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Research Article

On the Weak Solution of a Semilinear Boundary Value Problem without the Landesman-Lazer Condition

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We prove the existence of weak solution to a semilinear boundary value problem without the Landesman-Lazer condition.

1. Introduction

We consider the nonlinear boundary value problem

$$\Delta u + \lambda_k u + g(u) = h(x) \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, $h \in L^2(\Omega)$, λ_k is a simple eigenvalue of $-\Delta$ corresponding to the eigenvector ϕ_k, and the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$\left|g(u) - g(v)\right| \leq L|u - v|\left(\text{Lipschitz continuity}\right) \text{ for some constant } L > 0. \tag{H}$$

Landesman and Lazer [1] considered the problem (1.1)-(1.2) with continuous function g satisfying $g(-\infty) < g(\xi) < g(\infty)$, where $g(\pm\infty) = \lim_{s\to\pm\infty} g(s)$ exist and are finite. The authors showed that if ϕ_k is an eigenfunction corresponding to λ_k, $\Omega^+ = \{x \in \Omega : \phi_k > 0\}$ and

$\Omega^- = \{x \in \Omega : \phi_k < 0\}$, then the necessary and sufficient condition for the existence of weak solution of (1.1)-(1.2) is that

$$g(-\infty)\int_{\Omega^+}\phi_k dx + g(\infty)\int_{\Omega^-}\phi_k dx < \int_{\Omega} h\phi_k dx < g(\infty)\int_{\Omega^+}\phi_k dx + g(-\infty)\int_{\Omega^-}\phi_k dx. \quad (1.3)$$

The condition (1.3) is the well-known Landesman-Lazer condition, named after the authors. The result of the paper [1] has since been generalized by a number of authors which include [2–9], to mention a few.

We mention, briefly, few works without the assumption of the Landesman-Lazer condition. The perturbation of a second order linear elliptic problems by nonlinearity without Landesman-Lazer condition was investigated in [10]. The function $g(u)$ was assumed to be a bounded continuous function satisfying

$$g(t)t \leq 0, \quad t \in \mathbb{R}. \quad (1.4)$$

The nonhomogeneous term h was assumed to be an L^∞-function orthogonal to an eigenfunction ϕ in L^2, which corresponds to a simple eigenvalue λ_1. Ha [11] considered the solvability of an operator equation without the Landesman-Lazer condition. The author used a nonlinear Carathéodory function $g(x,u)$ which satisfies the conditions

$$\begin{gathered} |g(x,u)| \leq b(x), \\ ug(x,u) \geq 0, \end{gathered} \quad (1.5)$$

for almost all $x \in \Omega$ and all $u \in \mathbb{R}$, where $b \in L^2(\Omega)$. The solvability of the operator equation is proved under some hypotheses on $g(x,u)$. The nonhomogeneous term h was assumed to be an L^2-function. Iannacci and Nkashama proved existence of solutions to a class of semilinear two-point eigenvalue boundary value problems at resonance without the Landesman-Lazer condition, by imposing the same conditions as in [11] in conjunction with some other hypotheses on g and h. Furthermore, the existence of solution was proved only for the eigenvalue $\lambda = 1$. Assuming a Carathéodory function $f(x,u)$ with some growth restriction and assuming an L^2-function h, Santanilla [12] proved existence of solution to a nonlinear eigenvalue boundary value problem (for eigenvalue $\lambda = 1$) without Landesman-Lazer condition. Du [13] proved the existence of solution for nonlinear second-order two-point boundary value problems, by allowing the eigenvalue λ of the problem to change near the eigenvalues of $m^2\pi^2$ of the problem $y'' + m^2\pi^2 y = 0$, $y(0) = y(1) = 0$. The author did not use the Landesman-Lazer condition and imposed weaker conditions on $g(u)$ than in [12]. Recently, Sanni [14] proved the existence of solution to the same problem considered by Du [13] with $\lambda = m^2\pi^2$ exactly, without assuming the Landesman-Lazer condition. The author assumed that $|g'(u)| \leq C =$ constant and $h \in L^2(0,1)$. Other works without the assumption of Landesman-Lazer condition include [15–21]. We mention that most of the papers on this topic use the methods in [22] and [12]. The method of upper and lower solutions is used in [14]. For several other related resonance problems, we refer the reader to the book of Rădulescu [23].

The current work constitutes further deductions on the problem considered by Landesman and Lazer [1] and is motivated by previous works and by asking if it is possible to obtain a weak solution of (1.1)-(1.2) by setting $u := \phi_k v(x)$. The answer is in the affirmative. The substitution gives rise to a degenerate semilinear elliptic equation. Consequently, we prove the existence of weak solution to the degenerate semilinear elliptic equation in a ϕ_k^2-weight Sobolev's space, by using the Schaefer's fixed point theorem. For information on weighted Sobolev's spaces, the reader is referred to [24, 25]. The current work is significant in that the condition (H) enables a relaxation of the Landesman-Lazer condition (1.3), and the solution u to (1.1)-(1.2) is constructed using the eigenfunctions ϕ_k. Furthermore, the current analysis takes care of the situation where $g(\infty) = g(-\infty) = 0$.

The remaining part of this paper is organized as follows: the weighted Sobolev's spaces used are defined in Section 2. In addition, we use the substitution $u = \phi_k v$ to get the degenerate semilinear elliptic equation in v, from which we give a definition of a weak solution. Furthermore, we state two theorems used in the proof of the existence result. In Section 3, we prove the existence and uniqueness of solution to an auxiliary linear problem. In Section 4, we prove a necessary condition for the existence of solution to (1.1)-(1.2) before proving the existence of solution to (1.1)-(1.2). At the end of Section 4, we prove that $u := \phi_k v$ is in $H_0^1(\Omega)$, provided that $v \in X$. Finally, we give an illustrative example in Section 5 for which our result applies.

2. Preliminaries

We define the following weighted Sobolev's spaces used in this paper:

$$L^2\left(\Omega, \phi_k^2\right) := \left\{w : \Omega \longrightarrow \mathbb{R} \text{ such that } \|w\|_{L^2(\Omega,\phi_k^2)} < \infty\right\}, \tag{2.1}$$

where $\|w\|_{L^2(\Omega,\phi_k^2)} = \sqrt{\int_\Omega \phi_k^2 w^2 dx}$.

$$H^1\left(\Omega, \phi_k^2\right) := \left\{w : \Omega \longrightarrow \mathbb{R} \text{ such that } \|w\|_{H^1(\Omega,\phi_k^2)} < \infty\right\}, \tag{2.2}$$

where $\|w\|_{H^1(\Omega,\phi_k^2)} = \sqrt{\int_\Omega \phi_k^2 w^2 dx + \int_\Omega \phi_k^2 |\nabla w|^2 dx}$.

For brevity, we set $X = H^1(\Omega, \phi_k^2)$.

Set $u := \phi_k v(x)$ in (1.1) to deduce

$$-(\Delta\phi_k + \lambda_k\phi_k)v - \phi_k\Delta v - 2\nabla\phi_k \cdot \nabla v = g(\phi_k v) - h(x) \quad \text{in } \Omega. \tag{2.3}$$

Note that the first term on the left of (2.3) vanishes, multiply (2.3) by ϕ_k and use (1.2) to deduce

$$\begin{gathered} -\nabla \cdot \left(\phi_k^2 \nabla v\right) = \phi_k g(\phi_k v) - \phi_k h(x) \quad \text{in } \Omega, \\ \phi_k v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.4}$$

Thus, if we can prove the existence of solution to (2.4), then $u := \phi_k v$ solves (1.1)-(1.2). Indeed, we will prove that the solution u belongs to the Sobolev space $H_0^1(\Omega)$.

Definition 2.1. We say that $v \in X$ is a weak solution of the problem (2.4) provided

$$\int_{\Omega} \phi_k^2 \nabla v \cdot \nabla \zeta \, dx = \int_{\Omega} \phi_k \zeta g(\phi_k v) dx - \int_{\Omega} \phi_k \zeta h \, dx, \tag{2.5}$$

for each $\zeta \in X$.

Definition 2.2. Let X be a Banach space and $A : X \to X$ a nonlinear mapping. A is called compact provided for each bounded sequence $\{u_k\}_{k=1}^{\infty}$ the sequence $\{A[u_k]\}_{k=1}^{\infty}$ is precompact; that is, there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $\{A[u_{k_j}]\}_{j=1}^{\infty}$ converges in X (see [26]).

The following theorems are applied in this paper.

Theorem 2.3 (Bolzano-Weierstrass). *Every bounded sequence of real numbers has a convergent subsequence (see [27]).*

Theorem 2.4 (Schaefer's Fixed Point Theorem). *Let X be a Banach space and*

$$A : X \longrightarrow X \tag{2.6}$$

a continuous and compact mapping. Suppose further that the set

$$\{u \in X \mid u = \tau A[u] \text{ for some } 0 \leq \tau \leq 1\} \tag{2.7}$$

is bounded. Then A has a fixed point (see [26]).

3. Auxiliary Linear Problem

Consider the linear boundary value problem:

$$Lv := -\nabla \cdot \left(\phi_k^2 \nabla v\right) + \mu \phi_k^2 v = \mu \phi_k^2 s + \phi_k g(\phi_k s) - \phi_k h \quad \text{in } \Omega, \tag{3.1}$$

$$\phi_k v = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

where μ is a strictly positive constant; $s \in L^2(\Omega, \phi_k^2)$, $g(\phi_k s)$, and h are functions of x only.

Theorem 3.1 (a priori estimates). *Let v be a solution of* (3.1)-(3.2). *Then $v \in X$ and we have the estimate*

$$\|v\|_X^2 \leq C\left(\|s\|_{L^2(\Omega,\phi_k^2)}^2 + \|h\|_{L^2(\Omega)}^2 + 1\right) < \infty, \tag{3.3}$$

for some appropriate constant $C > 0$.

Proof. Multiply (3.1) by v, integrate by parts and apply (3.2) to get

$$\begin{aligned}
&\int_\Omega \phi_k^2|\nabla v|^2dx + \mu\int_\Omega \phi_k^2 v^2 dx \\
&\quad = \mu\int_\Omega \phi_k^2 v s\, dx + \int_\Omega v\phi_k g(\phi_k s)dx - \int_\Omega v\phi_k h\, dx \\
&\quad \le \mu\left(\int_\Omega \phi_k^2 v^2 dx\right)^{1/2}\left(\int_\Omega \phi_k^2 s^2 dx\right)^{1/2} \\
&\qquad + \left(\int_\Omega \phi_k^2 v^2 dx\right)^{1/2}\left(\int_\Omega |g(\phi_k s)|^2 dx\right)^{1/2} \\
&\qquad + \left(\int_\Omega \phi_k^2 v^2 dx\right)^{1/2}\left(\int_\Omega h^2 dx\right)^{1/2} \quad \text{(by Hölder's inequality)} \\
&\quad \le 3\epsilon\int_\Omega \phi_k^2 v^2 dx + \frac{1}{4\epsilon}\left(\mu^2\int_\Omega \phi_k^2 s^2 dx + \int_\Omega |g(\phi_k s)|^2 dx + \int_\Omega h^2 dx\right) \\
&\qquad\qquad \text{(by Cauchy's inequality with } \epsilon\text{)}.
\end{aligned} \tag{3.4}$$

Using H, the second term in the bracket on the right side of (3.4) may be estimated as

$$\begin{aligned}
&|g(\phi_k s) - g(0)|^2 \le L^2|\phi_k s|^2 \quad \text{or} \\
&\quad |g(\phi_k s)|^2 \le -|g(0)|^2 + 2|g(\phi_k s)||g(0)| + L^2|\phi_k s|^2 \\
&\qquad \le -|g(0)|^2 + \frac{1}{2}|g(\phi_k s)|^2 + 2|g(0)|^2 + L^2|\phi_k s|^2 \\
&\qquad\qquad \text{(by Young's inequality)}.
\end{aligned} \tag{3.5}$$

Simplifying (3.5), we deduce

$$|g(\phi_k s)| \le C(1 + |\phi_k s|), \tag{3.6}$$

(see [26]) for some constant $C = C(L, |g(0)|)$. Notice that (3.6) implies that

$$\int_\Omega |g(\phi_k s)|^2 dx \le C\left(1 + \|s\|_{L^2(\Omega,\phi_k^2)}\right)^2 < \infty, \tag{3.7}$$

so that $g(\phi_k s) \in L^2(\Omega)$.

Using (3.7) and choosing $\epsilon > 0$ sufficiently small in (3.4) and simplifying, we deduce (3.3). □

Definition 3.2. (i) The bilinear form $B[\cdot,\cdot]$ associated with the elliptic operator L defined by (3.1) is

$$B[v,\zeta] := \int_\Omega \phi_k^2 \nabla v \cdot \nabla \zeta \, dx + \mu \int_\Omega \phi_k^2 v \zeta \, dx, \tag{3.8}$$

for $v, \zeta \in X$,

(ii) $v \in X$ is called a weak solution of the boundary value problem (3.1)-(3.2) provided

$$B[u,\zeta] = \left(\mu\phi_k^2 s + \phi_k g(\phi_k s) - \phi_k h, \zeta\right), \tag{3.9}$$

for all $\zeta \in X$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

Theorem 3.3. *$B[u,v]$ satisfies the hypotheses of the Lax-Milgram theorem precisely. In other words, there exists constants α, β such that*

(i) $|B[v,\zeta]| \leq \alpha \|v\|_X \|\zeta\|_X$,

(ii) $\beta \|v\|_X^2 \leq B[v,v]$,

for all $v, \zeta \in X$.

Proof. We have

$$\begin{aligned} |B[v,\zeta]| = \left| \int_\Omega \phi_k^2 \nabla v \cdot \nabla \zeta \, dx + \mu \int_\Omega \phi_k^2 v \zeta \, dx \right| &\leq \mu \left(\int_\Omega \phi_k^2 v^2 dx \right)^{1/2} \left(\int_\Omega \phi_k^2 \zeta^2 dx \right)^{1/2} \\ &+ \left(\int_\Omega \phi_k^2 |\nabla v|^2 dx \right)^{1/2} \left(\int_\Omega \phi_k^2 |\nabla \zeta|^2 dx \right)^{1/2} \quad \text{(by Hölder's inequality)} \\ &\leq \alpha \|v\|_X \|\zeta\|_X, \end{aligned} \tag{3.10}$$

for appropriate constant $\alpha > 0$. This proves (i).

We now proof (ii). We readily check that

$$\beta \|v\|_X^2 \leq \int_\Omega \phi_k^2 |\nabla v|^2 dx + \mu \int_\Omega \phi_k^2 v^2 dx = B[v,v], \tag{3.11}$$

for some constant $\beta > 0$. We can for example take $\beta = \min\{1, \mu\}$. □

Theorem 3.4. *There exists unique weak solution to the degenerate linear boundary value problem* (3.1)-(3.2).

Proof. The hypothesis on h and (3.7) imply that $g(\phi_k s) - h \in L^2(\Omega)$. For fixed $g(\phi_k s) - h$, set $\langle \mu\phi_k^2 s + \phi_k g(\phi_k s) - \phi_k h, \zeta \rangle := (\mu\phi_k^2 s + \phi_k g(\phi_k s) - \phi_k h, \zeta)_{L^2(\Omega)}$ for all $\zeta \in X$ (where $\langle,\cdot,\rangle$ denotes the pairing of X with its dual). This is a bounded linear functional on $L^2(\Omega)$ and thus on X. Lax-Milgram theorem (see, e.g., [26]) can be applied to find a unique function $v \in X$

satisfying

$$B[v,\zeta] = \left\langle \mu\phi_k^2 s + \phi_k g(\phi_k s) - \phi_k h, \zeta \right\rangle, \tag{3.12}$$

for all $\zeta \in X$. Consequently, v is the unique weak solution of the problem (3.1)-(3.2). □

4. Main Results

Theorem 4.1. *The necessary condition that $u \in H_0^1(\Omega)$ be a weak solution to* (1.1)-(1.2) *is that*

$$\int_\Omega g(u)\phi_k dx = \int_\Omega h\phi_k dx. \tag{4.1}$$

Proof. Suppose $u \in H_0^1(\Omega)$ is a weak solution of (1.1)-(1.2). For a test function ϕ_k, using integration by parts, we have:

$$\begin{aligned}
&\int_\Omega \Delta u\phi_k dx + \lambda_k \int_\Omega u\phi_k dx + \int_\Omega g(u)\phi_k dx \\
&\quad = -\int_\Omega \nabla u \cdot \nabla\phi_k + \lambda_k \int_\Omega u\phi_k dx + \int_\Omega g(u)\phi_k dx \\
&\quad = \int_\Omega u(\Delta\phi_k + \lambda_k\phi_k) dx + \int_\Omega g(u)\phi_k dx = \int_\Omega h\phi_k dx,
\end{aligned} \tag{4.2}$$

from which (4.1) follows, since $\Delta\phi_k + \lambda_k\phi_k = 0$. □

Theorem 4.2. *Let the condition* (4.1) *of Theorem 4.1 holds. Then there exists a weak solution to the problem* (2.4).

Proof. The proof is split in seven steps.

Step 1. A fixed point argument to (2.4) is

$$\begin{gathered}
-\nabla \cdot \left(\phi_k^2 \nabla w\right) + \mu\phi_k^2 w = \mu\phi_k^2 v + \phi_k g(\phi_k v) - \phi_k h(x) \quad \text{in } \Omega, \\
\phi_k w = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{4.3}$$

Define a mapping

$$A : X \longrightarrow X \tag{4.4}$$

by setting $A[v] = w$ whenever w is derived from v via (4.3). We claim that A is a continuous and compact mapping. Our claim is proved in the next two steps.

Step 2. Choose $v, \tilde{v} \in X$, and define $A[v] = w$, $A[\tilde{v}] = \tilde{w}$. For two solutions $w, \tilde{w} \in X$ of (4.3), we have

$$\begin{gathered} -\nabla \cdot \left[\phi_k^2 \nabla (w - \tilde{w})\right] + \mu \phi_k^2 (w - \tilde{w}) = \mu \phi_k^2 (v - \tilde{v}) + \phi_k g(\phi_k v) - \phi_k g(\phi_k \tilde{v}) \quad \text{in } \Omega, \\ \phi_k (w - \tilde{w}) \ = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.5}$$

Using (4.5), we obtain an analogous estimate to (3.4), namely:

$$\begin{aligned} &\int_\Omega \phi_k |\nabla w - \nabla \tilde{w}|^2 dx + \mu \int_\Omega \phi_k^2 |w - \tilde{w}|^2 \\ &\quad \le 3\epsilon \int_\Omega \phi_k^2 |w - \tilde{w}|^2 dx \\ &\qquad + \frac{1}{4\epsilon} \left(\mu^2 \int_\Omega \phi_k |v - \tilde{v}|^2_{L^2(\Omega, \phi_k^2)} + \int_\Omega \left| g(\phi_k v) - g(\phi_k \tilde{v}) \right|^2 dx \right). \end{aligned} \tag{4.6}$$

Now

$$\int_\Omega \left| g(\phi_k v) - g(\phi_k \tilde{v}) \right|^2 dx \le \int_\Omega \phi_k^2 L^2 |v - \tilde{v}|^2 dx, \tag{4.7}$$

using the condition (H). We may now use (4.7) in (4.6) and simplify to deduce

$$\|A[v] - A[\tilde{v}]\|_X = \|w - \tilde{w}\|_X \le C \|v - \tilde{v}\|_{L^2(\Omega, \phi_k^2)} \le C \|v - \tilde{v}\|_X, \tag{4.8}$$

for some constant $C > 0$. Thus, the mapping A is Lipschitz continuous, and hence continuous.

Step 3. Let $\{v_k\}_{k=1}^\infty$ be a bounded sequence in X. By Bolzano-Weierstrass theorem, it has a convergent subsequence, say $\{v_{k_j}\}_{j=1}^\infty$. Define

$$v := \lim_{k_j \to \infty} v_{k_j}. \tag{4.9}$$

Using (4.8)-(4.9), we deduce

$$\lim_{k_j \to} \left\| A\left[v_{k_j}\right] - A[v] \right\|_X \le \lim_{k_j \to \infty} C \left\| v_{k_j} - v \right\|_X = 0. \tag{4.10}$$

Thus, $A[v_{k_j}] \to A[v]$ in X. Therefore, A is compact.

Step 4. Define a set $K := \{p \in X : p = \tau A[p]$ for some $0 \leq \tau \leq 1\}$. We will show that K is a bounded set. Let $v \in K$. Then $v = \tau A[v]$ for some $\tau \in [0,1]$. Thus, we have $v/\tau = A[v]$. By the definition of the mapping A, $w = v/\tau$ is the solution of the problem

$$\begin{gathered} -\nabla \cdot \left[\phi_k^2 \nabla\left(\frac{v}{\tau}\right)\right] + \mu\phi_k^2 \frac{v}{\tau} = \mu\phi_k^2 v + \phi_k g(\phi_k v) - \phi_k h(x) \quad \text{in } \Omega, \\ \phi_k \frac{v}{\tau} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.11}$$

Now, (4.11) are equivalent to

$$\begin{gathered} -\nabla \cdot \left(\phi_k^2 \nabla v\right) + \mu\phi_k^2 v = \mu\tau\phi_k^2 v + \tau\phi_k g(\phi_k v) - \tau\phi_k h(x) \quad \text{in } \Omega, \\ \phi_k v = 0 \ \text{ on } \ \partial\Omega. \end{gathered} \tag{4.12}$$

Using (4.12) we have an analogous estimate to (3.3) of Theorem 3.1, namely:

$$\|v\|_X^2 \leq \tau C\left(\|v\|^2_{L^2(\Omega,\phi_k^2)} + \|h\|^2_{L^2(\Omega)} + 1\right). \tag{4.13}$$

Choosing $\tau \in [0,1]$ sufficiently small in (4.13) and simplifying, we conclude that

$$\|v\|_X \leq C\sqrt{\|h\|^2_{L^2(\Omega)} + 1} < \infty \tag{4.14}$$

for some constant $C > 0$. Equation (4.14) implies that the set K is bounded, since v was arbitrarily chosen.

Since the mapping A is continuous and compact and the set K is bounded, by Schaefer's fixed point theorem (see, e.g., [26]), the mapping A has a fixed point in X.

Step 5. Write $\phi_k v_0 = \phi_k v|_{\partial\Omega} = 0$. For $m = 0, 1, 2, \ldots$, inductively define $v_{m+1} \in X$ to be the unique weak solution of the linear boundary value problem

$$-\nabla \cdot \left(\phi_k^2 \nabla v_{m+1}\right) + \mu\phi_k^2 v_{m+1} = \mu\phi_k^2 v_m + \phi_k g(\phi_k v_m) - \phi_k h(x) \quad \text{in } \Omega, \tag{4.15}$$

$$\phi_k v_{m+1} = 0 \quad \text{on } \partial\Omega. \tag{4.16}$$

Clearly, our definition of $v_{m+1} \in X$ as the unique weak solution of (4.15)-(4.16) is justified by Theorem 3.4. Hence, by the definition of the mapping A, we have for $m = 0, 1, 2, \ldots$:

$$v_{m+1} = A[v_m]. \tag{4.17}$$

Since A has a fixed point in X, there exists $v \in X$ such that

$$\lim_{m\to\infty} v_{m+1} = \lim_{m\to\infty} A[v_m] = A[v] = v. \tag{4.18}$$

Step 6. Using (4.15)-(4.16), we obtain an analogous estimate to (3.3), namely:

$$\begin{aligned}\|v_{m+1}\|_X^2 &\le C\Big(\|v_m\|^2_{L^2(\Omega,\phi_k^2)} + \|h\|^2_{L^2(\Omega)} + 1\Big)\\ &\le C\Big(\|v_m\|_X^2 + \|h\|^2_{L^2(\Omega)} + 1\Big)\end{aligned} \tag{4.19}$$

for some appropriate constant $C > 0$. Using (4.18), we take the limit on the right side of (4.19) to deduce that

$$\sup_m \|v_m\|_X < \infty. \tag{4.20}$$

Equation (4.20) implies the existence of a subsequence $\{v_{m_j}\}_{j=1}^{\infty}$ converging weakly in X to $v \in X$.

Furthermore, using (3.7), we deduce

$$\int_\Omega \big|g(\phi_k v_m)\big|^2 dx \le C\Big(1 + \|v_m\|^2_{L^2(\Omega,\phi_k^2)}\Big)^2. \tag{4.21}$$

Again, we use (4.18) to obtain the limit on the right side of (4.21) to deduce that

$$\sup_m \big\|g(\phi_k v_m)\big\|_{L^2(\Omega)} < \infty. \tag{4.22}$$

Equation (4.22) implies the existence of a subsequence $\{g(\phi_k v_{m_j})\}_{j=1}^{\infty}$ converging weakly in $L^2(\Omega)$ to $g(\phi_k v)$ in $L^2(\Omega)$.

Step 7. Finally, we verify that v is a weak solution of (2.4). For brevity, we take the subsequences of the last step as $\{v_m\}_{m=1}^{\infty}$ and $\{g(\phi_k v_m)\}_{m=1}^{\infty}$. Fix $\zeta \in X$. Multiply (4.15) by ζ, integrate by parts and apply (4.16) to get

$$\int_\Omega \phi_k^2 \nabla v_{m+1} \cdot \nabla\zeta\, dx + \mu \int_\Omega \phi_k^2 v_{m+1}\zeta\, dx = \mu \int_\Omega \phi_k^2 v_m \zeta\, dx + \int_\Omega \zeta\phi_k g(\phi_k v_m) dx - \int_\Omega \zeta\phi_k h\, dx. \tag{4.23}$$

Using the deductions of the last step, we let $m \to \infty$ in (4.23) to obtain

$$\int_\Omega \phi_k \nabla v \cdot \nabla\zeta\, dx + \mu \int_\Omega \phi_k^2 v\zeta\, dx = \mu \int_\Omega \phi_k^2 v \zeta\, dx + \int_\Omega \zeta\phi_k g(\phi_k v) dx - \int_\Omega \zeta\phi_k h\, dx, \tag{4.24}$$

from which canceling the terms in μ, we obtain (2.5) as desired. □

Theorem 4.3. *Let $v \in X$ be the solution of* (3.1)-(3.2)*. Then, the solution $u := \phi_k v$ of* (1.1)-(1.2) *belongs to $H_0^1(\Omega)$, and we have the estimate*

$$\|u\|_{H_0^1(\Omega)} \leq C\|v\|_X, \tag{4.25}$$

for some constant $C > 0$.

Proof. We split the proof in two steps.

Step 1. Recall that ϕ_k satisfies the equations:

$$\Delta\phi_k + \lambda_k\phi_k = 0 \quad \text{in } \Omega \in \mathbb{R}^n, \tag{4.26}$$

$$\phi_k = 0 \quad \text{on } \partial\Omega. \tag{4.27}$$

Multiplying (4.26) by $v^2\phi_k$, integrating by parts and applying (4.27) we compute

$$\begin{gathered}\int_\Omega v^2\phi_k\Delta\phi_k dx + \lambda_k\int_\Omega v^2\phi_k^2 dx = 0 \quad \text{or}\\ \int_\Omega \nabla\left(v^2\phi_k\right)\cdot\nabla\phi_k dx = \lambda_k\int_\Omega v^2\phi_k^2 dx \quad \text{or}\\ \int_\Omega \left|\nabla\phi_k\right|^2 v^2 dx = \lambda_k\int_\Omega v^2\phi_k^2 dx - 2\int_\Omega \phi_k v\nabla v\cdot\nabla\phi_k dx\end{gathered} \tag{4.28}$$

$$\leq \lambda_k\int_\Omega v^2\phi_k^2 dx + \epsilon\int_\Omega \left|\nabla\phi_k\right|^2 v^2 dx + \frac{1}{\epsilon}\int_\Omega \phi_k^2|\nabla v|^2 dx, \tag{4.29}$$

by Cauchy's inequality with ϵ. Choosing $\epsilon > 0$ sufficiently small in (4.29) and simplifying, we deduce

$$\int_\Omega \left|\nabla\phi_k\right|^2 v^2 dx \leq C\|v\|_X^2, \tag{4.30}$$

for some constant $C > 0$.

Step 2. We have

$$\begin{aligned}\int_\Omega u^2 dx &= \int_\Omega \phi_k^2 v^2 dx,\\ \int_\Omega |\nabla u|^2 dx &= \int_\Omega \left|\nabla\left(\phi_k v\right)\right|^2 dx = \int_\Omega \left|\nabla\phi_k v + \phi_k\nabla v\right|^2 dx\\ &\leq 2\int_\Omega \left|\nabla\phi_k\right|^2 v^2 dx + 2\int_\Omega \phi_k^2|\nabla v|^2 dx \leq C\|v\|_X^2, \ (\text{using } (4.30))\end{aligned} \tag{4.31}$$

for some constant $C > 0$. Thus, $u \in H^1(\Omega)$. Hence, by a Sobolev's embedding theorem (see [26, page 269]), we have that $u \in H_0^1(\Omega)$, since $u|_{\partial\Omega} = 0$. □

5. Illustrative Example

Consider the following special case for $n = 1$:

$$u'' + u - 2u = 1 \quad \text{in } (0, \pi), \qquad u(0) = u(\pi) = 0. \tag{5.1}$$

In this case, the eigenfunction $\phi_k = \sin x$, $g(u) = -2u$, and $h = 1$. Clearly $g(u)$ is Lipschitz continuous and $h \in L^2(\Omega)$. Provided the necessary condition

$$-2\int_0^\pi u \sin x \, dx = \int_0^\pi \sin x \, dx \tag{5.2}$$

is satisfied; Theorems 4.2 and 4.3 ensure the existence of a solution $u := \phi_k v(x) \in H_0^1(\Omega)$ of the problem (5.1). Now, the problem (5.1) admits the solution

$$u = \frac{\sinh(\pi - x) + \sinh x}{\sinh \pi} - 1. \tag{5.3}$$

Using (5.3) in (5.2), it is not difficult to verify that the necessary condition

$$-2\int_0^\pi \left(\frac{\sinh(\pi - x) + \sinh x}{\sinh \pi} - 1 \right) \sin x \, dx = \int_0^\pi \sin x \, dx = 2 \tag{5.4}$$

is satisfied.

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