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Research Article

On the Difference Equation $x_{n+1} = x_n x_{n-k}/(x_{n-k+1}(a + b x_n x_{n-k}))$

Stevo Stević,[1] Josef Diblík,[2,3] Bratislav Iričanin,[4] and Zdeněk Šmarda[3]

[1] *Mathematical Institute of the Serbian Academy of Sciences and Arts, Knez Mihailova 36/III, 11000 Belgrade, Serbia*
[2] *Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, 60200 Brno, Czech Republic*
[3] *Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, 61600 Brno, Czech Republic*
[4] *Faculty of Electrical Engineering, Belgrade University, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia*

Correspondence should be addressed to Stevo Stević, sstevic@ptt.rs

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We show that the difference equation $x_{n+1} = x_n x_{n-k}/x_{n-k+1}(a + b x_n x_{n-k}), n \in \mathbb{N}_0$, where $k \in \mathbb{N}$, the parameters a, b and initial values x_{-i}, $i = \overline{0,k}$ are real numbers, can be solved in closed form considerably extending the results in the literature. By using obtained formulae, we investigate asymptotic behavior of well-defined solutions of the equation.

1. Introduction

Recently, there has been some reestablished interest in difference equations which can be solved, as well as in their applications, see, for example, [1–16]. For some old results in the topic see, for example, the classical book [17].

In recently accepted paper [18] are given formulae for the solutions of the following four difference equations:

$$x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}(\pm 1 \pm x_n x_{n-3})}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

and some of these formulae are proved by induction.

Here, we show that the formulae obtained in [18] follow from known results in a natural way. Related idea was exploited in paper [7].

Moreover, we will consider here the following more general equation:

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}(a + b x_n x_{n-k})}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k \in \mathbb{N}$ and the parameters a, b as well as initial values x_{-i}, $i = \overline{0, k}$ are real numbers, and describe the behaviour of all well-defined solutions of the equation.

For a solution $(x_n)_{n \geq -k}$, $k \in \mathbb{N}$ of the difference equation

$$x_n = f(x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N}_0, \tag{1.3}$$

is said that it is eventually periodic with period p, if there is an $n_1 \geq -k$ such that

$$x_{n+p} = x_n, \quad \text{for } n \geq n_1. \tag{1.4}$$

If $n_1 = -k$, then it is said that the solution is periodic with period p. For some results in this area see, for example, [19–26] and the references therein.

2. Solutions of Equation (1.2)

By using the change of variables

$$y_n = \frac{1}{x_n x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

equation (1.2) is transformed into the following linear first-order difference equation:

$$y_{n+1} = a y_n + b, \quad n \in \mathbb{N}_0, \tag{2.2}$$

for which it is known (and easy to see) that

$$y_n = y_0 a^n + b \frac{1 - a^n}{1 - a} = \frac{b + a^n (y_0(1 - a) - b)}{1 - a}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

if $a \neq 1$, and

$$y_n = y_0 + bn, \quad n \in \mathbb{N}_0, \tag{2.4}$$

if $a = 1$.

From (2.1), we have

$$x_n = \frac{1}{y_n x_{n-k}} = \frac{y_{n-k}}{y_n} x_{n-2k}, \tag{2.5}$$

for $n \geq k$, from which it follows that

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{y_{(2j-1)k+i}}{y_{2jk+i}}, \tag{2.6}$$

for every $m \in \mathbb{N}_0$ and $i \in \{k, k+1, \ldots, 3k-1\}$.

Using (2.3) and (2.4) in (2.6), we get

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1 - a - bx_0x_{-k})}, \tag{2.7}$$

if $a \neq 1$, and

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{x_0x_{-k}b((2j-1)k+i) + 1}{x_0x_{-k}b(2jk+i) + 1}, \tag{2.8}$$

if $a = 1$, for every $m \in \mathbb{N}_0$ and $i \in \{k, k+1, \ldots, 3k-1\}$.

By using formulae (2.7) and (2.8), the behavior of well-defined solutions of equation (1.2) can be obtained. This is done in the following section.

Remark 2.1. It is easy to check that the formulae in Theorems 2.1 and 4.1 from [18] are direct consequences of formula (2.8), whereas formulae in Theorem 3.1 and Theorem 5.1 from [18] are direct consequences of formula (2.7).

Remark 2.2. Note that from formula (2.8), it follows that in the case $a = 1$, $b = 0$, all well-defined solutions of equation (1.2) are periodic with period $2k$. This can be also obtained from (1.2), without knowing explicit formulae for its solutions. Namely, in this case, (1.2) can be written as follows:

$$x_{n+1}x_{n-k+1} = x_nx_{n-k}, \tag{2.9}$$

since we assume $x_n \neq 0$, for all $n \geq -k$, from which it follows that the sequence x_nx_{n-k} is constant, that is, $x_nx_{n-k} = c, n \in \mathbb{N}_0$ for some $c \in \mathbb{R} \setminus \{0\}$. Hence,

$$x_n = \frac{c}{x_{n-k}} = x_{n-2k}, \quad n \geq k, \tag{2.10}$$

as claimed.

Remark 2.3. Note also that in the case $a \notin \{0, 1\}$, $b = 0$, from (2.7), we have

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{a^{(2j-1)k+i}}{a^{2jk+i}} = \frac{x_{i-2k}}{a^{k(m+1)}}, \tag{2.11}$$

for every $m \in \mathbb{N}_0$ and $i \in \{k, k+1, \dots, 3k-1\}$, from which the behaviour of the solutions in the case easily follows.

3. Asymptotic Behavior of Well-Defined Solutions of Equation (1.2)

In this section, we derive some results on asymptotic behavior of well-defined solutions of equation (1.2). We will use well-known asymptotic formulae as follows:

$$\begin{gathered}\ln(1+x) = x - \frac{x^2}{2} + O\left(x^3\right), \\ (1+x)^{-1} = 1 - x + O\left(x^2\right),\end{gathered} \tag{3.1}$$

for $x \to 0$, where O is the Landau "big-oh" symbol.

Theorem 3.1. *Let $a = 1$ and $b \neq 0$ in* (1.2). *Then, every well-defined solution $(x_n)_{n \geq -k}$ of equation* (1.2) *converges to zero.*

Proof. By formula (2.8), we have

$$\begin{aligned}\lim_{m\to\infty} x_{2km+i} &= \lim_{m\to\infty} x_{i-2k} \prod_{j=0}^{m} \frac{x_0 x_{-k} b\left((2j-1)k+i\right)+1}{x_0 x_{-k} b\left(2jk+i\right)+1} \\ &= x_{i-2k} \lim_{m\to\infty} \prod_{j=0}^{m} \left(1 - \frac{x_0 x_{-k} bk}{1 + x_0 x_{-k} b\left(2jk+i\right)}\right) \\ &= x_{i-2k} \lim_{m\to\infty} C(m_0) \prod_{j=m_0+1}^{m} \left(1 - \frac{x_0 x_{-k} bk}{1 + x_0 x_{-k} b\left(2jk+i\right)}\right),\end{aligned} \tag{3.2}$$

where m_0 is sufficiently large so that (3.1) can be applied below, and

$$C(m_0) = \prod_{j=0}^{m_0} \left(1 - \frac{x_0 x_{-k} bk}{1 + x_0 x_{-k} b\left(2jk+i\right)}\right). \tag{3.3}$$

Since

$$\sum_{j=m_0+1}^{\infty} \frac{k}{2jk+i} = +\infty, \tag{3.4}$$

we conclude, using (3.1), that

$$\begin{aligned}
\lim_{m\to\infty}\prod_{j=m_0+1}^{m}&\left(1-\frac{x_0x_{-k}bk}{1+x_0x_{-k}b(2jk+i)}\right)\\
&=\lim_{m\to\infty}\prod_{j=m_0+1}^{m}\exp\left[\ln\left(1-\frac{x_0x_{-k}bk}{1+x_0x_{-k}b(2jk+i)}\right)\right]\\
&=\lim_{m\to\infty}\prod_{j=m_0+1}^{m}\exp\left[-\frac{x_0x_{-k}bk}{1+x_0x_{-k}b(2jk+i)}-\frac{1}{2}\left(\frac{x_0x_{-k}bk}{1+x_0x_{-k}b(2jk+i)}\right)^2\right.\\
&\qquad\qquad\left.+O\left(\left(\frac{x_0x_{-k}bk}{1+x_0x_{-k}b(2jk+i)}\right)^3\right)\right]\\
&=\lim_{m\to\infty}\prod_{j=m_0+1}^{m}\exp\left[-\frac{k}{2jk+i}+O\left(\frac{1}{j^2}\right)\right]\\
&=\lim_{m\to\infty}\exp\left[-\sum_{j=m_0+1}^{m}\frac{k}{2jk+i}\right]\prod_{j=m_0+1}^{m}\exp\left[O\left(\frac{1}{j^2}\right)\right]=0.
\end{aligned}\tag{3.5}$$

Therefore,

$$\lim_{m\to\infty}x_{2km+i}=0,\tag{3.6}$$

for every $i\in\{k,k+1,\ldots,3k-1\}$, and consequently, $\lim_{n\to\infty}x_n=0$, as claimed. □

Before we formulate and prove our next result, we will prove an auxiliary result which is incorporated in the lemma that follows.

Lemma 3.2. *If $a\neq 1$, then equation* (1.2) *has $2k$-periodic solutions.*

Proof. Let

$$z_n=x_nx_{n-k},\quad n\in\mathbb{N}_0.\tag{3.7}$$

Then

$$z_{n+1}=\frac{z_n}{a+bz_n},\quad n\in\mathbb{N}_0.\tag{3.8}$$

Since $a\neq 1$, we see that equation (3.8) has equilibrium solution as follows:

$$z_n=\overline{z}=\frac{1-a}{b},\quad n\in\mathbb{N}_0.\tag{3.9}$$

For this solution of equation (3.8), we have

$$x_n = \frac{\overline{z}}{x_{n-k}} = x_{n-2k}, \quad n \geq k, \tag{3.10}$$

from which the lemma follows. □

Remark 3.3. Note that $2k$-periodic solutions in the previous lemma could be prime $2k$-periodic. For this it is enough to choose initial conditions $x_{-i}, i = \overline{0, k}$, such that the string

$$\left(x_{-k}, x_{-k+1}, \ldots, x_{-1}, \frac{\overline{z}}{x_{-k}}, \frac{\overline{z}}{x_{-k+1}}, \ldots, \frac{\overline{z}}{x_{-1}}\right) \tag{3.11}$$

is not periodic with a period less then $2k$.

Theorem 3.4. *Let* $|a| < 1$ *and* $b \neq 0$ *in* (1.2). *Then, every well-defined solution* $(x_n)_{n\geq -k}$ *of equation* (1.2) *converges to a, not necessarily prime,* $2k$*-periodic solution of the equation.*

Proof. First note that by Lemma 3.2, in this case, there are $2k$-periodic solutions of the equation. We know that in this case well-defined solutions of the equation are given by formula (2.7). From this and by using asymptotic formulae (3.1), we obtain that for sufficiently large m_1

$$\begin{aligned}
x_{2km+i} &= x_{i-2k}\prod_{j=0}^{m}\frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1 - a - bx_0x_{-k})} \\
&= x_{i-2k}C(m_1)\prod_{j=m_1+1}^{m}\frac{1 + a^{(2j-1)k+i}((1 - a - bx_0x_{-k})/bx_0x_{-k})}{1 + a^{2jk+i}((1 - a - bx_0x_{-k})/bx_0x_{-k})} \\
&= x_{i-2k}C(m_1)\prod_{j=m_1+1}^{m}\left(1 + a^{(2j-1)k+i}\left(1 - a^k\right)\frac{(1 - a - bx_0x_{-k})}{bx_0x_{-k}} + O\left(a^{4jk}\right)\right) \\
&= x_{i-2k}C(m_1)\exp\left(\left(1 - a^k\right)\frac{(1 - a - bx_0x_{-k})}{bx_0x_{-k}}\sum_{j=m_1+1}^{m}\left(a^{(2j-1)k+i} + O\left(a^{4jk}\right)\right)\right),
\end{aligned} \tag{3.12}$$

where

$$C(m_1) = \prod_{j=0}^{m_1}\frac{bx_0x_{-k} + a^{(2j-1)k+i}(1 - a - bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1 - a - bx_0x_{-k})}. \tag{3.13}$$

From (3.12) and since $|a| < 1$, it easily follows that the sequences $(x_{2km+i})_{m\in\mathbb{N}_0}$ are convergent for each $i \in \{k, k+1, \ldots, 3k-1\}$, from which the theorem follows. □

Theorem 3.5. *Let* $|a| > 1$ *and* $b \neq 0$ *in* (1.2). *Then, every well-defined solution* $(x_n)_{n\geq -k}$ *of equation* (1.2) *converges to zero.*

Proof. In this case, well-defined solutions of equation (1.2) are also given by formula (2.7). Further note that for each $i \in \{k, k+1, \dots, 3k-1\}$ holds

$$\lim_{j\to\infty} \frac{bx_0x_{-k} + a^{(2j-1)k+i}(1-a-bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1-a-bx_0x_{-k})} = \frac{1}{a^k}. \tag{3.14}$$

Now note that $1/|a|^k < 1$, due to the assumption $|a| > 1$. Using this fact and (3.14), it follows that for sufficiently large j, say $j \geq m_2$ we have

$$\left|\frac{bx_0x_{-k} + a^{(2j-1)k+i}(1-a-bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1-a-bx_0x_{-k})}\right| \leq \frac{1}{2}\left(1+\frac{1}{|a|^k}\right). \tag{3.15}$$

From this, we have

$$\begin{aligned}
|x_{2km+i}| &= |x_{i-2k}|C(m_2)\prod_{j=m_2+1}^{m}\left|\frac{bx_0x_{-k} + a^{(2j-1)k+i}(1-a-bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1-a-bx_0x_{-k})}\right| \\
&\leq |x_{i-2k}|C(m_2)\prod_{j=m_2+1}^{m}\left(\frac{1}{2}\left(1+\frac{1}{|a|^k}\right)\right) \\
&= |x_{i-2k}|C(m_2)\left(\frac{1}{2}\left(1+\frac{1}{|a|^k}\right)\right)^{m-m_2} \longrightarrow 0,
\end{aligned} \tag{3.16}$$

as $m \to \infty$, where

$$C(m_2) = \prod_{j=0}^{m_2}\left|\frac{bx_0x_{-k} + a^{(2j-1)k+i}(1-a-bx_0x_{-k})}{bx_0x_{-k} + a^{2jk+i}(1-a-bx_0x_{-k})}\right|, \tag{3.17}$$

from which the theorem follows. □

Theorem 3.6. *Let $a = -1$, $b \neq 0$, and k be even in* (1.2). *Then, every well-defined solution $(x_n)_{n\geq -k}$ of equation* (1.2) *is eventually periodic with, not necessarily prime, period $4k$.*

Proof. From (2.2), in this case, we have

$$y_n = -y_{n-1} + b = y_{n-2}, \quad n \geq 2, \tag{3.18}$$

which means that the sequence y_n is two-periodic, and consequently the sequence x_nx_{n-k} is two-periodic. Hence

$$x_{2n}x_{2n-k} = x_0x_{-k}, \qquad x_{2n+1}x_{2n-k+1} = x_1x_{-k+1}, \tag{3.19}$$

from which it follows that

$$x_{2n+i} = \frac{x_i x_{-k+i}}{x_{2n-k+i}} = x_{2n-2k+i}, \quad n \geq \left[\frac{k-i+1}{2}\right], \quad i \in \{0,1\}, \tag{3.20}$$

that is, the sequences x_{2n} and x_{2n+1} are $2k$-periodic from which the result easily follows. □

Theorem 3.7. *Let $a = -1$, $b \neq 0$, k be odd in* (1.2)*, and $(x_n)_{n \geq -k}$ be a well-defined solution of equation* (1.2)*. Then the following statements are true.*

(a) *If $x_0 x_{-k} = 2/b$, then the solution is $4k$-periodic.*

(b) *If $|bx_0 x_{-k} - 1| < 1$, then $x_{2n} \to 0$ and $|x_{2n+1}| \to \infty$, as $n \to \infty$.*

(c) *If $|bx_0 x_{-k} - 1| > 1$, then $x_{2n+1} \to 0$ and $|x_{2n}| \to \infty$, as $n \to \infty$.*

Proof. As in Theorem 3.6, we obtain (3.18) and consequently (3.19) holds. If $k = 2l+1$ for some $l \in \mathbb{N}_0$, then from (1.2) and (3.19) we have

$$\begin{aligned} x_{2n} x_{2n-2l-1} &= x_0 x_{-(2l+1)}, \\ x_{2n+1} x_{2n-2l} &= x_1 x_{-2l} = \frac{x_0 x_{-(2l+1)}}{b x_0 x_{-(2l+1)} - 1}. \end{aligned} \tag{3.21}$$

From (3.21) we obtain

$$\begin{aligned} x_{2n} &= \frac{x_0 x_{-(2l+1)}}{x_{2n-(2l+1)}} = \left(b x_0 x_{-(2l+1)} - 1\right) x_{2n-(4l+2)}, \\ x_{2n+1} &= \frac{x_1 x_{-2l}}{x_{2n-2l}} = \frac{x_{2n+1-(4l+2)}}{b x_0 x_{-(2l+1)} - 1}. \end{aligned} \tag{3.22}$$

From relation (3.22), the statements in this theorem easily follow. □

Remark 3.8. Note that the case $bx_0 x_{-k} - 1 = -1$ is not possible in Theorem 3.7. Namely, in this case $x_0 x_{-k} = 0$, due to the assumption $b \neq 0$ and, as we can see from (1.2), the solution is not well-defined.

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