Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 571795, 7 pages
doi:10.1155/2011/571795

Research Article

Bounds of Solutions of Integrodifferential Equations

Zdeněk Šmarda

Department of Mathematics, Faculty of Electrical Engineering and Communication, Technická 8, Brno University of Technology, 61600 Brno, Czech Republic

Correspondence should be addressed to Zdeněk Šmarda, smarda@feec.vutbr.cz

Received 20 January 2011; Accepted 24 February 2011

Academic Editor: Miroslava Růžičková

Some new integral inequalities are given, and bounds of solutions of the following integro-differential equation are determined: $x'(t) - \digamma(t, x(t), \int_0^t k(t, s, x(t), x(s))ds) = h(t)$, $x(0) = x_0$, where $h : R_+ \to R$, $k : R_+^2 \times R^2 \to R$, $\digamma : R_+ \times R^2 \to R$ are continuous functions, $R_+ = [0, \infty)$.

1. Introduction

Ou Yang [1] established and applied the following useful nonlinear integral inequality.

Theorem 1.1. *Let u and h be nonnegative and continuous functions defined on R_+ and let $c \geq 0$ be a constant. Then, the nonlinear integral inequality*

$$u^2(t) \leq c^2 + 2\int_0^t h(s)u(s)ds, \quad t \in R_+ \tag{1.1}$$

implies

$$u(t) \leq c + \int_0^t h(s)ds, \quad t \in R_+. \tag{1.2}$$

This result has been frequently used by authors to obtain global existence, uniqueness, boundedness, and stability of solutions of various nonlinear integral, differential, and integrodifferential equations. On the other hand, Theorem 1.1 has also been extended and generalized by many authors; see, for example, [2–19]. Like Gronwall-type inequalities, Theorem 1.1 is also used to obtain *a priori* bounds to unknown functions. Therefore, integral inequalities of this type are usually known as Gronwall-Ou Yang type inequalities.

In the last few years there have been a number of papers written on the discrete inequalities of Gronwall inequality and its nonlinear version to the Bihari type, see [13, 16, 20]. Some applications discrete versions of integral inequalities are given in papers [21–23].

Pachpatte [11, 12, 14–16] and Salem [24] have given some new integral inequalities of the Gronwall-Ou Yang type involving functions and their derivatives. Lipovan [7] used the modified Gronwall-Ou Yang inequality with logarithmic factor in the integrand to the study of wave equation with logarithmic nonlinearity. Engler [5] used a slight variant of the Haraux's inequality for determination of global regular solutions of the dynamic antiplane shear problem in nonlinear viscoelasticity. Dragomir [3] applied his inequality to the stability, boundedness, and asymptotic behaviour of solutions of nonlinear Volterra integral equations.

In this paper, we present new integral inequalities which come out from above-mentioned inequalities and extend Pachpatte's results (see [11, 16]) especially. Obtained results are applied to certain classes of integrodifferential equations.

2. Integral Inequalities

Lemma 2.1. *Let u, f, and g be nonnegative continuous functions defined on R_+. If the inequality*

$$u(t) \le u_0 + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau\right)ds \tag{2.1}$$

holds where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \le u_0\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right] \tag{2.2}$$

for $t \in R_+$.

Proof. Define a function $v(t)$ by the right-hand side of (2.1)

$$v(t) = u_0 + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau\right)ds. \tag{2.3}$$

Then, $v(0) = u_0$, $u(t) \le v(t)$ and

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_0^t g(s)(u(t) + u(s))ds \\ &\le f(t)v(t) + f(t)\int_0^t g(s)(v(t) + v(s))ds. \end{aligned} \tag{2.4}$$

Define a function $m(t)$ by

$$m(t) = v(t) + \int_0^t g(s)v(s)ds + v(t)\int_0^t g(s)ds, \tag{2.5}$$

then $m(0) = v(0) = u_0$, $v(t) \le m(t)$,

$$v'(t) \le f(t)m(t), \tag{2.6}$$

$$\begin{aligned} m'(t) &= 2g(t)v(t) + v'(t)\left(1 + \int_0^t g(s)ds\right) \\ &\le m(t)\left[2g(t) + f(t)\left(1 + \int_0^t g(s)ds\right)\right]. \end{aligned} \tag{2.7}$$

Integrating (2.7) from 0 to t, we have

$$m(t) \le u_0 \exp\left(\int_0^t \left(2g(s) + f(s)\left(1 + \int_0^s g(\sigma)d\sigma\right)\right)ds\right). \tag{2.8}$$

Using (2.8) in (2.6), we obtain

$$v'(t) \le u_0 f(t) \exp\left(\int_0^t \left(2g(s) + f(s)\left(1 + \int_0^s g(\sigma)d\sigma\right)\right)ds\right). \tag{2.9}$$

Integrating from 0 to t and using $u(t) \le v(t)$, we get inequality (2.2). The proof is complete. □

Lemma 2.2. *Let u, f, and g be nonnegative continuous functions defined on R_+, $w(t)$ be a positive nondecreasing continuous function defined on R_+. If the inequality*

$$u(t) \le w(t) + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau\right)ds, \tag{2.10}$$

holds, where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \le w(t)\left[1 + \int_0^t f(s) \exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \tag{2.11}$$

where $t \in R_+$.

Proof. Since the function $w(t)$ is positive and nondecreasing, we obtain from (2.10)

$$\frac{u(t)}{w(t)} \le 1 + \int_0^t f(s)\left(\frac{u(s)}{w(s)} + \int_0^s g(\tau)\left(\frac{u(s)}{w(s)} + \frac{u(\tau)}{w(\tau)}\right)d\tau\right)ds. \tag{2.12}$$

Applying Lemma 2.1 to inequality (2.12), we obtain desired inequality (2.11). □

Lemma 2.3. *Let u, f, g, and h be nonnegative continuous functions defined on R_+, and let c be a nonnegative constant.*

If the inequality

$$u^2(t) \le c^2 + 2\left[\int_0^t f(s)u(s)\left(u(s) + \int_0^s g(\tau)(u(\tau) + u(s))d\tau\right) + h(s)u(s)\right]ds \tag{2.13}$$

holds for $t \in R_+$, then

$$u(t) \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \tag{2.14}$$

where

$$p(t) = c + \int_0^t h(s)ds. \tag{2.15}$$

Proof. Define a function $z(t)$ by the right-hand side of (2.13)

$$z(t) = c^2 + 2\left[\int_0^t f(s)u(s)\left(u(s) + \int_0^s g(\tau)(u(\tau) + u(s))d\tau\right) + h(s)u(s)\right]ds. \tag{2.16}$$

Then $z(0) = c^2$, $u(t) \le \sqrt{z(t)}$ and

$$\begin{aligned} z'(t) &= 2\left[f(t)u(t)\left(u(t) + \int_0^t g(s)(u(t) + u(s))ds\right) + h(t)u(t)\right] \\ &\le 2\sqrt{z(t)}\left[f(t)\left(\sqrt{z(t)} + \int_0^t g(s)\left(\sqrt{z(t)} + \sqrt{z(s)}\right)ds\right) + h(t)\right]. \end{aligned} \tag{2.17}$$

Differentiating $\sqrt{z(t)}$ and using (2.17), we get

$$\begin{aligned} \frac{d}{dt}\left(\sqrt{z(t)}\right) &= \frac{z'(t)}{2\sqrt{z(t)}} \\ &\le f(t)\left(\sqrt{z(t)} + \int_0^t g(s)\left(\sqrt{z(t)} + \sqrt{z(s)}\right)ds\right) + h(t). \end{aligned} \tag{2.18}$$

Integrating inequality (2.18) from 0 to t, we have

$$\sqrt{z(t)} \le p(t) + \int_0^t f(s)\left(\sqrt{z(s)} + \int_0^s g(\tau)\left(\sqrt{z(s)} + \sqrt{z(\tau)}\right)d\tau\right)ds, \tag{2.19}$$

where $p(t)$ is defined by (2.15), $p(t)$ is positive and nondecreasing for $t \in R_+$. Now, applying Lemma 2.2 to inequality (2.19), we get

$$\sqrt{z(t)} \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right]. \tag{2.20}$$

Using (2.20) and the fact that $u(t) \le \sqrt{z(t)}$, we obtain desired inequality (2.14). □

3. Application of Integral Inequalities

Consider the following initial value problem

$$x'(t) - \digamma\left(t, x(t), \int_0^t k(t, s, x(t), x(s))ds\right) = h(t), \quad x(0) = x_0, \tag{3.1}$$

where $h : R_+ \to R$, $k : R_+^2 \times R^2 \to R$, $\digamma : R_+ \times R^2 \to R$ are continuous functions. We assume that a solution $x(t)$ of (3.1) exists on R_+.

Theorem 3.1. *Suppose that*

$$\begin{gathered} |k(t, s, u_1, u_2)| \le f(t)g(s)(|u_1| + |u_2|) \quad \text{for } (t, s, u_1, u_2) \in R_+^2 \times R^2, \\ |\digamma(t, u_1, v_1)| \le f(t)|u_1| + |v_1| \quad \text{for } (t, u_1, v_1) \in R_+ \times R^2, \end{gathered} \tag{3.2}$$

where f, g are nonnegative continuous functions defined on R_+. Then, for the solution $x(t)$ of (3.1) the inequality

$$\begin{gathered} |x(t)| \le r(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \\ r(t) = |x_0| + \int_0^t |h(t)|dt \end{gathered} \tag{3.3}$$

holds on R_+.

Proof. Multiplying both sides of (3.1) by $x(t)$ and integrating from 0 to t we obtain

$$x^2(t) = x_0^2 + 2\int_0^t \left[x(s)\digamma\left(s, x(s), \int_0^s k(s, \tau, x(s), x(\tau))d\tau\right) + x(s)h(s)\right]ds. \tag{3.4}$$

From (3.2) and (3.4), we get

$$|x(t)|^2 \leq |x_0|^2 + 2\int_0^t \left[f(s)|x(s)| \times \left(|x(s)| + \int_0^s g(\tau)(|x(s)| + |x(\tau)|)d\tau \right) + |h(s)||x(s)| \right] ds. \tag{3.5}$$

Using inequality (2.14) in Lemma 2.3, we have

$$|x(t)| \leq r(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma \right) \right) d\tau \right) ds \right], \tag{3.6}$$

where

$$r(t) = |x_0| + \int_0^t |h(t)|dt, \tag{3.7}$$

which is the desired inequality (3.3). □

Remark 3.2. It is obvious that inequality (3.3) gives the bound of the solution $x(t)$ of (3.1) in terms of the known functions.

Acknowledgment

This author was supported by the Council of Czech Government grant MSM 00216 30503 and MSM 00216 30529 and by the Grant FEKTS-11-2-921 of Faculty of Electrical Engineering and Communication.

References

[1] L. Ou Yang, "The boundedness of solutions of linear differential equations $y' + A(t)y = 0$," *Advances in Mathematics*, vol. 3, pp. 409–415, 1957.

[2] D. Baınov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[3] S. S. Dragomir, "On Volterra integral equations with kernels of L-type," *Analele Universit a tii din Timi soara. Seria Stiin te Matematice*, vol. 25, no. 2, pp. 21–41, 1987.

[4] S. S. Dragomir and Y.-H. Kim, "On certain new integral inequalities and their applications," *JIPAM: Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 4, article 65, p. 8, 2002.

[5] H. Engler, "Global regular solutions for the dynamic antiplane shear problem in nonlinear viscoelasticity," *Mathematische Zeitschrift*, vol. 202, no. 2, pp. 251–259, 1989.

[6] A. Haraux, *Nonlinear Evolution Equations. Global Behavior of Solutions*, vol. 841 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1981.

[7] O. Lipovan, "A retarded integral inequality and its applications," *Journal of Mathematical Analysis and Applications*, vol. 285, no. 2, pp. 436–443, 2003.

[8] Q. H. Ma and L. Debnath, "A more generalized Gronwall-like integral inequality wit applications," *International Journal of Mathematics and Mathematical Sciences*, vol. 15, pp. 927–934, 2003.

[9] Q.-H. Ma and E.-H. Yang, "On some new nonlinear delay integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 864–878, 2000.

[10] F. W. Meng and W. N. Li, "On some new integral inequalities and their applications," *Applied Mathematics and Computation*, vol. 148, no. 2, pp. 381–392, 2004.

[11] B. G. Pachpatte, "On some new inequalities related to certain inequalities in the theory of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 1, pp. 128–144, 1995.

[12] B. G. Pachpatte, "On some integral inequalities similar to Bellman-Bihari inequalities," *Journal of Mathematical Analysis and Applications*, vol. 49, pp. 794–802, 1975.

[13] B. G. Pachpatte, "On certain nonlinear integral inequalities and their discrete analogues," *Facta Universitatis. Series: Mathematics and Informatics*, no. 8, pp. 21–34, 1993.

[14] B. G. Pachpatte, "On some fundamental integral inequalities arising in the theory of differential equations," *Chinese Journal of Contemporary Mathematics*, vol. 22, pp. 261–273, 1994.

[15] B. G. Pachpatte, "On a new inequality suggested by the study of certain epidemic models," *Journal of Mathematical Analysis and Applications*, vol. 195, no. 3, pp. 638–644, 1995.

[16] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, Mathematics in Science and Engineering 197, Academic Press, San Diego, Calif, USA, 2006.

[17] Z. Šmarda, "Generalization of certain integral inequalities," in *Proceedings of the 8th International Conference on Applied Mathematics (APLIMAT '09)*, pp. 223–228, Bratislava, Slovakia, 2009.

[18] E. H. Yang, "On asymptotic behaviour of certain integro-differential equations," *Proceedings of the American Mathematical Society*, vol. 90, no. 2, pp. 271–276, 1984.

[19] C.-J. Chen, W.-S. Cheung, and D. Zhao, "Gronwall-Bellman-type integral inequalities and applications to BVPs," *Journal of Inequalities and Applications*, vol. 2009, Article ID 258569, 15 pages, 2009.

[20] E. H. Yang, "On some new discrete generalizations of Gronwall's inequality," *Journal of Mathematical Analysis and Applications*, vol. 129, no. 2, pp. 505–516, 1988.

[21] J. Baštinec and J. Diblík, "Asymptotic formulae for a particular solution of linear nonhomogeneous discrete equations," *Computers & Mathematics with Applications*, vol. 45, no. 6–9, pp. 1163–1169, 2003.

[22] J. Diblík, E. Schmeidel, and M. Růžičková, "Existence of asymptotically periodic solutions of system of Volterra difference equations," *Journal of Difference Equations and Applications*, vol. 15, no. 11-12, pp. 1165–1177, 2009.

[23] J. Diblík, E. Schmeidel, and M. Růžičková, "Asymptotically periodic solutions of Volterra system of difference equations," *Computers & Mathematics with Applications*, vol. 59, no. 8, pp. 2854–2867, 2010.

[24] S. Salem, "On some systems of two discrete inequalities of gronwall type," *Journal of Mathematical Analysis and Applications*, vol. 208, no. 2, pp. 553–566, 1997.