

*Research Article*

# Oscillation Criteria for a Class of Second-Order Neutral Delay Dynamic Equations of Emden-Fowler Type

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We establish some new oscillation criteria for the second-order neutral delay dynamic equations of Emden-Fowler type,  $[a(t)(x(t) + r(t)x(\tau(t)))^\Delta]^\Delta + p(t)x^\gamma(\delta(t)) = 0$ , on a time scale unbounded above. Here  $\gamma > 0$  is a quotient of odd positive integers with  $a$  and  $p$  being real-valued positive functions defined on  $\mathbb{T}$ . Our results in this paper not only extend and improve the results in the literature but also correct an error in one of the references.

## 1. Introduction

The study of dynamic equations on time scales, which goes back to its founder Hilger [1], is an area of mathematics that has recently received a lot of attention. It was partly created in order to unify the study of differential and difference equations. Many results concerning differential equations are carried over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies and helps avoid proving results twice—once for differential equations and once again for difference equations.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [2]), that is, when

$\mathbb{T} = \mathbb{R}$ ,  $\mathbb{T} = \mathbb{N}$ , and  $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ , where  $q > 1$ . Many other interesting time scales exist, and they give rise to many applications (see [3]). Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in, for example, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [3]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in *New Scientist* [4] discusses several possible applications. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [5] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [3], summarizes and organizes much of time scale calculus; see also the book by Bohner and Peterson [6] for advances results of dynamic equations on time scales.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales unbounded above and neutral differential equations; we refer the reader to the papers [7–19]. Some authors are especially interested in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of first and second-order linear and nonlinear neutral functional dynamic equations on time scales; we refer to the articles [20–28].

Agarwal et al. [7] considered the second-order delay dynamic equations

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{T} \quad (1.1)$$

and established some sufficient conditions for oscillation of (1.1). Şahiner [11] studied the second-order nonlinear delay dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T} \quad (1.2)$$

and obtained some sufficient conditions for oscillation by employing Riccati transformation technique. Zhang and Zhu [13] examined the second-order dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x(t - \tau)) = 0, \quad t \in \mathbb{T}, \quad (1.3)$$

and by using comparison theorems, they proved that oscillation of (1.3) is equivalent to the oscillation of the nonlinear dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0, \quad t \in \mathbb{T} \quad (1.4)$$

and established some sufficient conditions for oscillation by applying the results established in [15]. Erbe et al. [16] investigated the oscillation of the second-order nonlinear delay dynamic equations

$$\left(r(t)x^\Delta(t)\right)^\Delta + p(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T} \quad (1.5)$$

and by employing the generalized Riccati technique, they established some new sufficient conditions which ensure that every solution of (1.5) oscillates or converges to zero. Mathsen et al. [20] investigated the first-order neutral delay dynamic equations

$$[y(t) - r(t)y(\tau(t))]^\Delta + p(t)y(\delta(t)) = 0, \quad t \in \mathbb{T} \quad (1.6)$$

and established some new oscillation criteria which as a special case involve some well-known oscillation results for first-order neutral delay differential equations. Zhu and Wang [21] studied the nonoscillatory solutions to neutral dynamic equations

$$[y(t) + p(t)y(g(t))]^\Delta + f(t, x(h(t))) = 0, \quad t \in \mathbb{T} \quad (1.7)$$

and gave a classification scheme for the eventually positive solutions of (1.7). Agarwal et al. [22], Şahiner [23], Saker et al. [24–26], Wu et al. [27], and Zhang and Wang [28] considered the second-order nonlinear neutral delay dynamic equations

$$\left(r(t)\left((y(t) + p(t)y(\tau(t)))^\Delta\right)^\gamma\right)^\Delta + f(t, y(\delta(t))) = 0, \quad t \in \mathbb{T}, \quad (1.8)$$

where  $\gamma > 0$  is a quotient of odd positive integers, the delay function  $\tau$  and  $\delta$  satisfy  $\tau : \mathbb{T} \rightarrow \mathbb{T}$  and  $\delta : \mathbb{T} \rightarrow \mathbb{T}$  for all  $t \in \mathbb{T}$ , and  $r$  and  $p$  are real-valued positive functions defined on  $\mathbb{T}$ , and

$$(h_1) \quad r(t) > 0, \int_{t_0}^{\infty} (1/r(t))^{1/\gamma} \Delta t = \infty, \text{ and } 0 \leq p(t) < 1;$$

$$(h_2) \quad f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous function such that } uf(u) > 0 \text{ for all } u \neq 0, \text{ and there exists a nonnegative function } q \text{ defined on } \mathbb{T} \text{ such that } |f(t, u)| \geq q(t)|u|^\gamma.$$

By employing different Riccati transformation technique, the authors established some oscillation criteria for all solutions of (1.8).

Recently, some authors have been interested in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of Emden-Fowler type dynamic equations on time scales, differential equations, and difference equations; see, for example, [29–47].

Han et al. [32] studied the second-order Emden-Fowler delay dynamic equations

$$x^{\Delta\Delta}(t) + p(t)x^\gamma(\tau(t)) = 0, \quad t \in \mathbb{T} \quad (1.9)$$

and established some sufficient conditions for oscillation of (1.9) and extended the results given in [7].

Saker [34] studied the second-order superlinear neutral delay dynamic equation of Emden-Fowler type

$$\left[a(t)(y(t) + r(t)y(\tau(t)))^\Delta\right]^\Delta + p(t)|y(\delta(t))|^\gamma \text{sign } y(\delta(t)) = 0 \quad (1.10)$$

on a time scale  $\mathbb{T}$ .

The author assumes that

- (A<sub>1</sub>)  $\gamma > 1$ ;
- (A<sub>2</sub>) the delay functions  $\tau$  and  $\delta$  satisfy  $\tau : \mathbb{T} \rightarrow \mathbb{T}$ ,  $\delta : \mathbb{T} \rightarrow \mathbb{T}$ ,  $\tau(t) \leq t$ ,  $\delta(t) \leq t$  for all  $t \in \mathbb{T}$ , and  $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \delta(t) = \infty$ ;
- (A<sub>3</sub>)  $a, r$  and  $p$  are positive rd-continuous functions defined on  $\mathbb{T}$  such that  $a^\Delta(t) \geq 0$ ,  $\int_{t_0}^\infty (\Delta t/a(t)) = \infty$ , and  $0 \leq r(t) < 1$ .

The main result for the oscillation of (1.10) in [34] is the following.

**Theorem 1.1** (see, [34, Theorem 3.1]). *Assume that (A<sub>1</sub>)–(A<sub>3</sub>) hold. Furthermore, assume that*

$$\int_{t_0}^\infty p(t)(1 - r(\delta(t)))^\gamma \delta^\gamma(t) \Delta t = \infty, \quad (1.11)$$

and there exists a  $\Delta$ -differentiable function  $\eta$  such that for all constants  $M > 0$ ,

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[ \eta(s)p(s)(1 - r(\delta(s)))^\gamma \left( \frac{\delta(s)}{s} \right)^\gamma - \frac{a(s)(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1} \eta(s)} \right] \Delta s = \infty. \quad (1.12)$$

Then every solution of (1.10) is oscillatory.

We note that in [34], the author gave an open problem, that is, how to establish oscillation criteria for (1.10) when  $\gamma < 1$ .

In [35], the author examined the oscillation of the second-order neutral delay dynamic equations

$$(x(t) - rx(\tau(t)))^{\Delta\Delta} + H(t, x(h_1(t))) = 0, \quad t \in \mathbb{T}. \quad (1.13)$$

The author assumes that

- (H<sub>1</sub>)  $\tau$  and  $h_1 \in C_{\text{rd}}(\mathbb{T}, \mathbb{T})$ ,  $\tau(t) < t$ ,  $\tau(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ,  $h_1(t) < t$ ,  $h_1(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , and  $0 \leq r < 1$ ;
- (H<sub>2</sub>)  $H \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$  for each  $t \in \mathbb{T}$  which are nondecreasing in  $u$ , and  $H(t, u) > 0$ , for  $u > 0$ ;
- (H<sub>3</sub>)  $|H(t, u)| \geq \alpha(t)|u|^\lambda$ , where  $\alpha(t) \geq 0$ , and  $0 \leq \lambda = p/q < 1$  with  $p, q$  being odd integers.

The main result for the oscillation of (1.13) in [35] is the following.

**Theorem 1.2** (see, [35, Theorem 3.4]). *Assume that (H<sub>1</sub>)–(H<sub>3</sub>) hold. If for all sufficiently large  $t_1 \geq t_0$ ,*

$$\int_{t_1}^\infty \alpha(s)(\tau(h_1(s)))^\lambda \Delta s = \infty, \quad (1.14)$$

then (1.13) oscillates.

We find that the conclusion of this theorem is wrong. The following is a counter example of this theorem.

*Counter Example.* Consider the second-order differential equation

$$\left(x(t) - \frac{1}{3}x\left(\frac{t}{3}\right)\right)'' + \left(\frac{1}{27}e^{-1/3} - e^{-1/3}e^{-2t/3}\right)x^{1/3}(t-1) = 0, \quad t \geq t_0. \quad (1.15)$$

Let  $\alpha(t) = e^{-1/3}/27 - e^{-1/3}e^{-2t/3}$ ,  $r(t) = 1/3$ ,  $\tau(t) = t/3$ , and  $h_1(t) = t - 1$ ,  $\lambda = 1/3$ . For all sufficiently large  $t_1 \geq t_0$ , we find that

$$\int_{t_1}^{\infty} \alpha(s)(\tau(h_1(s)))^\lambda \Delta s = \int_{t_1}^{\infty} \alpha(s)(\tau(h_1(s)))^\lambda ds = \int_{t_1}^{\infty} \left(\frac{1}{27}e^{-1/3} - e^{-1/3}e^{-2s/3}\right) \left(\frac{s-1}{3}\right)^{1/3} ds. \quad (1.16)$$

It is easy to see that

$$\begin{aligned} \int_{t_1}^{\infty} \frac{1}{27}e^{-1/3} \left(\frac{s-1}{3}\right)^{1/3} ds &= \infty, \\ \int_{t_1}^{\infty} e^{-2s/3} \left(\frac{s-1}{3}\right)^{1/3} ds &\leq \int_{t_1}^{\infty} e^{-2s/3} s^{1/3} ds. \end{aligned} \quad (1.17)$$

Integrating by parts, we obtain

$$\int_{t_1}^{\infty} e^{-2s/3} s^{1/3} ds = -t_1^{1/3} \left(\frac{3}{2}e^{-2t_1/3}\right) + \frac{1}{2} \int_{t_1}^{\infty} e^{-2s/3} s^{-2/3} ds < \infty. \quad (1.18)$$

Hence

$$\int_{t_1}^{\infty} \alpha(s)(\tau(h_1(s)))^\lambda ds = \infty. \quad (1.19)$$

Therefore, by the above theorem, (1.15) is oscillatory. However,  $x(t) = e^{-t}$  is a positive solution of (1.15). Therefore, the above theorem is wrong. Tracing the error to its source, we find that the following false assertion was used in the proof of the aforementioned theorem.

*Assertion A*

If  $x$  is an eventually positive solution of (1.13), then  $z(t) = x(t) - r(t)x(\tau(t))$  is eventually positive.

Abdalla [37] studied the second-order superlinear neutral delay differential equations

$$\left[ a(t)(y(t) + r(t)y(\tau(t)))' \right]' + p(t)|y(\delta(t))|^{\gamma} \text{sign } y(\delta(t)) = 0, \quad t \in [t_0, \infty). \quad (1.20)$$

Most of the oscillation criteria are unsatisfactory since additional assumptions have to be imposed on the unknown solutions. Also, the author proved that if

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} = \int_{t_0}^{\infty} p(t)dt = \infty, \quad (1.21)$$

then every solution of (1.20) oscillates for every  $r(t) > 0$ , but one can easily see that this result cannot be applied when  $p(t) = t^{-\alpha}$  for  $\alpha > 1$ .

Lin [38] considered the second-order nonlinear neutral differential equations

$$[x(t) - p(t)x(t - \tau)]'' + q(t)f(x(t - \sigma)) = 0, \quad t \geq 0, \quad (1.22)$$

where  $0 \leq p(t) \leq 1$ ,  $q(t) \geq 0$ ,  $\tau, \sigma > 0$ . The author investigated the oscillation for (1.22) when  $f$  is superlinear.

Wong [46, 47] studied the second-order neutral differential equations

$$[y(t) - py(t - \tau)]'' + q(t)f(y(t - \sigma)) = 0, \quad t \geq 0, \quad (1.23)$$

$q \in C[0, \infty)$ ,  $q(t) \geq 0$ ,  $f \in C^1(-\infty, \infty)$ ,  $yf(y) > 0$  whenever  $y \neq 0$ ,  $f'(y) \geq 0$  for all  $y$ , and  $0 < p < 1$ ,  $\tau > 0$ ,  $\sigma > 0$  are constants.

The main results for the oscillation of (1.23) in [46, 47] are the following.

**Theorem 1.3** (see, [46, 47, Theorem 1]). *Suppose that  $f$  is superlinear. Then a solution of (1.23) is either oscillatory or tends to zero if and only if*

$$\int_{t_0}^{\infty} tq(t)dt = \infty. \quad (1.24)$$

**Theorem 1.4** (see, [46, 47, Theorem 2]). *Suppose that  $f$  is sublinear and in addition satisfies*

$$f(uv) \geq f(u)f(v), \quad uv \geq 0. \quad (1.25)$$

*Then a solution of (1.23) is either oscillatory or tends to zero if and only if*

$$\int_{t_0}^{\infty} f(t)q(t)dt = \infty. \quad (1.26)$$

Li and Saker [40] investigated the second-order sublinear neutral delay difference equations

$$\Delta(a_n \Delta(x_n + p_n x_{n-\tau})) + q_n x_{n-\sigma}^\gamma = 0, \quad (1.27)$$

where  $0 < \gamma < 1$  is a quotient of odd positive integers,  $a_n > 0$ ,  $\Delta a_n \geq 0$ ,  $\sum_{n=0}^{\infty} 1/a_n = \infty$ ,  $0 \leq p_n < 1$ , for all  $n \geq 0$  and  $q_n \geq 0$ .

The main result for the oscillation of (1.27) in [40] is the following.

**Theorem 1.5** (see, [40, Theorem 2.1]). *Assume that there exists a positive sequence  $\{\rho_n\}$  such that for every  $\alpha \geq 1$ ,*

$$\limsup_{n \rightarrow \infty} \sum_{l=0}^n \left[ \rho_l Q_l - \frac{a_{l-\sigma} (\alpha(l+1-\sigma))^{1-\gamma} (\Delta \rho_l)^2}{4\gamma \rho_l} \right] = \infty, \tag{1.28}$$

where  $Q_n = q_n(1 - p_{n-\sigma})^\gamma$ . Then every solution of (1.27) oscillates.

Yildiz and Öcalan [41] studied the higher-order sublinear neutral delay difference equations of the type

$$\Delta^m (y_n + p_n y_{n-1}) + q_n y_{n-k}^\alpha = 0, \quad n \in \mathbb{N}, \tag{1.29}$$

where  $0 < \alpha < 1$  is a ratio of odd positive integers. The authors established some oscillation criteria of (1.29).

The main results for the oscillation of (1.29) when  $m = 2$  in [41] are the following.

**Theorem 1.6** (see, [41, Theorem 2.1(a),  $m = 2$ ]). *Assume that  $0 \leq p_n < 1$ , and*

$$\sum_{n=0}^{\infty} q_n [(1 - p_{n-k})n]^\alpha = \infty. \tag{1.30}$$

Then all solutions of (1.29) are oscillatory.

**Theorem 1.7** (see, [41, Theorem 2.2,  $m = 2$ ]). *Assume that  $-1 < -p_2 \leq p_n \leq 0$ , where  $p_2 > 0$  is a constant, and*

$$\sum_{n=0}^{\infty} q_n n^\alpha = \infty. \tag{1.31}$$

Then every solution of (1.29) either oscillates or tends to zero as  $n \rightarrow \infty$ .

Cheng [42] considered the oscillation of the second-order nonlinear neutral difference equations

$$\Delta (p_n (\Delta (x_n + c_n x_{n-\tau}))^\gamma) + q_n x_{n-\sigma}^\beta = 0 \tag{1.32}$$

and established some oscillation criteria of (1.32) by means of Riccati transformation techniques.

Following this trend, in this paper, we are concerned with oscillation of the second-order neutral delay dynamic equations of Emden-Fowler type

$$\left[ a(t)(x(t) + r(t)x(\tau(t)))^\Delta \right]^\Delta + p(t)x^\gamma(\delta(t)) = 0, \quad t \in \mathbb{T}. \tag{1.33}$$

As we are interested in oscillatory behavior, we assume throughout this paper that the given time scales  $\mathbb{T}$  are unbounded above; that is, it is a time scale interval of the form  $[t_0, \infty)$  with  $t_0 \in \mathbb{T}$ .

We assume that  $\gamma > 0$  is a quotient of odd positive integers, the delay functions  $\tau$  and  $\delta$  satisfy  $\tau : \mathbb{T} \rightarrow \mathbb{T}$ ,  $\delta : \mathbb{T} \rightarrow \mathbb{T}$ ,  $\tau(t) \leq t$ ,  $\delta(t) \leq t$  for all  $t \in \mathbb{T}$ , and  $\lim_{t \rightarrow \infty} \tau(t) = \lim_{t \rightarrow \infty} \delta(t) = \infty$ ;  $a, r$  and  $p$  are real-valued rd-continuous functions defined on  $\mathbb{T}$ ,  $a(t) > 0$ ,  $p(t) > 0$ ,  $\int_{t_0}^{\infty} \Delta t/a(t) = \infty$ .

We note that if  $\mathbb{T} = \mathbb{R}$ , then  $\sigma(t) = t$ ,  $\mu(t) = 0$ ,  $x^\Delta(t) = x'(t)$ , and (1.33) becomes the second-order nonlinear delay differential equation

$$[a(t)(x(t) + r(t)x(\tau(t)))']' + p(t)x^\gamma(\delta(t)) = 0, \quad t \in \mathbb{R}. \quad (1.34)$$

If  $\mathbb{T} = \mathbb{Z}$ , then  $\sigma(t) = t + 1$ ,  $\mu(t) = 1$ ,  $x^\Delta(t) = \Delta x(t) = x(t + 1) - x(t)$ , and (1.33) becomes the second-order nonlinear delay differential equation

$$\Delta[a(t)\Delta(x(t) + r(t)x(\tau(t)))] + p(t)x^\gamma(\delta(t)) = 0, \quad t \in \mathbb{Z}. \quad (1.35)$$

In the case of  $\gamma > 1$ , (1.33) is the prototype of a wide class of nonlinear dynamic equations called Emden-Fowler sublinear dynamic equations, and if  $\gamma < 1$ , (1.33) is the prototype of dynamic equations called Emden-Fowler superlinear dynamic equations. It is interesting to study (1.33) because the continuous version, that is, (1.34), has several physical applications; see, for example, [1, 39], and when  $t$  is a discrete variable, it is (1.35), and it is also important in applications.

## 2. Main Results

In this section, we give some new oscillation criteria of (1.33). In order to prove our main results, we will use the formula

$$((x(t))^\gamma)^\Delta = \gamma \int_0^1 [hx^\sigma(t) + (1-h)x(t)]^{\gamma-1} x^\Delta(t) dh, \quad (2.1)$$

which is a simple consequence of Keller's chain rule [3, Theorem 1.90]. Also, we need the following auxiliary results.

For the sake of convenience, we assume that

$$z(t) = x(t) + r(t)x(\tau(t)), \quad R(t, t_*) = a(t) \int_{t_*}^t \frac{\Delta s}{a(s)}, \quad \alpha(t, t_*) = \frac{\int_{t_*}^{\delta(t)} \Delta s/a(s)}{\int_{t_*}^t \Delta s/a(s)}, \quad t_* \geq t_0. \quad (2.2)$$

**Lemma 2.1.** *Assume that (1.11) holds,  $a^\Delta(t) \geq 0$ , and  $0 \leq r(t) < 1$ . Then an eventually positive solution  $x$  of (1.33) eventually satisfies that*

$$z(t) \geq tz^\Delta(t) > 0, \quad z^{\Delta\Delta}(t) < 0, \quad (a(t)z^\Delta(t))^\Delta < 0, \quad \frac{z(t)}{t} \text{ is nonincreasing.} \quad (2.3)$$



*Proof.* From (1.11), the proof is similar to that of Saker et al. [24, Lemma 2.1], so it is omitted.  $\square$

**Lemma 2.2.** *Assume that*

$$\int_{t_0}^{\infty} p(t)\delta^{\gamma}(t)\Delta t = \infty, \tag{2.4}$$

$a^{\Delta}(t) \geq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . Then an eventually positive solution  $x$  of (1.33) eventually satisfies that

$$z(t) \geq tz^{\Delta}(t) > 0, \quad z^{\Delta\Delta}(t) < 0, \quad \left(a(t)z^{\Delta}(t)\right)^{\Delta} < 0, \quad \frac{z(t)}{t} \text{ is nonincreasing}, \tag{2.5}$$

or  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* Let  $x$  be an eventually positive solution of (1.33). Then there exists  $t_1 \geq t_0$  such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1$ . Assume that  $\lim_{t \rightarrow \infty} x(t) \neq 0$ , that is,  $\limsup_{t \rightarrow \infty} x(t) > 0$ . Then, we have to show that (2.5) holds. It follows from (1.33) that

$$\left(a(t)z^{\Delta}(t)\right)^{\Delta} = -p(t)x^{\gamma}(\delta(t)) < 0, \quad t \geq t_1, \tag{2.6}$$

which implies that  $az^{\Delta}$  is nonincreasing on  $[t_1, \infty)_{\mathbb{T}}$ . Since the function  $a$  is nondecreasing,  $z^{\Delta}$  must be nonincreasing on  $[t_1, \infty)_{\mathbb{T}}$ , that is,  $z^{\Delta}$  is eventually either positive or negative. In both cases,  $z$  is eventually monotonic, so that  $z$  has a limit at infinity (finite or infinite). This implies that  $\lim_{t \rightarrow \infty} z(t) \neq 0$ ; that is,  $z$  is eventually positive (see [19, Lemma 3]). Then we proceed as in the proof of [24, Lemma 2.1] to obtain (2.5). The proof is complete.  $\square$

**Lemma 2.3.** *Assume that  $0 \leq r(t) < 1$ . Further,  $x$  is an eventually positive solution of (1.33). Then there exists a  $t_* \geq t_0$  such that for  $t \geq t_*$ ,*

$$z^{\Delta}(t) > 0, \quad \left(a(t)z^{\Delta}(t)\right)^{\Delta} < 0, \quad z(t) \geq R(t, t_*)z^{\Delta}(t), \quad z(\delta(t)) \geq \alpha(t, t_*)z(t). \tag{2.7}$$

*Proof.* Let  $x$  be an eventually positive solution of (1.33). Then there exists  $t_1 \geq t_0$  such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1$ . It follows from (1.33) that (2.6) holds. From (2.6), we know that  $a(t)z^{\Delta}(t)$  is an eventually decreasing function. We claim that  $z^{\Delta}(t) > 0$  eventually. Otherwise, if there exists a  $t_2 \geq t_1$  such that  $z^{\Delta}(t) < 0$ , by (2.6), we have

$$a(t)z^{\Delta}(t) \leq a(t_2)z^{\Delta}(t_2) = b < 0, \quad t \geq t_2. \tag{2.8}$$

Thus

$$z^{\Delta}(t) \leq b \frac{1}{a(t)}. \tag{2.9}$$

Integrating the above inequality from  $t_2$  to  $t$  leads to  $\lim_{t \rightarrow \infty} z(t) = -\infty$ , which contradicts  $z(t) > 0$ . Hence,  $z^\Delta(t) > 0$  on  $[t_2, \infty)_{\mathbb{T}}$ . Therefore,

$$z(t) > z(t) - z(t_2) = \int_{t_2}^t \frac{a(s)z^\Delta(s)}{a(s)} \Delta s \geq \left( a(t)z^\Delta(t) \right) \int_{t_2}^t \frac{\Delta s}{a(s)}, \quad (2.10)$$

which yields

$$z(t) \geq \left( a(t) \int_{t_2}^t \frac{\Delta s}{a(s)} \right) z^\Delta(t). \quad (2.11)$$

Since  $a(t)z^\Delta(t)$  is strictly decreasing, we have

$$z(t) - z(\delta(t)) = \int_{\delta(t)}^t \frac{a(s)z^\Delta(s)}{a(s)} \Delta s \leq a(\delta(t))z^\Delta(\delta(t)) \int_{\delta(t)}^t \frac{\Delta s}{a(s)}, \quad (2.12)$$

and so

$$\frac{z(t)}{z(\delta(t))} \leq 1 + \frac{a(\delta(t))z^\Delta(\delta(t))}{z(\delta(t))} \int_{\delta(t)}^t \frac{\Delta s}{a(s)}. \quad (2.13)$$

Also, we have that for large  $t$ ,

$$z(\delta(t)) \geq z(\delta(t)) - z(t_2) = \int_{t_2}^{\delta(t)} \frac{a(s)z^\Delta(s)}{a(s)} \Delta s \geq a(\delta(t))z^\Delta(\delta(t)) \int_{t_2}^{\delta(t)} \frac{\Delta s}{a(s)}, \quad (2.14)$$

so we obtain

$$\frac{a(\delta(t))z^\Delta(\delta(t))}{z(\delta(t))} \leq \left( \int_{t_2}^{\delta(t)} \frac{\Delta s}{a(s)} \right)^{-1}. \quad (2.15)$$

Therefore, from (2.13), we have

$$z(\delta(t)) \geq \alpha(t, t_2)z(t). \quad (2.16)$$

This completes the proof.  $\square$

**Lemma 2.4.** *Assume that  $-1 < -r_0 \leq r(t) \leq 0$ ,  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . Then an eventually positive solution  $x$  of (1.33) satisfies that, for sufficiently large  $t_* \geq t_0$ ,*

$$z^\Delta(t) > 0, \quad \left( a(t)z^\Delta(t) \right)^\Delta < 0, \quad z(t) \geq R(t, t_*)z^\Delta(t), \quad z(\delta(t)) \geq \alpha(t, t_*)z(t), \quad t \geq t_*, \quad (2.17)$$

or  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* The proof is similar to that of the proof Lemmas 2.2 and 2.3, so we omit the details.  $\square$

**Theorem 2.5.** *Assume that (1.11) holds,  $a^\Delta(t) \geq 0$ , and  $0 \leq r(t) < 1$ . Then every solution of (1.33) oscillates if the inequality*

$$y^\Delta(t) + A(t)y^r(\delta(t)) \leq 0, \quad (2.18)$$

where

$$A(t) = p(t)(1 - r(\delta(t)))^Y \frac{(\delta(t))^Y}{(a(\delta(t)))^Y}, \quad (2.19)$$

has no eventually positive solution.

*Proof.* Suppose to the contrary that (1.33) has a nonoscillatory solution  $x$ . We may assume without loss of generality that there exists  $t_1 \geq t_0$  such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$  and  $x(\delta(t)) > 0$  for all  $t \geq t_1$ . From Lemma 2.1, there is some  $t_2 \geq t_1$  such that

$$x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) - r(t)z(\tau(t)) \geq (1 - r(t))z(t), \quad t \geq t_2. \quad (2.20)$$

From (1.33), there exists a  $t_3 \geq t_2$  such that

$$\left(a(t)z^\Delta(t)\right)^\Delta + p(t)(1 - r(\delta(t)))^Y (z(\delta(t)))^Y \leq 0, \quad t \geq t_3. \quad (2.21)$$

By Lemma 2.1, there exists a  $t_4 \geq t_3$  such that

$$z(\delta(t)) \geq \delta(t)z^\Delta(\delta(t)). \quad (2.22)$$

Substituting the last inequality in (2.21) we obtain for  $t \geq t_4$  that

$$\left(a(t)z^\Delta(t)\right)^\Delta + p(t)(1 - r(\delta(t)))^Y (\delta(t))^Y \left(z^\Delta(\delta(t))\right)^Y \leq 0. \quad (2.23)$$

Set  $y(t) = a(t)z^\Delta(t)$ . Then from (2.23),  $y$  is positive and satisfies the inequality (2.18), and this contradicts the assumption of our theorem. Thus every solution of (1.33) oscillates. This completes the proof.  $\square$

By [41, Lemma 1.1] and Theorem 2.5 in this paper, we have the following result.

**Corollary 2.6.** *If  $\mathbb{T} = \mathbb{Z}$ ,  $a(t) = 1$ ,  $\delta(t) = t - l$ ,  $l$  is a positive integer, and  $0 \leq r(t) < 1$ , then every solution of (1.33) oscillates if*

$$\sum_{t=n_0}^{\infty} t^Y p(t)(1 - r(\delta(t)))^Y = \infty. \quad (2.24)$$

**Theorem 2.7.** Assume that (2.4) holds, and  $a^\Delta(t) \geq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . Then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$  if the inequality

$$y^\Delta(t) + B(t)y^\gamma(\delta(t)) \leq 0, \quad (2.25)$$

where

$$B(t) = p(t) \frac{(\delta(t))^\gamma}{(a(\delta(t)))^\gamma}, \quad (2.26)$$

has no eventually positive solution.

*Proof.* Suppose to the contrary that (1.33) has a nonoscillatory solution  $x$ . We may assume without loss of generality that there exists  $t_1 \geq t_0$  such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1$ .

From Lemma 2.2, if (i) holds, there is some  $t_2 \geq t_1$  such that

$$x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) > 0, \quad t \geq t_2. \quad (2.27)$$

From (1.33), there exists a  $t_3 \geq t_2$  such that

$$(a(t)z^\Delta(t))^\Delta + p(t)(z(\delta(t)))^\gamma \leq 0, \quad t \geq t_3. \quad (2.28)$$

By Lemma 2.2, there exists a  $t_3 \geq t_2$  such that

$$z(\delta(t)) \geq \delta(t)z^\Delta(\delta(t)). \quad (2.29)$$

Substituting the last inequality in (2.28), we obtain for  $t \geq t_3$  that

$$(a(t)z^\Delta(t))^\Delta + p(t)(\delta(t))^\gamma (z^\Delta(\delta(t)))^\gamma \leq 0. \quad (2.30)$$

Set  $y(t) = a(t)z^\Delta(t)$ . Then from (2.30),  $y$  is positive and satisfies the inequality (2.25), and this contradicts the assumption of our theorem.

If (ii) holds, by Lemma 2.2, we have  $\lim_{t \rightarrow \infty} x(t) = 0$ . This completes the proof.  $\square$

By [41, Lemma 1.1] and Theorem 2.7 in this paper, we have the following result.

**Corollary 2.8.** Assume that  $\mathbb{T} = \mathbb{Z}$ ,  $a(t) = 1$ ,  $\delta(t) = t - l$ ,  $l$  is a positive integer,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r > -1$ . Then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$  if

$$\sum_{t=n_0}^{\infty} t^\gamma p(t) = \infty. \quad (2.31)$$

*Remark 2.9.* Theorems 2.5 and 2.7 reduce the question of (1.33) to the absence of eventually positive solution (the oscillatory) of the differential inequalities (2.18) and (2.25).

*Remark 2.10.* From Theorem 2.5, Theorem 2.7, and the results given in [7–9, 12, 14], we can obtain some oscillation criteria for (1.33) in the case when  $\gamma = 1$ ,  $a^\Delta(t) \geq 0$ .

**Theorem 2.11.** *Assume that (1.11) holds,  $\gamma < 1$ ,  $a^\Delta(t) \geq 0$ , and  $0 \leq r(t) < 1$ . Then every solution of (1.33) oscillates if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^\gamma} (1 - r(\delta(s)))^\gamma (\delta(s))^\gamma \Delta s = \infty. \tag{2.32}$$

*Proof.* We assume that (1.33) has a nonoscillatory solution such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1 \geq t_0$ . By proceeding as in the proof of Theorem 2.5, we get (2.21). By Lemma 2.1, note that  $(a(t)z^\Delta(t))^\Delta < 0$ , and from Keller’s chain rule, we obtain

$$\begin{aligned} \left( (a(t)z^\Delta(t))^{1-\gamma} \right)^\Delta &= (1-\gamma) \int_0^1 \left[ h(a(t)z^\Delta(t))^\sigma + (1-h)a(t)z^\Delta(t) \right]^{-\gamma} (a(t)z^\Delta(t))^\Delta dh \\ &\leq (1-\gamma) \int_0^1 \left[ ha(t)z^\Delta(t) + (1-h)a(t)z^\Delta(t) \right]^{-\gamma} (a(t)z^\Delta(t))^\Delta dh \\ &= (1-\gamma) (a(t)z^\Delta(t))^{-\gamma} (a(t)z^\Delta(t))^\Delta < 0, \end{aligned} \tag{2.33}$$

so

$$(a(t)z^\Delta(t))^{-\gamma} (a(t)z^\Delta(t))^\Delta \geq \frac{\left( (a(t)z^\Delta(t))^{1-\gamma} \right)^\Delta}{1-\gamma}. \tag{2.34}$$

Using (2.21), we have

$$\begin{aligned} 0 &\geq \frac{(a(t)z^\Delta(t))^\Delta + p(t)(1-r(\delta(t)))^\gamma (z(\delta(t)))^\gamma}{(a(t)z^\Delta(t))^\gamma} \\ &= (a(t)z^\Delta(t))^{-\gamma} (a(t)z^\Delta(t))^\Delta + p(t)(1-r(\delta(t)))^\gamma \left( \frac{z(\delta(t))}{a(t)z^\Delta(t)} \right)^\gamma \\ &\geq \frac{\left( (a(t)z^\Delta(t))^{1-\gamma} \right)^\Delta}{1-\gamma} + \frac{p(t)}{(a(\delta(t)))^\gamma} (1-r(\delta(t)))^\gamma (\delta(t))^\gamma. \end{aligned} \tag{2.35}$$

Hence,

$$\frac{p(t)}{(a(\delta(t)))^\gamma} (1-r(\delta(t)))^\gamma (\delta(t))^\gamma \leq \frac{\left( (a(t)z^\Delta(t))^{1-\gamma} \right)^\Delta}{\gamma-1}. \tag{2.36}$$

Upon integration we arrive at

$$\int_{t_1}^t \frac{p(s)}{(a(\delta(s)))^\gamma} (1 - r(\delta(s)))^\gamma (\delta(s))^\gamma \Delta s \leq \int_{t_1}^t \frac{\left( (a(s)z^\Delta(s))^{1-\gamma} \right)^\Delta}{\gamma - 1} \Delta s \leq \frac{(a(t_1)z^\Delta(t_1))^{1-\gamma}}{1 - \gamma}. \quad (2.37)$$

This contradicts (2.32) and finishes the proof.  $\square$

**Theorem 2.12.** *Assume that (2.4) holds, and  $\gamma < 1$ ,  $a^\Delta(t) \geq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . Then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$  if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^\gamma} (\delta(s))^\gamma \Delta s = \infty. \quad (2.38)$$

*Proof.* By Lemma 2.2, the proof is similar to that of the proof of Theorem 2.11, so we omit the details.  $\square$

**Theorem 2.13.** *Assume that  $\gamma < 1$  and  $0 \leq r(t) < 1$ . Then every solution of (1.33) oscillates if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^\gamma} (1 - r(\delta(s)))^\gamma (R(\delta(s), t_*))^\gamma \Delta s = \infty \quad (2.39)$$

*holds for all sufficiently large  $t_*$ .*

*Proof.* By Lemma 2.3, the proof is similar to that of the proof Theorem 2.11, so we omit the details.  $\square$

**Theorem 2.14.** *Assume that  $\gamma < 1$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . Then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$  if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^\gamma} (R(\delta(s), t_*))^\gamma \Delta s = \infty \quad (2.40)$$

*holds for all sufficiently large  $t_*$ .*

*Proof.* By using Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.11, so we omit the details.  $\square$

**Theorem 2.15.** *Assume that (1.11) holds,  $\gamma \geq 1$ ,  $a^\Delta(t) \geq 0$ , and  $0 \leq r(t) < 1$ . Then every solution of (1.33) oscillates if*

$$\limsup_{t \rightarrow \infty} \left\{ \frac{t}{a(t)} \int_t^{\infty} p(s) (1 - r(\delta(s)))^\gamma \left( \frac{\delta(s)}{s} \right)^\gamma \Delta s \right\} = \infty. \quad (2.41)$$

*Proof.* Suppose to the contrary that (1.33) has a nonoscillatory solution  $x$ . We may assume without loss of generality that there exists  $t_1 \geq t_0$  such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and

$x(\delta(t)) > 0$  for all  $t \geq t_1$ . By proceeding as in the proof of Theorem 2.5, we get (2.21). Thus from Lemma 2.1, we have for  $T \geq t \geq t_1$ ,

$$\int_t^T p(s)(1 - r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \leq - \int_t^T \left( a(s)z^\Delta(s) \right)^\Delta \Delta s = a(t)z^\Delta(t) - a(T)z^\Delta(T), \quad (2.42)$$

and hence

$$\int_t^T p(s)(1 - r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \leq a(t)z^\Delta(t). \quad (2.43)$$

This and Lemma 2.1 provide, for sufficiently large  $t \in \mathbb{T}$ ,

$$\begin{aligned} z(t) &\geq tz^\Delta(t) \geq \frac{t}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \\ &\geq \frac{t}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma \left( \frac{\delta(s)}{s} \right)^\gamma z^\gamma(s) \Delta s \\ &\geq z^\gamma(t) \left\{ \frac{t}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma \left( \frac{\delta(s)}{s} \right)^\gamma \Delta s \right\}. \end{aligned} \quad (2.44)$$

So

$$\left\{ \frac{t}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma \left( \frac{\delta(s)}{s} \right)^\gamma \Delta s \right\} \leq \left( \frac{1}{z(t)} \right)^{\gamma-1}. \quad (2.45)$$

We note that  $\gamma \geq 1$  and  $z^\Delta(t) > 0$  imply

$$\left\{ \frac{t}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma \left( \frac{\delta(s)}{s} \right)^\gamma \Delta s \right\} \leq \left( \frac{1}{z(t_1)} \right)^{\gamma-1}. \quad (2.46)$$

This contradicts (2.41) and completes the proof.  $\square$

**Theorem 2.16.** *Assume that (2.4) holds, and  $\gamma \geq 1$ ,  $a^\Delta(t) \geq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . Then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$  if*

$$\limsup_{t \rightarrow \infty} \left\{ \frac{t}{a(t)} \int_t^\infty p(s) \left( \frac{\delta(s)}{s} \right)^\gamma \Delta s \right\} = \infty. \quad (2.47)$$

*Proof.* By using Lemma 2.2 and (2.28), the proof is similar to that of the proof of Theorem 2.15, so we omit the details.  $\square$

**Theorem 2.17.** Assume that  $\gamma \geq 1$ ,  $0 \leq r(t) < 1$ . Then every solution of (1.33) oscillates if

$$\limsup_{t \rightarrow \infty} \left\{ \frac{R(t, t_*)}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma \Delta s \right\} = \infty \quad (2.48)$$

holds for all sufficiently large  $t_*$ .

*Proof.* Suppose to the contrary that (1.33) has a nonoscillatory solution  $x$ . We may assume without loss of generality that there exists  $t_1 \geq t_0$  such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1$ . By proceeding as in the proof of Theorem 2.5, we obtain (2.21). Thus from Lemma 2.3, we have, for  $T \geq t \geq t_1$ ,

$$\int_t^T p(s)(1 - r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \leq - \int_t^T \left( a(s)z^\Delta(s) \right)^\Delta \Delta s = a(t)z^\Delta(t) - a(T)z^\Delta(T), \quad (2.49)$$

and hence

$$\int_t^T p(s)(1 - r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \leq a(t)z^\Delta(t). \quad (2.50)$$

This and Lemma 2.3 provide, for sufficiently large  $t \in \mathbb{T}$ ,

$$\begin{aligned} z(t) &\geq R(t, t_*)z^\Delta(t) \geq \frac{R(t, t_*)}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \\ &\geq \frac{R(t, t_*)}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma z^\gamma(s) \Delta s \\ &\geq z^\gamma(t) \left\{ \frac{R(t, t_*)}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma \Delta s \right\}. \end{aligned} \quad (2.51)$$

So

$$\left\{ \frac{R(t, t_*)}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma \Delta s \right\} \leq \left( \frac{1}{z(t)} \right)^{\gamma-1}. \quad (2.52)$$

We note that  $\gamma \geq 1$  and  $z^\Delta(t) > 0$  imply

$$\left\{ \frac{R(t, t_*)}{a(t)} \int_t^\infty p(s)(1 - r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma \Delta s \right\} \leq \left( \frac{1}{z(t_1)} \right)^{\gamma-1}. \quad (2.53)$$

This contradicts (2.48) and completes the proof.  $\square$



**Theorem 2.18.** Assume that (2.4) holds, and  $\gamma \geq 1$ ,  $a^\Delta(t) \geq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r > -1$ . Then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$  if

$$\limsup_{t \rightarrow \infty} \left\{ \frac{R(t, t_*)}{a(t)} \int_t^\infty p(s) (\alpha(s, t_*))^\gamma \Delta s \right\} = \infty \tag{2.54}$$

holds for all sufficiently large  $t_*$ .

*Proof.* By using Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.17, so we omit the details. □

**Theorem 2.19.** Assume that (1.11) holds,  $\gamma > 1$ ,  $a^\Delta(t) \geq 0$ , and  $0 \leq r(t) < 1$ . Then every solution of (1.33) oscillates if

$$\int_{t_0}^\infty \sigma(s) \frac{p(s)}{a(s)} (1 - r(\delta(s)))^\gamma \left( \frac{\delta(s)}{\sigma(s)} \right)^\gamma \Delta s = \infty. \tag{2.55}$$

*Proof.* We assume that (1.33) has a nonoscillatory solution such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1 \geq t_0$ . By proceeding as in the proof of Theorem 2.5, we get (2.21). Define the function

$$\omega(t) = \frac{ta(t)z^\Delta(t)}{z^\gamma(t)}, \quad t \geq t_1. \tag{2.56}$$

By Lemma 2.1,  $\omega(t) > 0$ . We calculate

$$\omega^\Delta(t) = \left\{ a(t)z^\Delta(t) + \sigma(t) \left( a(t)z^\Delta(t) \right)^\Delta \right\} (z^{-\gamma}(t))^\sigma + ta(t)z^\Delta(t)(z^{-\gamma}(t))^\Delta. \tag{2.57}$$

From (2.21), we have

$$\omega^\Delta(t) \leq a(t)z^\Delta(t)(z^{-\gamma}(t))^\sigma - \sigma(t)p(t)(1 - r(\delta(t)))^\gamma \left( \frac{z(\delta(t))}{z(\sigma(t))} \right)^\gamma + ta(t)z^\Delta(t)(z^{-\gamma}(t))^\Delta, \tag{2.58}$$

and by Lemma 2.1, we have

$$\omega^\Delta(t) \leq a(t)z^\Delta(t)(z^{-\gamma}(t))^\sigma - \sigma(t)p(t)(1 - r(\delta(t)))^\gamma \left( \frac{\delta(t)}{\sigma(t)} \right)^\gamma, \tag{2.59}$$

because  $(z^{-\gamma}(t))^\Delta \leq 0$  due to Keller's chain rule. Since

$$\begin{aligned} \left( (z(t))^{1-\gamma} \right)^\Delta &= (1-\gamma) \int_0^1 [hz^\sigma(t) + (1-h)z(t)]^{-\gamma} z^\Delta(t) dh \\ &\leq (1-\gamma) \int_0^1 [hz^\sigma(t) + (1-h)z^\sigma(t)]^{-\gamma} z^\Delta(t) dh = (1-\gamma)(z^\sigma(t))^{-\gamma} z^\Delta(t), \end{aligned} \tag{2.60}$$

thus

$$\omega^\Delta(t) \leq a(t) \frac{\left((z(t))^{1-\gamma}\right)^\Delta}{1-\gamma} - \sigma(t)p(t)(1-r(\delta(t)))^\gamma \left(\frac{\delta(t)}{\sigma(t)}\right)^\gamma. \quad (2.61)$$

Upon integration we arrive at

$$\begin{aligned} & \int_{t_1}^t \sigma(s) \frac{p(s)}{a(s)} (1-r(\delta(s)))^\gamma \left(\frac{\delta(s)}{\sigma(s)}\right)^\gamma \Delta s \\ & \leq \int_{t_1}^t \left\{ \frac{\left((z(s))^{1-\gamma}\right)^\Delta}{1-\gamma} - \frac{\omega^\Delta(s)}{a(s)} \right\} \Delta s \\ & = \frac{(z(t))^{1-\gamma}}{1-\gamma} - \frac{(z(t_1))^{1-\gamma}}{1-\gamma} - \int_{t_1}^t \frac{\omega^\Delta(s)}{a(s)} \Delta s \\ & = \frac{(z(t))^{1-\gamma}}{1-\gamma} - \frac{(z(t_1))^{1-\gamma}}{1-\gamma} - \frac{\omega(t)}{a(t)} + \frac{\omega(t_1)}{a(t_1)} + \int_{t_1}^t \omega^\sigma(s) \left(\frac{1}{a(s)}\right)^\Delta \Delta s. \end{aligned} \quad (2.62)$$

Noting that  $(1/a(t))^\Delta \leq 0$ , we have

$$\int_{t_1}^t \sigma(s) \frac{p(s)}{a(s)} (1-r(\delta(s)))^\gamma \left(\frac{\delta(s)}{\sigma(s)}\right)^\gamma \Delta s \leq \frac{(z(t_1))^{1-\gamma}}{\gamma-1} + \frac{\omega(t_1)}{a(t_1)}. \quad (2.63)$$

This contradicts (2.55) and finishes the proof.  $\square$

**Theorem 2.20.** *Assume that (2.4) holds, and  $\gamma > 1$ ,  $a^\Delta(t) \geq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . Then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$  if*

$$\int_{t_0}^{\infty} \sigma(s)p(s) \left(\frac{\delta(s)}{\sigma(s)}\right)^\gamma \Delta s = \infty. \quad (2.64)$$

*Proof.* By using Lemma 2.2 and (2.28), the proof is similar to that of the proof of Theorem 2.19, so we omit the details.  $\square$

In the following, we use a Riccati transformation technique to establish new oscillation criteria for (1.33).

**Theorem 2.21.** *Assume that  $\gamma \geq 1$ , and  $0 \leq r(t) < 1$ . Furthermore, suppose that there exists a positive  $\Delta$ -differentiable function  $\eta$  such that for all sufficiently large  $t_*$ , and for all constants  $M > 0$ ,*

for  $t_1 \geq t_*$ ,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ \eta(s)p(s)(1-r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma - \frac{a(s)(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1}\eta(s)} \right] \Delta s = \infty. \quad (2.65)$$

Then every solution of (1.33) oscillates.

*Proof.* We assume that (1.33) has a nonoscillatory solution such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1 \geq t_0$ . By proceeding as in the proof of Theorem 2.5, we get (2.21). Define the function  $\omega$  by the Riccati substitution

$$\omega(t) = \eta(t) \frac{a(t)z^\Delta(t)}{z^\gamma(t)}, \quad t \geq t_1. \quad (2.66)$$

Then  $\omega(t) > 0$ . By the product rule and then the quotient rule

$$\begin{aligned} \omega^\Delta(t) &= \left( a(t)z^\Delta(t) \right)^\sigma \left[ \frac{\eta(t)}{z^\gamma(t)} \right]^\Delta + \frac{\eta(t)}{z^\gamma(t)} \left( a(t)z^\Delta(t) \right)^\Delta \\ &= \frac{\eta(t)}{z^\gamma(t)} \left( a(t)z^\Delta(t) \right)^\Delta + \left( a(t)z^\Delta(t) \right)^\sigma \left[ \frac{z^\gamma(t)\eta^\Delta(t) - \eta(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\sigma(t))^\gamma} \right]. \end{aligned} \quad (2.67)$$

In view of (2.21) and (2.66), we have

$$\omega^\Delta(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^\gamma \left( \frac{z(\delta(t))}{z(t)} \right)^\gamma + \frac{\eta^\Delta(t)}{\eta^\sigma(t)} \omega^\sigma(t) - \frac{\eta(t)(a(t)z^\Delta(t))^\sigma (z^\gamma(t))^\Delta}{z^\gamma(t)(z^\sigma(t))^\gamma}. \quad (2.68)$$

By the chain rule and  $\gamma \geq 1$ , we obtain

$$(z^\gamma(t))^\Delta \geq \gamma z^{\gamma-1}(t)z^\Delta(t) \geq \gamma M^{\gamma-1}z^\Delta(t), \quad (2.69)$$

where  $M = z(t_1) > 0$ . In view of  $(a(t)z^\Delta(t))^\Delta < 0$ , we have

$$a(t)z^\Delta(t) \geq \left( a(t)z^\Delta(t) \right)^\sigma, \quad (2.70)$$

and by Lemma 2.3, we see that

$$\omega^\Delta(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^\gamma (\alpha(t, t_*))^\gamma + \frac{\eta^\Delta(t)}{\eta^\sigma(t)} \omega^\sigma(t) - \frac{\gamma M^{\gamma-1}\eta(t)}{a(t)(\eta^\sigma(t))^2} (\omega^\sigma(t))^2. \quad (2.71)$$

Integrating (2.71) from  $t_1$  to  $t$ , we obtain

$$\begin{aligned} & \int_{t_1}^t \eta(s)p(s)(1-r(\delta(s)))^Y(\alpha(s,t_*))^Y \Delta s \\ & \leq - \int_{t_1}^t \omega^\Delta(s) \Delta s \\ & \quad + \int_{t_1}^t \frac{\eta^\Delta(s)}{\eta^\sigma(s)} \omega^\sigma(s) \Delta s - \int_{t_1}^t \frac{\gamma M^{\gamma-1} \eta(s)}{a(s)(\eta^\sigma(s))^2} (\omega^\sigma(s))^2 \Delta s. \end{aligned} \quad (2.72)$$

Hence

$$\int_{t_1}^t \left[ \eta(s)p(s)(1-r(\delta(s)))^Y(\alpha(s,t_*))^Y - \frac{a(s)(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1} \eta(s)} \right] \Delta s \leq \omega(t_1), \quad (2.73)$$

which contradicts condition (2.65). The proof is complete.  $\square$

**Theorem 2.22.** Assume that  $\gamma \geq 1$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . If there exists a positive  $\Delta$ -differentiable function  $\eta$  such that for all sufficiently large  $t_*$ , and for all constants  $M > 0$ , for  $t_1 \geq t_*$ ,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ \eta(s)p(s)(\alpha(s,t_*))^Y - \frac{a(s)(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1} \eta(s)} \right] \Delta s = \infty, \quad (2.74)$$

then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$ .

*Proof.* By Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.21, so we omit the details.  $\square$

**Theorem 2.23.** Assume that (1.11) holds,  $\gamma \leq 1$ ,  $a^\Delta(t) \geq 0$ , and  $0 \leq r(t) < 1$ . Furthermore, suppose that there exists a positive  $\Delta$ -differentiable function  $\eta$  such that for all sufficiently large  $t_1$ , and for all constants  $M > 0$ ,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ \eta(s)p(s)(1-r(\delta(s)))^Y \left( \frac{\delta(s)}{s} \right)^Y - \frac{a(s)(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1} (\sigma(s))^{\gamma-1} \eta(s)} \right] \Delta s = \infty. \quad (2.75)$$

Then every solution of (1.33) oscillates.

*Proof.* We assume that (1.33) has a nonoscillatory solution such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1 \geq t_0$ . By proceeding as in the proof of Theorem 2.5, we obtain (2.21).

Define the function  $\omega$  by the Riccati substitution as (2.66). Then  $\omega(t) > 0$ . By the product rule and then the quotient rule

$$\begin{aligned} \omega^\Delta(t) &= \left(a(t)z^\Delta(t)\right)^\sigma \left[\frac{\eta(t)}{z^\gamma(t)}\right]^\Delta + \frac{\eta(t)}{z^\gamma(t)} \left(a(t)z^\Delta(t)\right)^\Delta \\ &= \frac{\eta(t)}{z^\gamma(t)} \left(a(t)z^\Delta(t)\right)^\Delta + \left(a(t)z^\Delta(t)\right)^\sigma \left[\frac{z^\gamma(t)\eta^\Delta(t) - \eta(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\sigma(t))^\gamma}\right]. \end{aligned} \tag{2.76}$$

In view of (2.21) and (2.66), we have

$$\omega^\Delta(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^\gamma \left(\frac{z(\delta(t))}{z(t)}\right)^\gamma + \frac{\eta^\Delta(t)}{\eta^\sigma(t)} \omega^\sigma(t) - \frac{\eta(t)(a(t)z^\Delta(t))^\sigma (z^\gamma(t))^\Delta}{z^\gamma(t)(z^\sigma(t))^\gamma}. \tag{2.77}$$

From the chain rule and  $\gamma \leq 1$ , we get

$$(z^\gamma(t))^\Delta \geq \gamma z^{\gamma-1}(\sigma(t))z^\Delta(t). \tag{2.78}$$

Noting that  $z(t)/t$  is nonincreasing, and there exists a constant  $M > 0$ , such that  $z(t) \leq Mt$ , hence we have

$$(z^\gamma(t))^\Delta \geq \gamma z^{\gamma-1}(\sigma(t))z^\Delta(t) \geq \gamma M^{\gamma-1}(\sigma(t))^{\gamma-1}z^\Delta(t). \tag{2.79}$$

In view of  $(a(t)z^\Delta(t))^\Delta < 0$ , we have

$$a(t)z^\Delta(t) \geq \left(a(t)z^\Delta(t)\right)^\sigma, \tag{2.80}$$

and by Lemma 2.1, we see that

$$\omega^\Delta(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^\gamma \left(\frac{\delta(t)}{t}\right)^\gamma + \frac{\eta^\Delta(t)}{\eta^\sigma(t)} \omega^\sigma(t) - \frac{\gamma M^{\gamma-1}(\sigma(t))^{\gamma-1} \eta(t)}{a(t)(\eta^\sigma(t))^2} (\omega^\sigma(t))^2. \tag{2.81}$$

Integrating (2.81) from  $t_1$  to  $t$ , we obtain

$$\begin{aligned} &\int_{t_1}^t \eta(s)p(s)(1-r(\delta(s)))^\gamma \left(\frac{\delta(s)}{s}\right)^\gamma \Delta s \\ &\leq -\int_{t_1}^t \omega^\Delta(s) \Delta s + \int_{t_1}^t \frac{\eta^\Delta(s)}{\eta^\sigma(s)} \omega^\sigma(s) \Delta s - \int_{t_1}^t \frac{\gamma M^{\gamma-1}(\sigma(s))^{\gamma-1} \eta(s)}{a(s)(\eta^\sigma(s))^2} (\omega^\sigma(s))^2 \Delta s. \end{aligned} \tag{2.82}$$

Hence

$$\int_{t_1}^t \left[ \eta(s)p(s)(1-r(\delta(s)))^\gamma \left( \frac{\delta(s)}{s} \right)^\gamma - \frac{a(s)(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1}(\sigma(s))^{\gamma-1}\eta(s)} \right] \Delta s \leq \omega(t_1), \quad (2.83)$$

which contradicts condition (2.75). The proof is complete.  $\square$

**Theorem 2.24.** Assume that (2.4) holds,  $\gamma \leq 1$ ,  $a^\Delta(t) \geq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r_1 > -1$ . If there exists a positive  $\Delta$ -differentiable function  $\eta$  such that for all sufficiently large  $t_1$ , and for all constants  $M > 0$ ,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ \eta(s)p(s) \left( \frac{\delta(s)}{s} \right)^\gamma - \frac{a(s)(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1}(\sigma(s))^{\gamma-1}\eta(s)} \right] \Delta s = \infty, \quad (2.84)$$

then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$ .

*Proof.* By Lemma 2.2 and (2.28), the proof is similar to that of the proof of Theorem 2.23, so we omit the details.  $\square$

**Theorem 2.25.** Assume that  $\gamma \leq 1$ ,  $a^\Delta(t) \leq 0$ , and  $0 \leq r(t) < 1$ . Furthermore, suppose that there exists a positive  $\Delta$ -differentiable function  $\eta$  such that for all sufficiently large  $t_*$ , and for all constants  $M > 0$ , for  $t_1 \geq t_*$ ,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ \eta(s)p(s)(1-r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma - \frac{a(s)(\sigma(s))^{1-\gamma} (\eta^\Delta(s))^2}{4\gamma M^{\gamma-1} (a(\sigma(s)))^{1-\gamma} \eta(s)} \right] \Delta s = \infty. \quad (2.85)$$

Then every solution of (1.33) oscillates.

*Proof.* We assume that (1.33) has a nonoscillatory solution such that  $x(t) > 0$ ,  $x(\tau(t)) > 0$ , and  $x(\delta(t)) > 0$  for all  $t \geq t_1 \geq t_0$ . By proceeding as in the proof of Theorem 2.5, we have (2.21). Define the function  $\omega$  by the Riccati substitution as (2.66). Then  $\omega(t) > 0$ . By the product rule and then the quotient rule

$$\begin{aligned} \omega^\Delta(t) &= \left( a(t)z^\Delta(t) \right)^\sigma \left[ \frac{\eta(t)}{z^\gamma(t)} \right]^\Delta + \frac{\eta(t)}{z^\gamma(t)} \left( a(t)z^\Delta(t) \right)^\Delta \\ &= \frac{\eta(t)}{z^\gamma(t)} \left( a(t)z^\Delta(t) \right)^\Delta + \left( a(t)z^\Delta(t) \right)^\sigma \left[ \frac{z^\gamma(t)\eta^\Delta(t) - \eta(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\sigma(t))^\gamma} \right]. \end{aligned} \quad (2.86)$$

In view of (2.21) and (2.66), we have

$$\omega^\Delta(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^\gamma \left( \frac{z(\delta(t))}{z(t)} \right)^\gamma + \frac{\eta^\Delta(t)}{\eta^\sigma(t)} \omega^\sigma(t) - \frac{\eta(t)(a(t)z^\Delta(t))^\sigma (z^\gamma(t))^\Delta}{z^\gamma(t)(z^\sigma(t))^\gamma}. \quad (2.87)$$

By the chain rule and  $\gamma \leq 1$ , we obtain

$$(z^\gamma(t))^\Delta \geq \gamma z^{\gamma-1}(\sigma(t))z^\Delta(t), \tag{2.88}$$

and noting that  $(a(t)z^\Delta(t))^\Delta < 0$  and there exists a constant  $L > 0$  such that  $a(t)z^\Delta(t) \leq L$ , so

$$z(t) = z(t_1) + \int_{t_1}^t z^\Delta(s) \Delta s \leq z(t_1) + \int_{t_1}^t \frac{L}{a(s)} \Delta s. \tag{2.89}$$

From  $a^\Delta(t) \leq 0$ , there exists a positive constant  $M$  such that

$$z(t) \leq z(t_1) + \frac{L}{a(t)}(t - t_1) = \frac{z(t_1)a(t) + L(t - t_1)}{a(t)} \leq \frac{Mt}{a(t)}. \tag{2.90}$$

Hence

$$(z^\gamma(t))^\Delta \geq \gamma z^{\gamma-1}(\sigma(t))z^\Delta(t) \geq \gamma M^{\gamma-1} \left( \frac{\sigma(t)}{a(\sigma(t))} \right)^{\gamma-1} z^\Delta(t). \tag{2.91}$$

In view of  $(a(t)z^\Delta(t))^\Delta < 0$ , we have

$$a(t)z^\Delta(t) \geq \left( a(t)z^\Delta(t) \right)^\sigma, \tag{2.92}$$

and by Lemma 2.3, we see that

$$\begin{aligned} \omega^\Delta(t) &\leq -\eta(t)p(t)(1 - r(\delta(t)))^\gamma (\alpha(t, t_*))^\gamma \\ &\quad + \frac{\eta^\Delta(t)}{\eta^\sigma(t)} \omega^\sigma(t) - \frac{\gamma M^{\gamma-1} \eta(t)}{a(t)(\eta^\sigma(t))^2} \left( \frac{\sigma(t)}{a(\sigma(t))} \right)^{\gamma-1} (\omega^\sigma(t))^2. \end{aligned} \tag{2.93}$$

Integrating (2.93) from  $t_1$  to  $t$ , we obtain

$$\begin{aligned} &\int_{t_1}^t \eta(s)p(s)(1 - r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma \Delta s \\ &\leq - \int_{t_1}^t \omega^\Delta(s) \Delta s + \int_{t_1}^t \frac{\eta^\Delta(s)}{\eta^\sigma(s)} \omega^\sigma(s) \Delta s - \int_{t_1}^t \frac{\gamma M^{\gamma-1} \eta(s)}{a(s)(\eta^\sigma(s))^2} \left( \frac{\sigma(s)}{a(\sigma(s))} \right)^{\gamma-1} (\omega^\sigma(s))^2 \Delta s. \end{aligned} \tag{2.94}$$

Thus

$$\int_{t_1}^t \left[ \eta(s)p(s)(1 - r(\delta(s)))^\gamma (\alpha(s, t_*))^\gamma - \frac{a(s)(\sigma(s))^{1-\gamma} (\eta^\Delta(s))^2}{4\gamma M^{\gamma-1} (a(\sigma(s)))^{1-\gamma} \eta(s)} \right] \Delta s \leq \omega(t_1), \tag{2.95}$$

which contradicts condition (2.85). The proof is complete.  $\square$

**Theorem 2.26.** Assume that  $\gamma \leq 1$ ,  $a^\Delta(t) \leq 0$ ,  $-1 < -r_0 \leq r(t) \leq 0$ , and  $\lim_{t \rightarrow \infty} r(t) = r > -1$ . If there exists a positive  $\Delta$ -differentiable function  $\eta$  such that for all sufficiently large  $t_*$ , and for all constants  $M > 0$ , for  $t_1 \geq t_*$ ,

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[ \eta(s)p(s)(\alpha(s, t_*))^\gamma - \frac{a(s)(\sigma(s))^{1-\gamma}(\eta^\Delta(s))^2}{4\gamma M^{\gamma-1}(a(\sigma(s)))^{1-\gamma}\eta(s)} \right] \Delta s = \infty, \quad (2.96)$$

then every solution of (1.33) either oscillates or tends to zero as  $t \rightarrow \infty$ .

*Proof.* By Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.25, so we omit the details.  $\square$

### 3. Conclusions

In this paper, we consider the oscillation of second-order Emden-Fowler neutral delay dynamic equations (1.33). In some sense, our results extend and improve the results in [7, 32, 34, 35, 40, 41]. For example, Theorems 2.5, 2.11, 2.13, and 2.23 give some answers for the open problem posed by [34] since these results can be applied to (1.33) when  $\gamma < 1$ , Theorems 2.7, 2.12, 2.14, 2.16, 2.18, 2.20, 2.22, 2.24, and 2.26 correct an error in [35]. Theorem 2.15 includes the results of [7, Theorem 4.4], [32, Theorem 3.1], Theorem 2.11 includes the result of [32, Theorem 3.5], Theorem 2.11 and Corollary 2.6 include the result of [41, Theorem 2.1(a),  $m = 2$ ], Corollary 2.8 includes result of [41, Theorem 2.2,  $m = 2$ ], Theorem 2.13 does not require the conditions  $a^\Delta(t) \geq 0$ , so it improves the results of [40], and Theorems 2.17 and 2.21 improve the results in [34] since these results can be applied when  $a^\Delta(t) \leq 0$ .

The main results in this paper require that  $\int_{t_0}^{\infty} \Delta t / a(t) = \infty$ ; it would be interesting to find another method to study (1.33) when  $\int_{t_0}^{\infty} \Delta t / a(t) < \infty$ . Additional examples may also be given; due to the limited space, we leave this to the interested reader.

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