provided that E is barreled. Following Wu and Zhang [32], we also define on E^* the topology of uniform convergence on compacts of E, denoted by $\mathcal{C}(E^*, E)$; more precisely, given a functional $x_0^* \in E^*$, the basis of open neighborhoods of x_0^* with respect to $\mathcal{C}(E^*, E)$ is given by $N(x_0^* : \mathbf{K}, \varepsilon) := \{x^* \in E^* : \sup_{x\in\mathbf{K}}|\langle x^* - x_0^*, x\rangle| < \varepsilon\}$, where $\mathbf{K}$ runs over all compacts of E and $\varepsilon > 0$. Then $(E^*, \mathcal{C}(E^*, E))$ is locally convex, complete and the topology $\mathcal{C}(E^*, E)$ is finer than the topology induced by the calibration $(|\cdot|_B)_{B\in\mathcal{B}}$.

Now we focus our attention to the adjoint type theorems for (local) k-regularized C-resolvent families. The proof of following theorem follows from the arguments given in the proofs of [26, Theorems 2.14 and 2.15]; because of that, we will omit it.

Theorem 3.6. *(i) Suppose* X *is barreled,* $\zeta > 0$, $(R(t))_{t\in[0,\tau)}$ *is a* k-*regularized* C-*resolvent family for* (1.1), *and* $\overline{\bigcap_{j=0}^{n-1} D(A_j)} = \overline{R(C)} = X$. *Then* $((g_\zeta * R(\cdot)^*)(t))_{t\in[0,\tau)}$ *is a* k-*regularized* C^*-*resolvent family for* (1.1), *with* A_j *replaced by* A_j^* $(0 \leq j \leq n-1)$.

(ii) Suppose X is barreled, $(R(t))_{t\in[0,\tau)}$ is a (local, global exponentially equicontinuous) k-regularized C-resolvent family for (1.1), *and* $\overline{\bigcap_{j=0}^{n-1} D(A_j)} = \overline{R(C)} = X$. *Put* $Z := \overline{\bigcap_{j=0}^{n-1} D(A_j^*)}$. *Then* $(R(t)^*_{|Z})_{t\in[0,\tau)}$, *is a (local, global exponentially equicontinuous) k-regularized $C^*_{|Z}$-resolvent family for* (1.1), *in Z.*

(iii) Suppose $(R(t))_{t\in[0,\tau)}$ is a locally equicontinuous k-regularized C-resolvent family for (1.1), *and* $\overline{\bigcap_{j=0}^{n-1} D(A_j)} = \overline{R(C)} = X$. *Then* $(R(t)^*)_{t\in[0,\tau)}$ *is a locally equicontinuous k-regularized C^*-resolvent family for* (1.1), *in* $(X^*, \mathcal{C}(X^*, X))$, *with A_j replaced by A_j^* $(0 \le j \le n-1)$. Furthermore, if $(R(t))_{t\ge 0}$ is exponentially equicontinuous, then $(R(t)^*)_{t\ge 0}$ is also exponentially equicontinuous.*

Notice here that a similar theorem can be proved for the class of k-regularized C-resolvent propagation families.

Let $f \in C([0,T] : X)$. Convoluting both sides of (1.1) with $g_{\alpha_n}(t)$, we get that

$$
\begin{aligned}
&u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j \left[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot) \right] \\
&\quad = g_{\alpha_n-\alpha} * A \left[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot) \right] + \left(g_{\alpha_n} * f\right)(\cdot), \quad t \in [0,T].
\end{aligned} \tag{3.21}
$$

In the subsequent theorem, whose proof follows from a slight modification of the proof of [24, Theorem 3.1(i)], we will analyze inhomogeneous Cauchy problem (3.21) in more detail.

Theorem 3.7. *Suppose $(E(t))_{t\in[0,\tau)}$ is a locally equicontinuous C_1-existence family for* (1.1), *$T \in (0,\tau)$, and $u_i \in \mathbf{D}_i$ for $0 \le i \le m_n - 1$. Let $f \in C([0,T] : X)$, let $g \in C([0,T] : Y)$ satisfy $C_1 g(t) = f(t)$, $t \in [0,T]$, and let $G \in C([0,T] : Y)$ satisfy $(g_{\alpha_n-m_n+1} * g)(t) = (g_1 * G)(t)$, $t \in [0,T]$. Then the function*

$$
\begin{aligned}
u(t) = {} & \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \left(g_{\alpha_n-\alpha_j} * E^{(m_n-1-i)}\right)(t) v_{i,j} \\
&+ \sum_{i=m}^{m_n-1} \left(g_{\alpha_n-\alpha} * E^{(m_n-1-i)}\right)(t) v_{i,0} + \int_0^t E(t-s) G(s)\, ds, \quad 0 \le t \le T
\end{aligned} \tag{3.22}
$$

is a mild solution of the problem (3.21) *on $[0,T]$, where $v_{i,j} \in Y$ satisfy $A_j u_i = C_1 v_{i,j}$ for $0 \le j \le n-1$. If, additionally, $g \in C^1([0,T] : Y)$ and $(E^{(m_n-1)}(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is locally equicontinuous, then the solution $u(t)$, given by* (3.22), *is a strong solution of* (1.1) *on $[0,T]$.*

Remark 3.8. Suppose that all conditions quoted in the first part of the above theorem hold, and the family $(E^{(m_n-1)}(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is locally equicontinuous. We assume, instead of condition $g \in C^1([0,T] : Y)$, that there exists a locally equicontinuous C_2-uniqueness family for (1.1) on $[0,\tau)$, as well as that there exist functions $h_j \in L^1([0,T] : Y)$ such that $A_j f(t) = C_1 h_j(t)$, $t \in [0,T]$, $0 \le j \le n-1$ (cf. also the formulation of [24, Theorem 3.1(ii)]). Using the functional equation for $(E(t))_{t\in[0,\tau)}$, one can simply prove that, for every $\sigma \in [0,T]$, the function

$$r_\sigma(\cdot) = E(\cdot)g(\sigma) - g_{m_n}(\cdot)f(\sigma) + \sum_{j=1}^{n-1}\Big(g_{\alpha_n-\alpha_j} * E(\cdot)h_j(\sigma)\Big)(\cdot) - \big(g_{\alpha_n-\alpha} * E(\cdot)h_0(\sigma)\big)(\cdot) \tag{3.23}$$

is a mild solution of the problem

$$u(t) + \sum_{j=1}^{n-1} A_j\Big(g_{\alpha_n-\alpha_j} * u\Big)(t) - A\big(g_{\alpha_n-\alpha} * u\big)(t) = 0, \quad t \in [0,T]. \tag{3.24}$$

By the uniqueness of solutions, we have that the following holds:

$$E(t-\sigma)g(\sigma) - g_{m_n}(t-\sigma)f(\sigma) + \sum_{l=1}^{n-1}\big(g_{\alpha_n-\alpha_l} * E(\cdot)h_l(\sigma)\big)(t-\sigma)\big(g_{\alpha_n-\alpha} * E(\cdot)h_0(\sigma)\big)(t-\sigma) = 0, \tag{3.25}$$

provided $0 \le t, \sigma \le T$ and $\sigma \le t$. Fix $i \in \mathbb{N}_{n-1}^0$. Then the above equality implies that, for every $j \in \mathbb{N}_{m_n-1}^0$ with $j \le \min(\lfloor \alpha_i + m_n - \alpha_{n-1} - 1 \rfloor, \lfloor \alpha_i + m_n - \alpha - 1 \rfloor)$, one has:

$$\begin{aligned} &A_i E^{(j)}(t-\sigma)g(\sigma) - g_{m_n-j}(t-\sigma)C_1 h_i(\sigma) + \sum_{l=1}^{n-1} A_i\Big(g_{\alpha_n-\alpha_l} * E^{(j)}(\cdot)h_l(\sigma)\Big)(t-\sigma) \\ &\quad - A_i\Big(g_{\alpha_n-\alpha} * E^{(j)}(\cdot)h_0(\sigma)\Big)(t-\sigma) = 0, \end{aligned} \tag{3.26}$$

provided $0 \le t, \sigma \le T$ and $\sigma \le t$. For such an index j, we conclude from (3.26) that the mapping $t \mapsto \int_0^t A_i E^{(j)}(t-\sigma)g(\sigma)d\sigma$, $t \in [0,T]$ is continuous. Observe now that the condition

$$\alpha_n - \alpha_i - m_n + \min(\lfloor \alpha_i + m_n - \alpha_{n-1} - 1 \rfloor, \lfloor \alpha_i + m_n - \alpha - 1 \rfloor) \ge 0, \quad i \in \mathbb{N}_{n-1}^0, \tag{3.27}$$

which holds in the case of abstract Cauchy problem (ACP_n), shows that the mapping $t \mapsto A_i[g_{\alpha_n-\alpha_i-m_n+j} * E^{(j)} * g](t)$, $t \in [0,T]$ is continuous as well as that the mapping $t \mapsto (d/dt)[E^{(m_n-1)} * g](t)$, $t \in [0,T]$ is continuous. Hence, the validity of condition (3.27) implies that the function $u(t)$, given by (3.22), is a strong solution of (1.1) on $[0,T]$.

4. Subordination Principles

The proof of following theorem can be derived by using Theorem 3.5 and the argumentation given in [10, Section 3].

Theorem 4.1. *Suppose $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective and $\gamma \in (0,1)$.*

(i) *Let $\omega \geq \max(0, \operatorname{abs}(k))$, and let the assumptions of Theorem 3.5*(i)-(b) *hold. Put*

$$W_\gamma(t) := \int_0^\infty t^{-\gamma}\Phi_\gamma(t^{-\gamma}s)W(s)y\,ds, \quad t > 0,\ y \in Y,\ W_\gamma(0) := W(0). \tag{4.1}$$

Define, for every $j \in \mathbb{N}_{n-1}^0$ and $t \geq 0$, $W_{j,\gamma}(t)$ by replacing $W(t)$ in (4.1) *with $W_j(t)$. Suppose that there exist a number $\nu > 0$ and a continuous kernel $k_\gamma(t)$ satisfying (P1) and $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma-1}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. Then there exists an exponentially bounded k_γ-regularized C_1-existence family $(E_\gamma(t))_{t\geq 0}$ for* (1.1), *with α_j replaced by $\alpha_j\gamma$ therein ($0 \leq j \leq n-1$). Furthermore, the family $\{(1+t^{\lceil\alpha_n\gamma\rceil-2})^{-1}e^{-\omega^{1/\gamma}t}E_\gamma(t) : t \geq 0\}$ is equicontinuous.*

(ii) *Let $\omega \geq 0$, let the assumptions of Theorem 3.5(ii) hold, and let $k(t) = k_\gamma(t) = 1$. Define, for every $j \in \mathbb{N}_{n-1}^0$ and $t \geq 0$, $V_{j,\gamma}(t)$ by replacing $W(t)$ in* (4.1) *with $V_j(t)$. Then, for every $j \in \mathbb{N}_{n-1}^0$, the family $\{e^{-\omega^{1/\gamma}t}V_{j,\gamma}(t) : t \geq 0\}$ is equicontinuous,*

$$\int_0^\infty e^{-\lambda t}V_{j,\gamma}(t)y\,dt = \lambda^{\alpha_j\gamma-\alpha_n\gamma-1}\mathbf{P}_{\lambda^\gamma}^{-1}A_jC_1y, \tag{4.2}$$

for every $\lambda \in \mathbb{C}$ with $\Re(\lambda^\gamma) > \omega$, and $y \in D(A_jC_1)$. Let $u_i \in \mathbf{D}_{i,\gamma}$ (defined in the obvious way), and let $C_1v_i = u_i$ for some $v_i \in Y$ ($0 \leq i \leq \lceil\alpha_n\gamma\rceil - 1$). Then, for every $p \in \circledast_X$, there exist $c_p > 0$ and $q_p \in \circledast_Y$ such that the corresponding solution $u(t)$ satisfies the following estimate:

$$\begin{aligned} p(u(t)) &\leq c_p e^{\omega^{1/\gamma}t}\sum_{i=0}^{\lceil\alpha_n\gamma\rceil-1} q_p(v_i), \quad t \geq 0,\ \text{if } \omega > 0, \\ p(u(t)) &\leq c_p g_{\lceil\alpha_n\gamma\rceil}(t)\sum_{i=0}^{\lceil\alpha_n\gamma\rceil-1} q_p(v_i), \quad t \geq 0,\ \text{if } \omega = 0. \end{aligned} \tag{4.3}$$

(iii) *Suppose $(U(t))_{t\geq 0}$ is a k-regularized C_2-uniqueness family for* (1.1), *and the family $\{e^{-\omega t}U(t) : t \geq 0\}$ is equicontinuous. Define, for every $t \geq 0$, $U_\gamma(t)$ by replacing $W(t)$ in* (4.1) *with $U(t)$. Suppose that there exist a number $\nu > 0$ and a continuous kernel $k_\gamma(t)$ satisfying (P1) and $\widetilde{k_\gamma}(\lambda) = \lambda^{\gamma(2-m_n)-2+\lceil\alpha_n\gamma\rceil}\tilde{k}(\lambda^\gamma)$, $\lambda > \nu$. Then there exists a k_γ-regularized C_2-uniqueness family for* (1.1), *with α_j replaced by $\alpha_j\gamma$ therein ($0 \leq j \leq n-1$). Furthermore, the family $\{e^{-\omega^{1/\gamma}t}U_\gamma(t) : t \geq 0\}$ is equicontinuous.*

Remark 4.2. (i) Consider the situation of Theorem 4.1(iii). Then we have the obvious equality $(k * g_{m_n-1})(0) = (k_\gamma * g_{\lceil\alpha_n\gamma\rceil-1})(0)$. If $\sigma \geq 1$, $k(t) = g_\sigma(t)$ and $(\sigma - 1 + m_n - 1)\gamma + 1 - \lceil\alpha_n\gamma\rceil \geq 0$ (this inequality holds provided $\sigma \geq 2$), then $k_\gamma(t) = g_{(\sigma-1+m_n-1)\gamma+2-\lceil\alpha_n\gamma\rceil}(t)$.

(ii) Let $b \in L^1_{\text{loc}}([0,\infty))$ be a kernel, and let $(U(t))_{t\in[0,\tau)}$ be a (local) k-regularized C_2-uniqueness family for (1.1). Then $((b * U)(t))_{t\in[0,\tau)}$ is a $(b * k)$-regularized C_2-uniqueness family for (1.1).

(iii) Concerning the analytical properties of k_γ-regularized C_1-existence families in Theorem 4.1(i), the following facts should be stated.

(a) The mapping $t \mapsto E_\gamma(t)$, $t > 0$ admits an extension to $\Sigma_{\min(((1/\gamma)-1)(\pi/2),\pi)}$ and, for every $y \in Y$, the mapping $z \mapsto E_\gamma(z)y$, $z \in \Sigma_{\min(((1/\gamma)-1)(\pi/2),\pi)}$ is analytic.

(b) Let $\varepsilon \in (0, \min(((1/\gamma)-1)(\pi/2),\pi))$, and let $(W(t))_{t\geq 0}$ be equicontinuous. Then $(E_\gamma(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic k_γ-regularized C_1-existence family of angle $\min(((1/\gamma)-1)(\pi/2),\pi)$, and for every $p \in \circledast_X$, there exist $M_{p,\varepsilon} > 0$ and $q_{p,\varepsilon} \in \circledast_Y$ such that

$$p\left(E_\gamma(z)y\right) \leq M_{p,\varepsilon} q_{p,\varepsilon}(y)\left(1+|z|^{\lceil \alpha_n \gamma \rceil -1}\right), \quad z \in \Sigma_{\min(((1/\gamma)-1)(\pi/2),\pi)-\varepsilon}. \tag{4.4}$$

(c) $(E_\gamma(t))_{t\geq 0}$ is an exponentially equicontinuous, analytic k_γ-regularized C_1-exis-tence family of angle $\min(((1/\gamma)-1)(\pi/2),\pi/2)$.

The similar statements hold for the k_γ-regularized C_2-uniqueness family $(U_\gamma(t))_{t\geq 0}$ in Theorem 4.1(iii).

The results on k-regularized (C_1, C_2)-existence and uniqueness families can be applied in the study of following abstract Volterra equation:

$$u(t) = f(t) + \sum_{j=0}^{n-1}\left(a_j * A_j u\right)(t), \quad t \in [0,\tau), \tag{4.5}$$

where $0 < \tau \leq \infty$, $f \in C([0,\tau) : X)$, $a_0, \ldots, a_{n-1} \in L^1_{\mathrm{loc}}([0,\tau))$, and $A_0, \ldots, A_{n-1}$ are closed linear operators on X. As in Definition 2.7, by a mild solution, respectively, strong solution, of (4.5), we mean any function $u \in C([0,\tau) : X)$ such that $A_j(a_j * u)(t) \in C([0,\tau) : X)$, $j \in \mathbb{N}^0_{n-1}$ and that

$$u(t) = f(t) + \sum_{j=0}^{n-1} A_j\left(a_j * u\right)(t), \quad t \in [0,\tau), \tag{4.6}$$

respectively, any function $u \in C([0,\tau) : X)$ such that $u(t) \in \bigcap_{j=0}^{n-1} D(A_j)$, $t \in [0,\tau)$ and that (4.5) holds.

We need the following definition.

Definition 4.3. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_1-existence family for (4.5) if and only if

$$E(t)y = k(t)C_1 y + \sum_{j=0}^{n-1} A_j\left(a_j * E\right)(t)y, \quad t \in [0,\tau), \; y \in Y. \tag{4.7}$$

(ii) A strongly continuous operator family $(U(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be a (local, if $\tau < \infty$) k-regularized C_2-uniqueness family for (4.5) if and only if

$$U(t)x = k(t)C_2 x + \sum_{j=0}^{n-1}\left(a_j * A_j U\right)(t)x, \quad t \in [0,\tau), \; x \in \bigcap_{j=0}^{n-1} D\left(A_j\right). \tag{4.8}$$

Notice also that one can introduce the classes of k-regularized (C_1, C_2)-existence and uniqueness families as well as k-regularized C-resolvent families for (4.5); compare Definition 3.1. The full analysis of k-regularized (C_1, C_2)-existence and uniqueness families for (4.5) falls out from the framework of this paper.

The following facts are clear.

(i) Suppose $(E(t))_{t\in[0,\tau)}$ is a k-regularized C_1-existence family for (4.5). Then, for every $y \in Y$, the function $u(t) = E(t)y$, $t \in [0,\tau)$, is a mild solution of (4.5) with $f(t) = k(t)C_1y$, $t \in [0,\tau)$.

(ii) Let $(U(t))_{t\in[0,\tau)}$ be a locally equicontinuous k-regularized C_2-uniqueness family for (4.5). Then there exists at most one mild (strong) solution of (4.5).

The proof of following subordination principle is standard and therefore omitted (cf. the proofs of [29, Theorem 4.1, page 101] and [24, Theorem 2.7]).

Theorem 4.4. *(i) Suppose there is an exponentially equicontinuous k-regularized C_1-existence family for* (1.1)*. Let $c(t)$ be completely positive, let $c(t)$, $k(t)$ and $k_1(t)$ satisfy (P1), and let $\omega_0 > 0$ be such that, for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, the following holds:*

$$\begin{aligned} \tilde{a}_j(\lambda) &= -\widetilde{k_1}(\lambda)\tilde{c}(\lambda)^{1+\alpha_n-\alpha_j}\frac{\lambda}{\tilde{k}(1/\tilde{c}(\lambda))}, \quad j \in \mathbb{N}_{n-1}, \\ \tilde{a}_0(\lambda) &= -\widetilde{k_1}(\lambda)\tilde{c}(\lambda)^{1+\alpha_n-\alpha}\frac{\lambda}{\tilde{k}(1/\tilde{c}(\lambda))}. \end{aligned} \tag{4.9}$$

Assume, additionally, that there exist a number $z \in \mathbb{C}$ and a function $k_2(t)$ satisfying (P1) so that, for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, one has:

$$\frac{\tilde{k}_1(\lambda)}{\tilde{k}(1/\tilde{c}(\lambda))} = z + \tilde{k}_2(\lambda). \tag{4.10}$$

Then there exists an exponentially equicontinuous k_1-regularized C_1-existence family for (4.5).

(ii) Suppose there is an exponentially equicontinuous k-regularized C_2-uniqueness family for (1.1)*. Let $c(t)$ be completely positive, let $c(t)$, $k(t)$ and $k_1(t)$ satisfy (P1), and let $\omega_0 > 0$ be such that, for every $\lambda > \omega_0$ with $\tilde{c}(\lambda) \neq 0$ and $\tilde{k}(1/\tilde{c}(\lambda)) \neq 0$, the following holds:*

$$\tilde{a}_j(\lambda) = \tilde{c}(\lambda)^{\alpha_n-\alpha_j}, j \in \mathbb{N}_{n-1}^0, \qquad \tilde{k}_1(\lambda) = \lambda^{-1}\tilde{c}(\lambda)^{m_n-2}\tilde{k}\left(\frac{1}{\tilde{c}(\lambda)}\right). \tag{4.11}$$

Then there exists an exponentially equicontinuous k_1-regularized C_2-uniqueness family for (4.5).

It is not difficult to reformulate Theorem 4.4 for the class of strong C-propagation families (cf. also Example 5.3 below).

Although our analysis tends to be exhaustive, we cannot cover, in this limited space, many interested subjects. For example, the characterizations of some special classes of q-exponentially equicontinuous k-regularized (C_1, C_2)-existence and uniqueness families in complete locally convex spaces. We also leave to the interested reader the problem of clarifying the Trotter-Kato type theorems for introduced classes.

5. Examples and Applications

We start this section with the following example.

Example 5.1. Suppose $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$ and, for every $i \in \mathbb{N}^0_{m_n-1}$ with $m-1 \geq i$, one has $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$ and $\sum_{j \in \mathbb{N}_{n-1} \setminus D_i} |c_j|^2 > 0$. Let $A_j = c_j I$ for $1 \leq j \leq n-1$.

(i) (a) Suppose $0 < \delta \leq 2$, $\sigma \geq 1$, $(\pi\delta/2(\alpha_n - \alpha)) - (\pi/2) > 0$, and A is a subgenerator of an exponentially equicontinuous (g_δ, g_σ)-regularized C-resolvent family $(R_\delta(t))_{t\geq 0}$ which satisfies the following equality:

$$A\int_0^t g_\delta(t-s)R_\delta(s)x\, ds = R_\delta(t)x - g_\sigma(t)Cx, \quad x \in E,\ t \geq 0. \tag{5.1}$$

Put $\sigma' := \max(1, 1+(\alpha_n-\alpha)(\sigma-1)\delta^{-1})$ and $\theta := \min(\pi/2, \pi\delta/2(\alpha_n-\alpha)-(\pi/2))$. By [26, Theorem 2.7], we have that, for every sufficiently small $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that $\omega_\varepsilon + \Sigma_{(\pi/2)\delta-\varepsilon} \subseteq \rho_C(A)$ and the family $\{|\lambda|^{(\delta-\sigma)/\delta}(1 + |\lambda|^{1/\delta})(\lambda - A)^{-1}C : \lambda \in \omega_\varepsilon + \Sigma_{(\pi/2)\alpha-\varepsilon}\}$ is equicontinuous. Notice also that

$$\begin{aligned}
&\arg\left(\lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} c_j\lambda^{\alpha_j-\alpha}\right)\\
&\quad= \arg\left(\lambda^{\alpha_n-\alpha}|\lambda|^{\alpha-((\alpha_{n-1}+\alpha_n)/2)} + \sum_{j=1}^{n-1} c_j\lambda^{\alpha_j-\alpha}|\lambda|^{\alpha-((\alpha_{n-1}+\alpha_n)/2)}\right)\\
&\quad\approx \arg\left(\lambda^{\alpha_n-\alpha}|\lambda|^{\alpha-((\alpha_{n-1}+\alpha_n)/2)}\right)\\
&\quad= (\alpha_n - \alpha)\arg(\lambda), \quad \lambda \longrightarrow \infty,\ \arg(\lambda) < \frac{\pi}{\alpha_n - \alpha}.
\end{aligned} \tag{5.2}$$

Due to the choice of θ, we have that, for every sufficiently small $\varepsilon > 0$, there exists $\omega_\varepsilon > 0$ such that, for every $\lambda \in \omega_\varepsilon + \Sigma_{(\pi/2)+\theta-\varepsilon}$, one has:

$$\arg\left(\lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} c_j\lambda^{\alpha_j-\alpha}\right) < \frac{\pi}{2}\delta - \varepsilon. \tag{5.3}$$

Therefore, we have the following: if the operator A is densely defined, then the above inequality in combination with Theorem 2.12 indicates that A is a subgenerator of an exponentially equicontinuous, analytic $(\sigma' - 1)$-times integrated C-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ being the angle of analyticity; if the operator A is not densely defined, then the above conclusion continues to hold with σ' replaced by any number $\sigma'' > \sigma'$.

(a$'$) Suppose $0 < \delta \leq 2$, $\sigma \geq 1$, $(\delta((\pi/2) + \gamma)/(\alpha_n - \alpha)) - (\pi/2) > 0$, A is a subgenerator of an exponentially equicontinuous, analytic (g_δ, g_σ)-regularized C-resolvent family $(R_\delta(t))_{t\geq 0}$ of angle $\gamma \in (0, \pi/2]$, and (5.1) holds. Put $\sigma_1 := \sigma'$ and $\theta_1 := \min(\pi/2, (\delta((\pi/2) + \gamma)/(\alpha_n - \alpha)) - (\pi/2))$. If the operator A is densely defined, then it follows from [26, Theorem 3.6] and the above analysis that the operator $C^{-1}AC$ is the integral generator of an exponentially equicontinuous, analytic $(\sigma_1 - 1)$-times integrated C-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ_1 being the angle of analyticity; if the operator A is not densely defined, then the above conclusion continues to hold with σ_1 replaced by any number $\sigma_2 > \sigma_1$. Now we will apply this result to the following fractional analogue of the telegraph equation:

$$\mathbf{D}_t^{\alpha_2} u(t,x) + c_1 \mathbf{D}_t^{\alpha_1} u(t,x) = D\Delta_x u(t,x), \quad t > 0, \ x \in \mathbb{R}^n, \tag{5.4}$$

where $c_1 > 0$, $D > 0$ and $0 < \alpha_1 \leq \alpha_2 < 2$. Let E be one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ and $0 \leq l \leq n$. Put $\mathbb{N}_0^l := \{\alpha \in \mathbb{N}_0^n : \alpha_{l+1} = \cdots = \alpha_n = 0\}$ and recall that the space E_l $(0 \leq l \leq n)$ is defined by $E_l := \{f \in E : f^{(\alpha)} \in E \text{ for all } \alpha \in \mathbb{N}_0^l\}$. The totality of seminorms $(q_\alpha(f) := \|f^{(\alpha)}\|_E, f \in E_l;\ \alpha \in \mathbb{N}_0^l)$ induces a Fréchet topology on E_l. Let T_l possess the same meaning as in [33], and let $A := D\Delta$ act with its maximal distributional domain. Suppose first $E \neq L^\infty(\mathbb{R}^n)$ and $E \neq C_b(\mathbb{R}^n)$. Then the operator A is the integral generator of an exponentially equicontinuous, analytic C_0-semigroup of angle $\pi/2$, which implies that A is the integral generator of an exponentially equicontinuous, analytic I-regularized resolvent propagation family $(R_0(t))_{t\geq 0}$, if $\alpha_2 \leq 1$, respectively, $((R_0(t))_{t\geq 0}, (R_1(t))_{t\geq 0})$ if $\alpha_2 > 1$, of angle $\zeta = \min(\pi/2, (\pi/\alpha_2) - (\pi/2))$; the established conclusion also holds in the Fréchet nuclear space Ξ which consists of those smooth functions on $\mathbb{R}^n$ with period 1 along each coordinate axis [26]. In this place, we would like to observe that it is not clear whether the angle of analyticity of constructed I-regularized resolvent propagation families, in the case that $\alpha_1 < \alpha_2 < 1$, can be improved by allowing that ζ takes the value $\min(\pi, (\pi/\alpha_2) - (\pi/2))$. Suppose now $E = L^\infty(\mathbb{R}^n)$ or $E = C_b(\mathbb{R}^n)$. Then, for every $\sigma' > 1$, the operator A is the integral generator of an exponentially equicontinuous, analytic $(\sigma' - 1)$-times integrated I-regularized resolvent propagation family $(R_0(t))_{t\geq 0}$, if $\alpha_2 \leq 1$, respectively, $((R_0(t))_{t\geq 0}, (R_1(t))_{t\geq 0})$ if $\alpha_2 > 1$, of angle $\min(\pi/2, (\pi/\alpha_2) - (\pi/2))$.

(b) Suppose $0 < \delta \leq 2$, $\sigma \geq 1$, $(\pi\delta/2(\alpha_n - \alpha)) - (\pi/2) > 0$, $a > 0$, $b \in (0,1)$, $k_{a,b}(t) := \mathcal{L}^{-1}(\exp(-a\lambda^b))(t)$, $t \geq 0$ and A is a subgenerator of an exponentially equicontinuous $(g_\delta, k_{a,b})$-regularized C-resolvent family $(R_{a,b}(t))_{t\geq 0}$ which satisfies the following equality:

$$A\int_0^t g_\delta(t-s)R_{a,b}(s)x\,ds = R_{a,k}(t)x - k_{a,b}(t)Cx, \quad x \in E,\ t \geq 0. \tag{5.5}$$

Let θ be defined as in (a). Then it is checked at once that $(\alpha_n - \alpha)b\delta^{-1} < 1$ and $(\alpha_n - \alpha)b\delta^{-1}((\pi/2)+\theta) < \pi/2$. Put $k_1(t) := k_{a_1,b_1}(t)$, $t \geq 0$, where $b_1 := (\alpha_n - \alpha)b\delta^{-1}$ and $a_1 > a(\cos((\alpha_n - \alpha)b\delta^{-1}((\pi/2)+\theta)))^{-1}$. It is clear that, for every $\theta' \in (0,\theta)$, there exists a sufficiently large $\omega_{\theta'} > 0$ such that, for every $\lambda \in \omega_{\theta'} + \Sigma_{(\pi/2)+\theta'}$,

$$\begin{aligned} &\frac{\left|\widetilde{k_1}(\lambda)\right|}{\left|\widetilde{k}\left(\left(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}c_j\lambda^{\alpha_j-\alpha}\right)^{1/\delta}\right)\right|} \\ &\quad \leq \left|\widetilde{k_1}(\lambda)\right|\exp\left(a|\lambda|^{b_1}+\sum_{j=1}^{n-1}\left|c_j\right||\lambda|^{(\alpha_j-\alpha)b/\delta}\right). \end{aligned} \tag{5.6}$$

Arguing as in (a), we reveal that A is a subgenerator of an exponentially equicontinuous, analytic k_1-regularized C-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ being the angle of analyticity.

(b$'$) Suppose $0 < \delta \leq 2$, $\sigma \geq 1$, $\delta(((\pi/2)+\gamma)/(\alpha_n - \alpha)) - (\pi/2) > 0$, $A_j = c_jI$ $(1 \leq j \leq n-1)$, $a > 0$, $b \in (0,1)$, A is a subgenerator of an exponentially equicontinuous, analytic $(g_\delta, k_{a,b})$-regularized C-resolvent family $(R_{a,b}(t))_{t\geq 0}$ of angle $\gamma \in (0,\pi/2]$, and (5.5) holds. Assume, additionally, that $b(1+(2\gamma/\pi)) \leq 1$. Define θ_1 as in (a)$'$, and $k_2(t) := k_{a_2,b_2}(t)$, $t \geq 0$, where $b_2 := (\alpha_n - \alpha)b\delta^{-1}$ and $a_2 > a(\cos((\alpha_n - \alpha)b\delta^{-1}((\pi/2)+\theta_1)))^{-1}$. Then one can simply verify that $(\alpha_n - \alpha)b < \delta$ and $(\alpha_n - \alpha)b\delta^{-1}((\pi/2)+\gamma) \leq \pi/2$. Making use of [26, Theorem 3.6] and the foregoing arguments, we obtain that the operator $C^{-1}AC$ is the integral generator of an exponentially equicontinuous, analytic k_2-regularized C-resolvent propagation family $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ for (1.1), with θ being the angle of analyticity. Before proceeding further, we would like to recommend for the reader [14, 20, 21, 26, 30, 34] for some examples of (nondensely defined, in general) differential operators generating various types of $(g_\sigma, k_{a,b})$-regularized C-resolvent families.

(ii) Suppose E is complete, $0 < \delta \leq 2$, $(\pi\delta/2(\alpha_n - \alpha)) - (\pi/2) > 0$, and A is the densely defined generator of a q-exponentially equicontinuous (g_δ, g_1)-regularized I-resolvent family $(R_\delta(t))_{t\geq 0}$ which satisfies that, for every $p \in ⊛$, there exist $M_p \geq 1$ and $\omega_p \geq 0$ such that $p(R_\delta(t)x) \leq M_p e^{\omega_p t}p(x)$, $t \geq 0$, $x \in E$. By [20, Theorem 3.1], we infer that A is a compartmentalized operator and that, for every $p \in ⊛$, the operator $\overline{A_p}$ is the integral generator of an exponentially bounded (g_δ, g_1)-regularized $\overline{I_p}$-resolvent family in $\overline{E_p}$. Then the first part of this example shows that $\overline{A_p}$ is the integral generator of an exponentially bounded, analytic $\overline{I_p}$-resolvent propagation family, with $\min(\pi/2, (\pi\delta/2(\alpha_n - \alpha)) - (\pi/2))$ being the angle of analyticity. By Theorem 2.13(ii), we obtain that A is the integral generator of a q-exponentially equicontinuous, analytic I-resolvent propagation family $((R_0(t))_{t\in[0,\tau)},\dots,(R_{m_n-1}(t))_{t\in[0,\tau)})$ for (1.1), and that the corresponding angle of analyticity is $\min(\pi/2,(\pi\delta/2(\alpha_n-\alpha))-(\pi/2))$. It can be simply shown that, for every $p\in\circledast$ and $i\in\mathbb{N}^0_{m_n-1}$, there exist $M_{p,i}\geq 1$ and $\omega_{p,i}\geq 0$ such that $p(R_i(t)x)\leq M_{p,i}e^{\omega_{p,i}t}p(x)$, $t\geq 0$, $x\in E$. In the continuation, we will also present some other applications of (a,k)-regularized C-resolvent families in the analysis of some special cases of (1.1); as already mentioned, this theory is inapplicable if some of initial values $u_0,\dots,u_{m_n-1}$ is a non-zero element of E. Consider the abstract Basset-Boussinesq-Oseen equation (1.2) and assume that E is complete. Set $a_\alpha(t):=\mathcal{L}^{-1}(\lambda^\alpha/(\lambda+1))(t)$, $t\geq 0$, $k_\alpha(t):=e^{-t}$, $t\geq 0$ and $\delta_\alpha:=\min(\pi/2,(\pi\alpha/2(1-\alpha)))$. Suppose A is the integral generator of a q-exponentially equicontinuous (g_1,g_1)-regularized I-resolvent family $(R(t))_{t\geq 0}$ satisfying (2.37); cf. [20, 25] for important examples of differential operators generating q-exponentially equicontinuous (g_δ,g_1)-regularized I-resolvent families. Then it has been proved in [20] that A is the integral generator of a q-exponentially equicontinuous, analytic (a_α,k_α)-regularized resolvent family of angle δ_α. Notice, finally, that the choice of function $a_\alpha(t)$ instead of $g_1(t)$ has some advantages.

Example 5.2. Suppose $1\leq p\leq\infty$, $E:=L^p(\mathbb{R})$, $m:\mathbb{R}\to\mathbb{C}$ is measurable, $a_j\in L^\infty(\mathbb{R})$, $(A_jf)(x):=a_j(x)f(x)$, $x\in\mathbb{R}$, $f\in E$ $(1\leq j\leq n-1)$ and $(Af)(x):=m(x)f(x)$, $x\in\mathbb{R}$, with maximal domain. Assume $s\in(1,2)$, $\delta=1/s$, $M_p=p!^s$ and $k_\delta(t)=\mathcal{L}^{-1}(e^{-\lambda^\delta})(t)$, $t\geq 0$. Denote by $M(t)$ the associated function of the sequence (M_p) [30] and put $\Lambda_{\alpha',\beta',\gamma'}:=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda\geq\gamma'^{-1}M(\alpha'\lambda)+\beta'\}$, $\alpha'>0$, $\beta'>0$, $\gamma'>0$. Clearly, there exists a constant $C_s>0$ such that $M(\lambda)\leq C_s|\lambda|^{1/s}$, $\lambda\in\mathbb{C}$. Hereafter we assume that the following condition holds:

(H) for every $\tau>0$, there exist $\alpha'>0$, $\beta'>0$ and $d>0$ such that $\tau\leq\cos(\delta\pi/2)/C_s\alpha'^{1/s}$ and

$$\left|\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)\right|\geq d,\quad x\in\mathbb{R},\ \lambda\in\Lambda_{\alpha',\beta',1}. \tag{5.7}$$

Notice that the above condition holds provided $n=2$, $\alpha_2-\alpha=2$, $\alpha_2-\alpha_1=1$ and $m(x)=(1/4)a_1^2(x)-(1/16)a_1^4(x)-1$, $x\in\mathbb{R}$ (cf. [31]), and that the validity of condition (H) does not imply, in general, the essential boundedness of the function $m(\cdot)$. We will prove that A is the integral generator of a global (not exponentially bounded, in general) k_δ-regularized I-resolvent propagation family $((R_0(t))_{t\geq 0},\dots,(R_{m_n-1}(t))_{t\geq 0})$ for (1.1). Clearly, it suffices to show that, for every $\tau\in(0,\infty)$, A is the integral generator of a local k_δ-regularized I-resolvent propagation family for (1.1) on $[0,\tau)$. Suppose that $\tau>0$ is given in advance, and that $\alpha'>0$, $\beta'>0$ and $d>0$ satisfy (H), for this τ. Let Γ denote the upwards oriented boundary of ultralogarithmic region $\Lambda_{\alpha',\beta',1}$. Put, for every $t\in[0,\tau)$, $f\in E$ and $x\in\mathbb{R}$,

$$(R_i(t)f)(x):=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}\,d\lambda, \tag{5.8}$$

if $m-1<i$, and

$$\left(R_i(t)f\right)(x):=\frac{(-1)}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\lambda^{\alpha_j-\alpha-i}a_j(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}\,d\lambda+\left(k_\delta*g_i\right)(t)f(x), \tag{5.9}$$

if $m-1\geq i$. It is clear that, for every $i\in\mathbb{N}^0_{m_n-1}$, $R_i(t)A_j\subseteq A_jR_i(t)$, $t\in[0,\tau)$, $j\in\mathbb{N}^0_{n-1}$ and that $(R_i(t))_{t\in[0,\tau)}\subseteq L(E)$ is strongly continuous. Furthermore, the Cauchy theorem implies that $R_i(0)=0=k_\delta(0)$, $i\in\mathbb{N}^0_{m_n-1}$. Now we will prove that the identity (2.6) holds provided $m-1<i$ and $C_2=I$. Let $f\in D(A)$. Then a straightforward computation involving Cauchy theorem shows that (2.6) holds, with x replaced by $f(\cdot)$ therein, if and only if:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&\quad+\sum_{j=1}^{n-1}\frac{1}{2\pi i}\int_\Gamma\left(\int_0^t g_{\alpha_n-\alpha_j}(t-s)e^{\lambda s}\,ds\right)e^{-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{l\in D_i}\lambda^{\alpha_l-\alpha-i}g_l(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{l=1}^{n-1}\lambda^{\alpha_l-\alpha}a_l(x)-m(x)}d\lambda\\
&\quad-\frac{1}{2\pi i}\int_\Gamma\left(\int_0^t g_{\alpha_n-\alpha}(t-s)e^{\lambda s}\,ds\right)e^{-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]m(x)f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\left[\lambda^{-i}f(x)+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha_n-i}a_j(x)f(x)\right]d\lambda.
\end{aligned}\tag{5.10}$$

Using [28, Lemma 5.5, page 23] and the Cauchy theorem, the above equality is equivalent with:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&\quad+\sum_{j=1}^{n-1}\frac{1}{2\pi i}\int_\Gamma\frac{e^{\lambda t-\lambda^\delta}}{\lambda^{\alpha_n-\alpha_j}}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{l\in D_i}\lambda^{\alpha_l-\alpha-i}g_l(x)\right]f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{l=1}^{n-1}\lambda^{\alpha_l-\alpha}a_l(x)-m(x)}d\lambda\\
&\quad-\frac{1}{2\pi i}\int_\Gamma\frac{e^{\lambda t-\lambda^\delta}}{\lambda^{\alpha_n-\alpha}}\frac{\left[\lambda^{\alpha_n-\alpha-i}+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha-i}a_j(x)\right]m(x)f(x)}{\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(x)-m(x)}d\lambda\\
&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t-\lambda^\delta}\left[\lambda^{-i}f(x)+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha_n-i}a_j(x)f(x)\right]d\lambda,
\end{aligned}\tag{5.11}$$

which is true because the integrands appearing on both sides of this equality are equal identically. One can similarly prove that the identity (2.6) holds provided $m-1\geq i$ and $C_2=I$, so that $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$, defined in the obvious way, is a k_δ-regularized I-resolvent propagation family for (1.1), with subgenerator A. Notice that the condition (H) implies $m(\cdot)/(\lambda^{\alpha_n-\alpha}+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha}a_j(\cdot)-m(\cdot))\in L^\infty(\mathbb{R})$ for all $\lambda\in\Lambda_{\alpha',\beta',1}$, which has as a further consequence that $R(R_i(t))\subseteq D(A)$, provided $t\geq 0$ and $m-1<i$, and that $R(R_i(t)-(k_\delta * g_i)(t))\subseteq D(A)$, provided $t\geq 0$ and $m-1\geq i$. The equality (2.5) holds for $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$, the integral generator of $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$, defined similarly as in the second section, coincides with the operator A, which is the unique subgenerator of $((R_0(t))_{t\geq0},\ldots,(R_{m_n-1}(t))_{t\geq0})$. Notice that, for every compact set $K\subseteq[0,\infty)$, there exists $h_K>0$ such that

$$\sup_{t\in K,p\in\mathbb{N}_0,i\in\mathbb{N}^0_{m_n-1}}\frac{\left\|h_k^p(d^p/dt^p)R_i(t)\right\|}{p!^s}<\infty,\tag{5.12}$$

and that one can similarly consider the generation of local $k_{1/2}$-regularized I-resolvent propagation families which oblige a modification of the property stated above with $s=2$. Now we would like to give an example of k_δ-regularized I-resolvent propagation family for (1.1) in which $A_j\notin L(E)$ for some $j\in\mathbb{N}_{n-1}$. Assume $n=2$, $\alpha_2-\alpha=2$, $\alpha_2-\alpha_1=1$, $a_1(x)=-2x$, $x\in\mathbb{R}$ and $m(x)=x^2-x^4-1$, $x\in\mathbb{R}$. Define A_1, A and $R_i(\cdot)$ as before ($i=0,1$). Then the established conclusions continue to hold since, for every $\tau>0$, there exist $\alpha'>0$, $\beta'>0$ and $d>0$ such that (H) holds as well as that:

$$\frac{x^2+(x^4-x^2+1)|\lambda|^{-2}}{\left|\lambda^2-2x\lambda+(x^4-x^2+1)\right|}\leq d,\quad x\in\mathbb{R},\quad \lambda\in\Lambda_{\alpha',\beta',1}.\tag{5.13}$$

Notice, finally, that it is not so difficult to construct examples of local k-regularized C-resolvent propagation families which cannot be extended beyond its maximal interval of existence.

Example 5.3. Suppose $1\leq p\leq\infty$, $X:=L^p(\mathbb{R})$, $a\in\mathbb{R}$, $r>0$, $\vartheta(\cdot)\in W^{1,\infty}(\mathbb{R})$, $1/2<\gamma\leq 1$, $T>0$, $f\in C([0,T]:X)$, and $(d/dt)(g_{2\gamma-1}*(d/dx)f(t,\cdot))\in C([0,T]:X)$. Put $A_1:=ad/dx$ and $Au:=r\Delta u-\vartheta(\cdot)u$ with maximal distributional domain. Now we will focus our attention to the following fractional analogue of damped Klein-Gordon equation:

$$\begin{gathered}\mathbf{D}_t^{2\gamma}u(t,x)+a\frac{\partial}{\partial x}\mathbf{D}_t^{\gamma}u(t,x)-r\Delta_x u(t,x)+\vartheta(x)u(t,x)=f(t,x),\quad t>0,\quad x\in\mathbb{R},\\ u(0,x)=\phi(x),\qquad u_t(0,x)=\psi(x),\quad x\in\mathbb{R}.\end{gathered}\tag{5.14}$$

The case $\gamma=1$ has been analyzed in [24, Example 4.1], showing that there exists an exponentially bounded I-uniqueness family for (5.14) and that, for every $\mu_0\in\rho(A_1)$, there exists an exponentially bounded $(\mu_0-A_1)^{-1}$-existence family for (5.14) with $Y=X$. It is worth noting that the estimates obtained in cited example enables one to simply verify that the conditions of Theorem 4.1(i)-(ii) hold with $k(t)=1$ and $C_1=(\mu_0-A_1)^{-1}$, and that the conditions of Theorem 4.1(iii) hold with $k(t)=t$ and $C_2=I$. This implies that there exists an exponentially bounded $g_{2\gamma}$-regularized I-uniqueness family $(U_\gamma(t))_{t\geq0}$ for (5.14) with $\alpha_j=j\gamma$, $j=0,1,2$, and that there exists an exponentially bounded $(\mu_0-A_1)^{-1}$-existence family $(E_\gamma(t))_{t\geq0}$ for (5.14) with $\alpha_j=j\gamma$, $j=0,1,2$. Applying Theorem 3.7, we obtain that, for every $\phi\in W^{3,p}(\mathbb{R})$ and $\psi\in W^{3,p}(\mathbb{R})$, there exists a unique mild solution $u(t,x)$ of the corresponding problem (3.21) as well as that there exist $M \geq 1$ and $\omega \geq 0$ such that the following estimate holds for each $t \geq 0$:

$$\|u(t,x)\|_{L^p(\mathbb{R})} \leq Me^{\omega t}\Bigg[\|\phi\|_{W^{1,p}(\mathbb{R})} + \|\psi\|_{W^{1,p}(\mathbb{R})} + \int_0^t (t-s)^{2\gamma-2}\|f(s,\cdot)\|_{L^p(\mathbb{R})}ds + \int_0^t \left\|\frac{d}{ds}\left(g_{2\gamma-1} * \frac{d}{dx}f(s,\cdot)\right)\right\|_{L^p(\mathbb{R})} ds\Bigg]. \quad (5.15)$$

It is checked at once that the solution $u(t,x)$ is analytically extensible to the sector $\Sigma_{((1/\gamma)-1)(\pi/2)}$, provided that $f(t,x) \equiv 0$. Suppose now $\vartheta(x) \equiv \vartheta > 0$, $\kappa \geq |1/2 - 1/p|$, provided $1 < p < \infty$, respectively, $\kappa > 1/2$, provided $p \in \{1,\infty\}$, $C := (1-\Delta)^{-(1/2)\kappa}$ and $f(t,x) \equiv 0$. Then there exists a strong C-propagation family $\{(S_0(t))_{t\geq 0}, (S_1(t))_{t\geq 0}\}$ for the problem (5.14) with $\gamma = 1$ (cf. [28, Example 5.8, page 130]). Using [10, (1.23), page 12; Theorems 3.1–3.3, pages 40–42] and [28, Proposition 5.3(iii), page 116], it readily follows that, for every $\phi \in W^{p,2}(\mathbb{R})$ and $\psi \in W^{p,2}(\mathbb{R})$, the function $u_\gamma(t,\cdot)$, $t > 0$, given by

$$u_\gamma(t,\cdot) := \int_0^\infty t^{-\gamma}\Phi(st^{-\gamma})\left[S_1(s)\phi + S_1'(s)\phi\right]ds + \int_0^t g_{1-\gamma}(t-s)\int_0^\infty s^{-\gamma}\Phi(rs^{-\gamma})S_1(r)\psi\, dr\, ds, \quad (5.16)$$

is a unique strong solution of the corresponding integral equation (3.21) with $u_0 = C\phi$ and $u_1 = C\psi$; obviously, this solution is analytically extensible to the sector $\Sigma_{((1/\gamma)-1)(\pi/2)}$. Notice also that one can similarly consider (cf. [24, Example 4.2] for more details) the results concerning the existence and uniqueness of mild solutions of the following time-fractional equation:

$$\mathbf{D}_t^{2\gamma}u(t,x) + \left(\rho_1\frac{\partial^3}{\partial x^3} - \rho_2\frac{\partial^2}{\partial x^2}\right)\mathbf{D}_t^{\gamma}u(t,x) + \left(c\frac{\partial^2}{\partial x^2} + a(x)\right)u(t,x) = f(t,x), \quad (5.17)$$

$$u(0,x) = \phi(x), \qquad u_t(0,x) = \psi(x), \quad (5.18)$$

and that Theorem 4.4 can be applied in the analysis of the following integral equation:

$$u(t,x) = a\int_0^t a_1(t-s)\frac{\partial}{\partial x}u(s,x)ds + \int_0^t a_2(t-s)\left[r\Delta_x u(s,x) - \vartheta(x)u(s,x)\right]ds + f(t,x), \quad (5.19)$$

for certain kernels $a_1(t)$ and $a_2(t)$. We leave details to the interested reader.

Consider now the following slight modification of (5.14):

$$\mathbf{D}_t^{2\gamma}u(t,x)+a\frac{\partial}{\partial x}\mathbf{D}_t^{\gamma}u(t,x)-re^{i(2-2\gamma)(\pi/2)}\Delta_x u(t,x)+\vartheta(x)u(t,x)=f(t,x),\quad t>0,\ x\in\mathbb{R},$$
$$u(0,x)=\phi(x),\qquad \left(\mathbf{D}_t^{\gamma}u(t,x)\right)_{|t=0}=\psi(x),\quad x\in\mathbb{R}. \tag{5.20}$$

Suppose now that $a\neq 0$ (for further information concerning the case $a=0$, [21, 23] may be of some importance). Although the equality $\mathbf{D}_t^{2\gamma}u(t,x)=\mathbf{D}_t^{\gamma}u(t,x)\mathbf{D}_t^{\gamma}u(t,x)$ does not hold in general, we would like to point out that the existence and uniqueness of mild solutions to the homogeneous counterpart of (5.20) cannot be so easily proved for initial values belonging to the Sobolev space $W^{k,p}(\mathbb{R})$, for some $k\in\mathbb{N}$. In order to better explain this, we will introduce the new function $v(t,x)$ by $v(t,x):=\mathbf{D}_t^{\gamma}u(t,x)$. Then (5.20) can be rewritten in the following equivalent matricial form:

$$\mathbf{D}_t^{\gamma}\left[u(t,x)\ \ v(t,x)\right]^T=\begin{bmatrix}0 & 1\\ -re^{i(2-2\gamma)\pi/2} & -aix\end{bmatrix}(A)\left[u(t,x)\ \ v(t,x)\right]^T,\quad t\geq 0, \tag{5.21}$$

where $A=-id/dx$; see, for example, [35, 36]. The characteristic values of associated polynomial matrix $P(x):=\left[\begin{smallmatrix}0 & 1\\ -re^{i(2-2\gamma)(\pi/2)} & -aix\end{smallmatrix}\right]$ are $\lambda_{1,2}(x)=(1/2)(-aix\pm\sqrt{a^2+4re^{i(2-2\gamma)(\pi/2)}})$, $x\in\mathbb{R}$, which implies that the condition of Petrovskii for systems of abstract time-fractional equations, that is, $\sup_{x\in\mathbb{R}}\Re((\lambda_{1,2}(x))^{1/\gamma})<\infty$, is not satisfied [36]. Notice, finally, that (1.1) cannot be converted to an equivalent matrix form, except for some very special values of $\alpha_0,\ldots,\alpha_n$.

Before proceeding further, we would like to observe that several examples of k-times integrated (C_1,C_2)-existence and uniqueness families, acting on products of possibly different Banach spaces ($k\in\mathbb{N}$), can be constructed following the consideration given in [37, Section 7].

Example 5.4. Let $s'>1$,

$$E:=\left\{f\in C^{\infty}[0,1];\ \|f\|:=\sup_{p\geq 0}\frac{\|f^{(p)}\|_{\infty}}{p!^{s'}}<\infty\right\},$$
$$A:=-\frac{d}{ds},\qquad D(A):=\left\{f\in E;\ f'\in E,f(0)=0\right\}. \tag{5.22}$$

Then $\rho(A)=\mathbb{C}$, and for every $\eta>1$, $\|R(\lambda:A)\|=O(e^{\eta|\lambda|})$, $\lambda\in\mathbb{C}$ [21]. Consider now the complex non-zero polynomials $P_j(z)=\sum_{l=0}^{n_j}a_{j,l}z^l$, $z\in\mathbb{C}$, $a_{j,n_j}\neq 0$ $(0\leq j\leq n-1)$, and define, for every $\lambda\in\mathbb{C}$ and $j\in\mathbb{N}_{n-1}^0$, the operator $P_j(A)$ by $D(P_j(A)):=D(A^{n_j})$ and $P_j(A)f:=\sum_{l=0}^{n_j}a_{j,l}A^lf$, $f\in D(P_j(A))$. Our intention is to analyze the smoothing properties of solutions of the equation (3.21) with $A_j:=p_j(A)$, $j\in\mathbb{N}_{n-1}^0$, $u_k=0$, $k\in\mathbb{N}_{m_n-1}^0$, and a suitable chosen function $f(t)$. In order to do that, set $N:=\max(dg\ (P_0),\ldots,dg\ (P_{n-1}))$,

$\mathcal{P}_\lambda(z) := 1+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha_n}P_j(z)-\lambda^{\alpha-\alpha_n}P_0(z)$ ($\lambda \in \mathbb{C}\setminus\{0\}$, $z\in\mathbb{C}$), and after that, $\Phi := \{\lambda\in\mathbb{C}\setminus\{0\} : dg\ (\mathcal{P}_\lambda(\cdot)) = N, \mathcal{P}_\lambda(0)\neq 0\}$. Then it is not difficult to prove (cf. [21, Example 2.10]) that, for every $\lambda\in\mathbb{C}\setminus\{0\}$, $\mathbf{P}_\lambda^{-1} = (I+\sum_{j=1}^{n-1}\lambda^{\alpha_j-\alpha_n}A_j - \lambda^{\alpha-\alpha_n}A)^{-1}\in L(E)$ and that

$$\mathbf{P}_\lambda^{-1} = (-1)^N g(\lambda)^{-1}R(z_{1,\lambda} : A)\cdots R(z_{N,\lambda} : A), \quad \lambda\in\Phi, \tag{5.23}$$

where $z_{1,\lambda},\ldots,z_{N,\lambda}$ denote the zeroes of $\mathcal{P}_\lambda(z)$ and $g(\lambda) := N!^{-1}\mathcal{P}_\lambda^{(N)}(0)$, $\lambda\in\Phi$. Suppose now that the following condition holds:

(H) there exist $\sigma\in(0,1)$, $\omega>0$ and $m>0$ such that, for every $j\in\mathbb{N}_{n-1}^0$, one has: $|z_{j,\lambda}|\leq m|\lambda|^\sigma$, $\lambda\in\Phi$, $\Re\lambda>\omega$.

It is well known from the elementary courses of numerical analysis [38] that the condition:

(H_1) there exist $\sigma\in(0,1)$, $\omega>0$ and $m>0$ such that, for every $j\in\mathbb{N}_{n-1}^0$, one has:

$$\left|\frac{N!\mathcal{P}_\lambda^{(j)}(0)}{j!\mathcal{P}_\lambda^{(N)}(0)}\right|^{1/(N-j)} \leq \frac{1}{2}m|\lambda|^\sigma, \quad \lambda\in\Phi,\ \Re\lambda>\omega, \tag{5.24}$$

implies (H). The validity of last condition can be simply verified in many concrete situations, and it seems that slightly better estimates can be obtained only in the case of very special equations of the form (1.1). We would also like to point out that the condition (H) need not to be satisfied, in general. Using (5.23), the inequality $\|A^lR(\mu_1 : A)\cdots R(\mu_l : A)\| \leq (1+|\mu_1|\|R(\mu_1 : A)\|)\cdots(1+|\mu_l|\|R(\mu_l : A)\|)$ ($l\in\mathbb{N},\mu_1,\ldots,\mu_l\in\mathbb{C}$), as well as the continuity of mappings $\lambda\mapsto\mathbf{P}_\lambda^{-1}$, $\Re\lambda>\omega$ and $\lambda\mapsto A_j\mathbf{P}_\lambda^{-1}$, $\Re\lambda>\omega$, for $0\leq j\leq n-1$, we obtain the existence of a positive polynomial $p(\cdot)$ such that

$$\left\|\mathbf{P}_\lambda^{-1}\right\| + \sum_{j=0}^{n-1}\left\|A_j\mathbf{P}_\lambda^{-1}\right\| \leq p(|\lambda|)e^{mN|\lambda|^\sigma}, \quad \Re\lambda>\omega. \tag{5.25}$$

In what follows, we will use the following family of kernels. Define, for every $l>0$, the entire function $\omega_l(\cdot)$ by $\omega_l(\lambda) := \prod_{p=1}^\infty(1+(l\lambda/p^s))$, $\lambda\in\mathbb{C}$, where $s := \sigma^{-1}$. Then it is clear that $|\omega_l(\lambda)| \geq \sup_{k\in\mathbb{N}}\prod_{p=1}^k|1+(l\lambda/p^s)| \geq \sup_{k\in\mathbb{N}}\prod_{p=1}^k l|\lambda|/p^s \geq \sup_{k\in\mathbb{N}}(l|\lambda|)^k/p!^s$, $\lambda\in\mathbb{C}$, $\Re\lambda\geq 0$. Hence, $|\omega_l(\lambda)|\geq e^{M(l|\lambda|)}$, $\lambda\in\mathbb{C}$, $\Re\lambda\geq 0$, where $M(\lambda) := \sup_{p\in\mathbb{N}_0}\ln|\lambda|^p/p!^s$, $\lambda\in\mathbb{C}\setminus\{0\}$ and $M(0) := 0$. It is also worth noting that, for every $\zeta\in(0,\pi/2)$, $p\in\mathbb{N}_0$ and $\lambda\in\Sigma_{(\pi/2)+\zeta}$, we have $|1+(l\lambda/p^s)|\geq l|\Im\lambda|/p^s \geq l(1+\tan\zeta)^{-1}|\lambda|/p^s$, and

$$|\omega_l(\lambda)| \geq e^{M(l(1+\tan\zeta)^{-1}|\lambda|)}, \quad \zeta\in\left(0,\frac{\pi}{2}\right),\ l>0,\ \lambda\in\Sigma_{(\pi/2)+\zeta}. \tag{5.26}$$

Put now $K_l(t) := \mathcal{L}^{-1}(1/\omega_l(\lambda))(t)$, $t\geq 0$, $l>0$. Then, for every $l>0$, $0\in\operatorname{supp}K_l$, $K_l(0)=0$ and $K_l(t)$ is infinitely differentiable for $t\geq 0$. By Theorem 3.5(i)-(b) and (iii), we easily infer from (5.25) that there exists $k>0$ such that, for every $l>k$, there exists an exponentially bounded K_l-regularized I-resolvent family $(E_l(t))_{t\geq 0}$ for (1.1), with $Y=X=E$. Furthermore, the mapping $t \mapsto E_l(t)$, $t \geq 0$ is infinitely differentiable in the uniform operator topology of $L(E)$ and, for every compact set $K \subseteq [0, \infty)$ and for every $l > k$, there exists $h_{K,l} > 0$ such that

$$\sup_{p \geq 0, t \in K} \frac{h_{K,l}^p \left\| E_l^{(p)}(t) \right\|}{p!^s} < \infty. \tag{5.27}$$

One can similarly construct examples of exponentially bounded, analytic K_l-regularized I-resolvent families.

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