Research Article

The Optimization of Solutions of the Dynamic Systems with Random Structure

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The paper deals with the class of jump control systems with semi-Markov coefficients. The control system is described as the system of linear differential equations. Every jump of the random process implies the random transformation of solutions of the considered system. Relations determining the optimal control to minimize the functional are derived using Lyapunov functions. Necessary conditions of optimization which enables the synthesis of the optimal control are established as well.

1. The Statement of the Problem

The optimal control theory as mathematical optimization method for deriving control policies plays an important role in the development of the modern mathematical control theory. The optimal control deals with the problem of finding such a control law for a given system that a certain optimality criterion is achieved. The background for the optimization method can be found in the work of Lev Pontryagin with his well-known Pontryagin's maximum principle. The optimal control has been applied in diverse fields, such as economics, bioengineering, process control, and many others. Some real-life problems are described by a continuous-time or discrete-time linear system of differential equations, but a lot of them are described by dynamic systems with random jumping changes, for example economics systems. The general theory of random structure systems can be found in the work of Artemiev and Kazakov [1]. The optimization of linear systems with random parameters are considered in many works, for example in [2–12]. Particularly, the original results concerning the stabilization of the systems with random coefficients and a random process are derived using moment equations and Lyapunov functions in [4]. These results create a more convenient technique for applying the method in practice using suitable software for engineering or economics investigation. Our aim is the expansion of the achieved results to a new

class of systems of linear differential equations with semi-Markov coefficients and random transformation of solutions performed simultaneously with jumps of semi-Markov process. We will focus on using the particular values of Lyapunov functions for the calculation of coefficients of the control vector which minimize the quality criterion. We will also establish the necessary conditions of the optimal solution which enables the synthesis of the optimal control for the considered class of systems.

Let us consider the linear control system

$$\frac{dX(t)}{dt} = A(t, \xi(t))X(t) + B(t, \xi(t))U(t) \tag{1.1}$$

on the probability basis $(\Omega, \mathfrak{T}, \mathbf{P}, \mathbf{F} \equiv \{\mathbf{F}_t : t \geq 0\})$ and together with (1.1) we consider the initial conditions

$$X(0) = \varphi(\omega), \quad \varphi : \Omega \longrightarrow \mathbb{R}^n. \tag{1.2}$$

The coefficients of the system are semi-Markov coefficients defined by the transition intensities $q_{\alpha k}(t)$, $\alpha, k = 1, 2, \ldots, n$, from state θ_k to state θ_α. We suppose that the vectors $U(t)$ belong to the set of control U and the functions $q_{\alpha k}(t)$, $\alpha, k = 1, 2, \ldots, n$, satisfy the conditions [13]:

$$q_{\alpha k}(t) \geq 0, \quad \int_0^\infty q_k(t)dt = 1, \quad q_k(t) \equiv \sum_{\alpha=1}^\infty q_{\alpha k}(t). \tag{1.3}$$

Definition 1.1. Let the matrices $Q(t, \xi(t))$, $L(t, \xi(t))$ with semi-Markov elements be symmetric and positive definite. The cost functional

$$J = \int_0^\infty \langle X^*(t)Q(t, \xi(t))X(t) + U^*(t)L(t, \xi(t))U(t) \rangle dt, \tag{1.4}$$

defined on the space $C^1 \times U$, where $\langle \cdot \rangle$ denotes mathematical expectation, is called *the quality criterion*.

Definition 1.2. Let $S(t, \xi(t))$ be a matrix with semi-Markov elements. The control vector

$$U(t) = S(t, \xi(t))X(t) \tag{1.5}$$

which minimizes the quality criterion $J(X, U)$ with respect to the system (1.1) is called *the optimal control*.

If we denote

$$G(t, \xi(t)) \equiv A(t, \xi(t)) + B(t, \xi(t))S(t, \xi(t)),$$

$$H(t, \zeta(t)) \equiv Q(t, \zeta(t)) + S^*(t, \zeta(t))L(t, \zeta(t))S(t, \zeta(t)), \tag{1.6}$$

then the system (1.1) can be rewritten to the form

$$\frac{dX(t)}{dt} = G(t, \xi(t))X(t), \tag{1.7}$$

and the functional (1.4) to the form

$$J = \int_0^\infty \langle X^*(t)H(t, \xi(t))X(t)\rangle dt. \tag{1.8}$$

We suppose also that, together with every jump of random process $\xi(t)$ in time t_j, the solutions of the system (1.7) submit to the random transformation

$$X(t_j + 0) = C_{sk}X(t_j - 0), \quad s, k = 1, 2, \ldots, n, \tag{1.9}$$

if the conditions $\xi(t_j + 0) = \theta_s$, $\xi(t_j - 0) = \theta_k$ hold.

Definition 1.3. Let $a_k(t)$, $k = 1, \ldots, n$, $t \geq 0$ be a selection of n different positive functions. If $\xi(t_j + 0) = \theta_s$, $\xi(t_j - 0) = \theta_k$, $s, k = 1, \ldots, n$, and for $t_j \leq t \leq t_{j+1}$ the equality $a(t, \xi(t) = \theta_s) = a_s(t - t_j)$ holds, then the function $a(t, \xi(t))$ is called *semi-Markov function*.

The application of semi-Markov functions makes it possible to use the concept of stochastic operator. In fact, the semi-Markov function $a(t, \xi(t))$ is an operator of the semi-Markov process $\xi(t)$, because the value of the semi-Markov function $a(t, \xi(t))$ is defined not only by the values t and $\xi(t)$, but it is also necessary to specify the function $a_s(t)$, $t \geq 0$ and the value of the jump of the process $\xi(t)$ in time t_j which precedes the moment of time t.

Our task is the construction of Lyapunov function for the new class of systems of linear differential equations with semi-Markov coefficients and then applying the function to solve the optimization problem which minimizes the quality criterion.

2. Auxiliary Results

In the proof of Theorem 3.1 in Section 3, we will employ two results concerning the construction of the Lyapunov function and the construction of the optimal control for the system of linear differential equations in a deterministic case. We will derive these auxiliary results in this part.

2.1. The Construction of the Lyapunov Function

Let us consider the system of linear differential equations

$$\frac{dX(t)}{dt} = A(t, \xi(t))X(t) \tag{2.1}$$

associated to the system (1.1).

Let us define a quadratic form

$$w(t, x, \xi(t)) = x^* B(t, \xi(t)) x, \quad B(t, \xi(t)) > 0, \tag{2.2}$$

where elements of the matrix $B(t, \xi(t))$ are the semi-Markov processes. The matrix $B(t, \xi(t))$ is defined by such a set of n different symmetric and positive definite matrices $B_k(t)$, $t \geq 0$, $k = 1, \ldots, n$, that the equality $\xi(t) = \theta_s$ for $t_j \leq t \leq t_j + 1$ implies

$$B(t, \xi(t)) = B_s(t - t_j), \quad s = 1, 2, \ldots, n. \tag{2.3}$$

Our purpose in this section is to express the value of the functional

$$v = \int_0^\infty \langle w(t, X(t), \xi(t)) \rangle dt \tag{2.4}$$

in a convenient form, which can help us to prove the L_2-stability of the trivial solution of the system (2.1).

At first, we introduce the particular Lyapunov functions

$$v_k(x) = \int_0^\infty \langle w(t, X(t), \xi(t)) \mid X(t) = x, \ \xi(0) = \theta_k \rangle dt, \quad k = 1, 2, \ldots, n. \tag{2.5}$$

If we can find the values of the particular Lyapunov functions in the form $v_k(x) = x^* C_k x$, $k = 1, 2, \ldots, n$, then value of the functional v can be expressed by the formula

$$v = \int_{E_n} \sum_{k=1}^n v_k(x) f_k(0, x) dx = \sum_{k=1}^n \int_{E_n} C_k \circ xx^* f_k(0, x) dx = \sum_{k=1}^n C_k \circ D_k(0), \tag{2.6}$$

where the scalar value

$$N \circ S = \sum_{k=1}^l \sum_{j=1}^m v_{kj} s_{kj} \tag{2.7}$$

is called the scalar product of the two matrices $N = (v_{kj})$, $S = (s_{kj})$ and has the property [14]

$$\frac{D(N \circ S)}{DS} = N. \tag{2.8}$$

The first auxiliary result contains two equivalent, necessary, and sufficient conditions for the L_2-stability (see in [4]) of the trivial solution of the system (2.1) and one sufficient condition for the stability of the solutions.

Theorem 2.1. *The trivial solution of the system* (2.1) *is L_2-stable if and only if any of the next two equivalent conditions hold:*

(1) *the system of equations*

$$C_k = H_k + \int_0^\infty \sum_{s=1}^n q_{sk}(t) N_k^*(t) C_{sk}^* C_s C_{sk} N_k(t) dt, \quad k = 1, 2, \ldots, n \tag{2.9}$$

has a solution $C_k > 0$, $k = 1, 2, \ldots, n$ for $H_k > 0$, $k = 1, 2, \ldots, n$,

(2) *the sequence of the approximations*

$$C_k^{(0)} = 0,$$

$$C_k^{(j+1)} = H_k + \int_0^\infty \sum_{s=1}^n q_{sk}(t) N_k^*(t) C_{sk}^* C_s^{(j)} C_{sk} N_k(t) dt, \quad k = 1, 2, \ldots, n, \ j = 0, 1, 2, \tag{2.10}$$

converges.

Moreover, the solutions of the system (2.1) *are L_2-stabile, if there exist symmetric and positive definite matrices $C_k > 0$, $k = 1, 2, \ldots, n$, such that the property*

$$C_k - \int_0^\infty \sum_{s=1}^n q_{sk}(t) N_k^*(t) C_{sk}^* C_s C_{sk} N_k(t) dt > 0, \quad k = 1, 2, \ldots, n \tag{2.11}$$

holds.

Proof. We will construct a system of equations, which will define the particular Lyapunov functions $v_k(x)$, $k = 1, 2, \ldots, n$. Let us introduce the auxiliary semi-Markov functions

$$u_k(t, x) = \langle w(t, X(t), \xi(t)) \mid X(0) = x, \ \xi(0) = \theta_k \rangle, \quad k = 1, 2, \ldots, n. \tag{2.12}$$

For the state $\xi(t) = \theta_k$, $t \geq 0$ of the random process $\xi(t)$, the equalities

$$X(t) = N_k(t)x, \quad X(0) = x \tag{2.13}$$

are true. Simultaneously, with the jumps of the random process $\xi(t)$, the jumps of solutions of (2.1) occurred, so in view of (2.12), we derive the equations

$$u_k(t, x) = \psi_k(t) w_k(t, N_k(t)x) + \int_0^t \sum_{s=1}^n q_{sk}(\tau) u_s(t - \tau, C_{sk} N_k(\tau)x) d\tau, \quad k = 1, 2, \ldots, n. \tag{2.14}$$

Further, if we introduce denoting

$$u_k(t, x) = x^* u_k(t)x, \quad k = 1, 2, \ldots, n, \tag{2.15}$$

then (2.14) can be rewritten as the system of integral equations for the matrix $u_k(t)$ in the form

$$
\begin{aligned}
u_k(t) = {}& \psi_k(t) N_k^*(t) B_k(t) N_k(t) \\
& + \int_0^t \sum_{s=1}^n q_{sk}(\tau) N_k^*(\tau) C_{sk}^* u_s(t-\tau) C_{sk} N_k(\tau) d\tau, \quad k = 1, 2, \ldots, n.
\end{aligned}
\tag{2.16}
$$

We define matrices C_k, $k = 1, 2, \ldots, n$ and functions $v_k(t)$, $k = 1, 2, \ldots, n$, with regard to (2.5) and (2.12), by formulas

$$
C_k = \int_0^\infty u_k(t) dt, \qquad v_k(x) = \int_0^\infty u_k(t, x) dt.
\tag{2.17}
$$

Integrating the system (2.16) from 0 to ∞, we get the system

$$
\begin{aligned}
C_k = {}& \int_0^\infty \psi_k(t) N_k^*(t) B_k(t) N_k(t) dt \\
& + \int_0^\infty \sum_{s=1}^n q_{sk}(\tau) N_k^*(\tau) C_{sk}^* C_s C_{sk} N_k(\tau) d\tau, \quad k = 1, 2, \ldots, n.
\end{aligned}
\tag{2.18}
$$

Similarly, integrating the system of (2.14), we get the system of equations determining the particular Lyapunov functions

$$
v_k(x) = \int_0^\infty \psi_k(t) w_k(t, N_k(t)x) dt + \int_0^\infty \sum_{s=1}^n q_{sk}(t) v_k(C_{sk} N_k(t)x) dt.
\tag{2.19}
$$

Let us denote

$$
H_k = \int_0^\infty \psi_k(t) N_k^*(t) B_k(t) N_k(t) dt, \quad k = 1, 2, \ldots, n.
\tag{2.20}
$$

If there exist such positive constants λ_1, λ_2 that

$$
\lambda_1 E \le B_k(t) \le \lambda_2 E,
\tag{2.21}
$$

or equivalent conditions

$$
\lambda_1 \|x\|^2 \le x^* B_k(t) x \le \lambda_2 \|x\|^2
\tag{2.22}
$$

hold, then the matrices H_k, $k = 1, 2, \ldots, n$ are symmetric and positive definite. Using (2.17), the system (2.18) can be rewritten to the form

$$
C_k = H_k + \sum_{s=1}^n L_{sk}^* C_s, \quad k = 1, 2, \ldots, n.
\tag{2.23}
$$

It is easy to see that the system (2.23) is conjugated to the system (2.9). Therefore, the existence of a positive definite solution $C_k > 0$, $k = 1, 2, \ldots, n$ of the system (2.23) is equivalent to the existence of a positive definite solution $B_k > 0$, $k = 1, 2, \ldots, n$ and it is equivalent to L_2-stability of the solution of the system (2.1). On the other hand, if the existence of the particular Lyapunov functions $v_k(x)$, $k = 1, 2, \ldots, n$ in (2.5) implies L_2-stability of the solutions of the system (2.1), then, in view of conditions (2.22) and the convergence of the integral (2.17), we get the inequality

$$\int_0^\infty \langle w(t, X(t), \xi(t)) \rangle dt \geq \int_0^\infty \left\langle \|X\|^2 \right\rangle dt. \tag{2.24}$$

The theorem is proved. □

Remark 2.2. If the system of linear differential equations (2.1) is a system with piecewise constant coefficients and the function $w(t, X(t), \xi(t))$ has the form

$$w(t, X(t), \xi(t)) = x^* B(\xi(t)) x, \qquad B_k \equiv B(\theta_k), \quad k = 1, 2, \ldots, n, \tag{2.25}$$

then the system (2.18) can be written in the form

$$C_k = \int_0^\infty \psi_k(t) e^{A_k^* t} B_k e^{A_k t} dt + \int_0^\infty \sum_{s=1}^n q_{sk}(t) e^{A_k^* t} C_{sk}^* C_s C_{sk} e^{A_k t} dt, \quad k = 1, 2, \ldots, n. \tag{2.26}$$

Particularly, if the semi-Markov process $\xi(t)$ is identical with a Markov process, then the system (2.26) has the form

$$C_k = \int_0^\infty e^{a_{kk} t} e^{A_k^* t} B_k e^{A_k t} dt + \int_0^\infty \sum_{\substack{s=1 \\ s \neq k}}^n a_{sk} e^{a_{kk} t} e^{A_k^* t} C_{sk}^* C_s C_{sk} e^{A_k t} dt, \quad k = 1, 2, \ldots, n, \tag{2.27}$$

or, more simply

$$C_k = \int_0^\infty e^{a_{kk} t} e^{A_k^* t} \left(B_k + \sum_{\substack{s=1 \\ s \neq k}}^n a_{sk} C_{sk}^* C_s C_{sk} \right) e^{A_k t} dt, \quad k = 1, 2, \ldots, n. \tag{2.28}$$

Moreover, under the assumption that the integral in (2.28) converges, the system (2.28) is equivalent to the system of matrices equations

$$(E a_{kk} + A_k^*) C_k + C_k A_K + B_k + \sum_{\substack{s=1 \\ s \neq k}}^n a_{sk} C_{sk}^* C_s C_{sk} = 0, \quad k = 1, 2, \ldots, n, \tag{2.29}$$

which can be written as the system

$$A_k^* C_k + C_k A_K + B_k + \sum_{\substack{s=1 \\ s \neq k}}^{n} a_{sk} C_{sk}^* C_s C_{sk} = 0, \quad k = 1, 2, \ldots, n, \tag{2.30}$$

if $C_{kk} = E$, $k = 1, 2, \ldots, n$.

Example 2.3. Let the semi-Markov process $\xi(t)$ take two states θ_1, θ_2 and let it be identical with the Markov process described by the system of differential equations

$$\frac{dp_1(t)}{dt} = -\lambda p_1(t) + \lambda p_2(t),$$

$$\frac{dp_2(t)}{dt} = \lambda p_1(t) - \lambda p_2(t). \tag{2.31}$$

We will consider the L_2-stability of the solutions of the differential equation

$$\frac{dx(t)}{dt} = a(\xi(t))x(t), \qquad a(\theta_k) \equiv a_k, \tag{2.32}$$

constructing a system of the type (2.26) related to (2.32). The system is

$$c_1 = 1 + \int_0^\infty e^{2a_2 t} \lambda e^{-\lambda t} c_2 dt, \qquad c_2 = 1 + \int_0^\infty e^{2a_1 t} \lambda e^{-\lambda t} c_1 dt, \tag{2.33}$$

and its solution is

$$c_1 = \frac{(\lambda - a_1)(\lambda - 2a_2)}{2a_1 a_2 - \lambda(a_1 + a_2)}, \qquad c_2 = \frac{(\lambda - a_2)(\lambda - 2a_1)}{2a_1 a_2 - \lambda(a_1 + a_2)}. \tag{2.34}$$

The trivial solution of (2.32) is L_2-stable, if $c_1 > 0$ and $c_2 > 0$. Let the intensities of semi-Markov process $\xi(t)$ satisfy the conditions

$$q_{11}(t) \approx 0, \qquad q_{22}(t) \approx 0, \qquad q_{21}(t) - \lambda e^{-\lambda t} \approx 0, \qquad q_{12}(t) - \lambda e^{-\lambda t} \approx 0. \tag{2.35}$$

Then, using the Theorem 2.1, the conditions

$$1 - c_1 \int_0^\infty q_{11}(t) e^{2a_1 t} dt - c_2 \int_0^\infty \left(q_{21}(t) - \lambda e^{-\lambda t} \right) e^{2a_2 t} dt > 0,$$

$$1 - c_1 \int_0^\infty \left(q_{12}(t) - \lambda e^{-\lambda t} \right) e^{2a_1 t} dt - c_2 \int_0^\infty q_{22}(t) e^{2a_2 t} dt > 0 \tag{2.36}$$

are sufficient conditions for the L_2-stability of solutions of (2.32).

2.2. The Construction of an Optimal Control for the System of Linear Differential Equations in the Deterministic Case

Let us consider the deterministic system of the linear equations

$$\frac{dX(t)}{dt} = A(t)X(t) + B(t)U(t) \tag{2.37}$$

in the boundary field G, where $X \in \mathbb{R}^m$, $U \in \mathbb{R}^l$, and together with (2.37) we consider the initial conditions

$$X(t) = x_0. \tag{2.38}$$

We assume that the vector $U(t)$ belongs to the control set U. The quality criterion has the form of the quadratic functional

$$I(t) = \frac{1}{2} \int_t^\infty [X^*(\tau)C(\tau)X(\tau) + U^*(\tau)\mathbf{D}(\tau)C(\tau)]d\tau,$$

$$C^*(t) = C(t), \qquad \mathbf{D}^*(t) = \mathbf{D}(t) \tag{2.39}$$

in the space $\mathbb{C}^1(G) \times U$. The control vector

$$U(t) = S(t)X(t), \quad \dim S(t) = l \times m, \tag{2.40}$$

which minimizes the quality criterion (2.39) is called the optimal control.

The optimization problem is the problem of finding the optimal control (2.40) from all feasible control U, or, in fact, it is the problem of finding the equation to determine $S(t)$, $\dim S(t) = l \times m$.

Theorem 2.4. *Let there exist the optimal control* (2.40) *for the system of* (2.37). *Then the control equations*

$$S = -\mathbf{D}^{-1}(t)B^*(t)\mathbf{\Psi}^*, \quad \mathbf{\Psi}^* = K(t)X(t), \tag{2.41}$$

where the matrix $K(t)$ satisfies the Riccati equation

$$\frac{dK(t)}{dt} = -C(t) - K(t)A(t) - A^*(t)K(t) + K^*(t)B(t)\mathbf{D}^{-1}(t)B^*(t)K(t), \tag{2.42}$$

determines the synthesis of the optimal control.

Proof. Let the control for the system (2.37) have the form (2.40), where the matrix $S(t)$ is unknown. Then, the minimum value of the quality criterion (2.39) is

$$\min_{S(t)} I(t) = \frac{1}{2}X^*(t)K(t)X(t) \equiv v(t, X(t)). \tag{2.43}$$

Under assumption that the vector $X(t)$ is known and using Pontryagin's maximum principle [1, 15], the minimum of the quality criterion (2.39) is written as

$$\min_{S(t)} I(t) = \frac{1}{2}\mathbf{\Psi}(t)X(t), \quad \tau \geq t, \tag{2.44}$$

where

$$\mathbf{\Psi}(t) = \frac{Dv(t, x)}{Dx} = X^*K(t) \tag{2.45}$$

is the row-vector. If we take Hamiltonian function [15] of the form

$$H(t, x, U, \mathbf{\Psi}) = \mathbf{\Psi}(A(t)x + B(t)U) + \frac{1}{2}(x^*Cx + U^*\mathbf{D}U), \quad U = Sx, \tag{2.46}$$

the necessary condition for optimality is

$$\frac{\partial H}{\partial s_{kj}} = 0, \quad k = 1, 2, \ldots, l, \ j = 1, 2, \ldots, m, \tag{2.47}$$

where s_{kj} are elements of the matrix S. The scalar value

$$\frac{dH}{dS} = \left\| \frac{\partial H}{\partial s_{kj}} \right\|, \quad k = 1, 2, \ldots, l, \ j = 1, 2, \ldots, m, \tag{2.48}$$

is called derivative of the matrix H with respect to the matrix S.

Employing the scalar product of the two matrices in our calculation, the Hamiltonian function (2.46) can be rewritten into the form

$$H = \mathbf{\Psi}A(t)x + \frac{1}{2}x^*C(t)x + B^*(t)\mathbf{\Psi}^*x^* \circ S + \frac{1}{2}\mathbf{D}(t) \cdot Sxx^* \circ S, \tag{2.49}$$

and its derivative with respect to the matrix S is

$$\frac{dH}{dS} = B^*(t)\mathbf{\Psi}^*x^* + \mathbf{D}(t)Sxx^* = 0. \tag{2.50}$$

Because the equality (2.50) holds for any value of x, the expression of the vector control U has the form

$$U = Sx = -\mathbf{D}^{-1}(t)B^*(t)\mathbf{\Psi}^* = -\mathbf{D}^{-1}(t)B^*(t)K(t)x, \tag{2.51}$$

which implies

$$S = -\mathbf{D}^{-1}(t)B^*(t)\mathbf{\Psi}^*. \tag{2.52}$$

If we put the expression of matrix S to (2.49), we obtain a new expression for the Hamiltonian function

$$H = \mathbf{\Psi}(t)A(t)x + \frac{1}{2}x^*C(t)x - \frac{1}{2}\mathbf{\Psi}B(t)\mathbf{D}^{-1}(t)B^*(t)\mathbf{\Psi}^*, \tag{2.53}$$

for which the canonical system of linear differential equations

$$\frac{dx}{dt} = \frac{DH}{D\mathbf{\Psi}}, \qquad \frac{d\mathbf{\Psi}}{dt} = \frac{DH}{Dx} \tag{2.54}$$

has the form

$$\frac{dx}{dt} = A(t)x - B(t)\mathbf{D}^{-1}(t)B^*(t)\mathbf{\Psi}^*,$$

$$\frac{d\mathbf{\Psi}^*}{dt} = -C(t)x - A^*(t)\mathbf{\Psi}^*. \tag{2.55}$$

In the end, we define the matrix $K(t)$ as the integral manifolds of solutions of the system equations

$$\mathbf{\Psi}^* = K(t)X(t). \tag{2.56}$$

If we derive the system (2.56) with respect to t regarding the system (2.55) and extract the vector $\mathbf{\Psi}^*$, then we obtain the matrix differential equation (2.40). This equation is known as Riccati equation in literature, see for example in [16, 17]. The solution $K_T(t)$ of (2.42) satisfying the initial condition

$$K_T(t) = 0, \quad T > 0 \tag{2.57}$$

determines the minimum of the functional

$$\min_{S(\tau)} \int_t^T [X^*(\tau)C(\tau)X(\tau) + U^*(\tau)\mathbf{D}(\tau)U(\tau)]d\tau = \frac{1}{2}X^*(t)K_T(t)X(t), \tag{2.58}$$

and $K(t)$ can be obtained as the limit of the sequence $\{K_T(t)\}_{T=1}^{\infty}$ of the successive approximations $K_T(t)$:

$$K(t) = \lim_{T \to \infty} K_T(t). \tag{2.59}$$

$\square$

Remark 2.5. Similar results can be obtained from the Bellman equation [18], where the function $v(t, x)$ satisfie

$$\min_{S(t)} \left\{ \frac{\partial v(t, x)}{\partial t} + \frac{Dv(t, x)}{Dx}[A(t) + B(t)S(t)]x + \frac{1}{2}x^*C(t)x + \frac{1}{2}x^*S^*(t)\mathbf{D}(t)S(t)x \right\} = 0. \tag{2.60}$$

3. The Main Result

Theorem 3.1. *Let the coefficients of the control system* (1.1) *be the semi-Markov functions and let them be defined by the equations*

$$\frac{dX_k(t)}{dt} = G_k(t)X_k(t), \quad G_k(t) \equiv A_k(t) + B_k(t)S_k(t), \quad k = 1, \dots, n. \tag{3.1}$$

Then the set of the optimal control is a nonempty subset of the control $\mathcal{U}$, *which is identical with the family of the solutions of the system*

$$U_s(t) = L_s^{-1}(t)B_s^*(t)R_s(t)X_s(t), \quad s = 1, \dots, n, \tag{3.2}$$

where the matrix $R_s(t)$ *is defined by the system of Riccati type of differential equations*

$$\frac{dR_s(t)}{dt} = -Q_s(t) - A_s^*(t)R_s(t) - R_s(t)A_s(t)$$

$$+ R_s(t)B_s(t)L_s^{-1}(t)B_s^*(t)R_s(t) - \frac{\Psi_s'}{\Psi_s(t)}R_s(t) \tag{3.3}$$

$$- \sum_{k=1}^{n} \frac{q_{ks}(t)}{\Psi_s(t)} C_{ks}^* R_k(0)C_{ks}, \quad s = 1, \dots, n.$$

3.1. The Proof of Main Result Using Lyapunov Functions

It should be recalled that the coefficients of the systems (1.1), (1.7) and of the functionals (1.4), (1.8) have the form

$$A(t, \xi(t)) = A_s(t - t_j), \qquad B(t, \xi(t)) = B_s(t - t_j),$$

$$Q(t, \xi(t)) = Q_s(t - t_j), \qquad L(t, \xi(t)) = L_s(t - t_j), \qquad S(t, \xi(t)) = S_s(t - t_j), \tag{3.4}$$

if $t_j \leq t < t_{j+1}$, $\xi(t) = \theta_s$. In addition to this, we have

$$G(t, \xi(t)) = G_s(t - t_j) \equiv A_s(t - t_j) + B_s(t - t_j)S_s(t - t_j),$$

$$H(t, \xi(t)) = H_s(t - t_j) \equiv Q_s(t - t_j) + S_s^*(t - t_j)L_s(t - t_j)S_s(t - t_j). \tag{3.5}$$

The formula

$$V = \sum_{k=1}^{n} C_k \circ D_k(0) = \sum_{k=1}^{n} \int_{E_m} v_k(x)f_k(0, x)dx \tag{3.6}$$

is useful for the calculation of the particular Lyapunov functions $v_k(x) \equiv x^* C_k x$, $k = 1,\ldots,n$ of the functional (1.8). We get

$$
v_k(x) \equiv x^* C_k x
$$
$$
= \int_0^\infty \langle X^*(t)H(t,\xi(t))X(t) \mid X(0) = x, \ \xi(0) = \theta_k \rangle dt, \quad k = 1,2,\ldots,n, \tag{3.7}
$$

or, the more convenient form

$$
v_k(x) \equiv x^* C_k x
$$
$$
= \int_0^\infty \left[X_k^*(t)\left(\Psi_k(t)Q_k(t) + \sum_{s=1}^n q_{sk}(t)C_{sk}^* C_s C_{sk} \right) U_k^*(t)\Psi_k(t)L_k(t)U_k(t) \right] dt, \tag{3.8}
$$
$$
k = 1,2,\ldots,n.
$$

Then the system (3.1) has the form

$$
\frac{dX_k(t)}{dt} = A_k(t)X_k(t) + B_k(t)U_k(t), \quad U_k(t) \equiv S_k(t)X_k(t), \ k = 1,\ldots,n. \tag{3.9}
$$

Let us assume that for the control system (1.1) the optimal control exists in the form (1.5) independent of the initial value $X(0)$. Regarding the formula (3.6), there exist minimal values of the particular Lyapunov functions $v_k(x)$, $k = 1,\ldots,n$, which are associated with the optimal control. It also follows from the fact that the functions $v_k(x)$, $k = 1,\ldots,n$ are particular values of the functional (3.6). Finding the minimal values $v_k(x)$, $k = 1,\ldots,n$ by choosing the optimal control $U_k(x)$ is a well-studied problem, for the main results see [16]. It is significant that all matrices C_s, $s = 1,\ldots,n$ of the integrand in the formula (3.8) are constant matrices, hence, solving the optimization problem they can be considered as matrices of parameters.

Therefore, the problem to find the optimal control (1.5) for the system (1.1) can be transformed to n problems to find the optimal control for the deterministic system (3.9), which is equivalent to the system of linear differential equations of type (2.37).

3.2. The Proof of the Main Result Using Lagrange Functions

In this part, we get one more proof of the Theorem 3.1 using the Lagrange function.

We are looking for the optimal control which reaches the minimum of quality criterion

$$
x^* C x = \int_0^T [(X^*(t)QA)X(t) + U^*(t)L(t)U(t)]dt. \tag{3.10}
$$

Let us introduce the Lagrange function

$$
I = \int_0^T \left[X^*(t)Q(t)X(t) + U^*(t)L(t)U(t) + 2Y^*(t)\left(A(t)X(t) + B(t)U(t) - \frac{dX(t)}{dt} \right) \right] dt, \tag{3.11}
$$

where $Y(t)$ is the column-vector of Lagrange multipliers. In accordance with Pontryagin's maximum principle, we put the first variations of the functionals ∂I_x, ∂I_y equal to zero and we obtain the system of linear differential equations

$$\frac{dX(t)}{dt} = A(t)X(t) - B(t)L^{-1}(t)B^*(t)Y(t),$$

$$\frac{dY(t)}{dt} = -Q(t)X(t) - A^*(t)Y(t). \tag{3.12}$$

Then the optimal control $U(t)$ can be expressed by

$$U(t) = L^{-1}(t)B^*(t)Y(t), \quad Y(T) = 0. \tag{3.13}$$

The synthesis of the optimal control needs to find the integral manifolds of the solutions of the system (3.12) in the form

$$Y(t) = K(t)X(t), \quad K(T) = 0. \tag{3.14}$$

According to the theory of integral manifolds [19] we construct the differential matrix equations of the Riccati type

$$\frac{dK(t)}{dt} = -Q(t) - A^*(t)K(t)A(t) - K(t)B(t)L^{-1}(t)B^*(t)K(t). \tag{3.15}$$

for the matrix $K(t)$. Integrating them from time $t = T$ to time $t = 0$ and using the initial condition $K(T) = 0$ we obtain Lagrange functions for the optimal control

$$U(t) = -L^{-1}(t)B^*(t)K(t)X(t). \tag{3.16}$$

We will prove that

$$\int_t^T [X^*(\tau)Q(\tau)X(\tau) + U^*(\tau)L(\tau)U(\tau)]d\tau = X^*(t)K(t)X(t). \tag{3.17}$$

Differentiating the equality (3.17) with respect to t we obtain the matrix equation

$$-X^*(t)Q(t)X(t) - U^*(t)L(t)U(t) = X^*(t)\frac{dK(t)}{dt}X(t) + X^*(t)K(t)(A(t)X(t) + B(t)U(t))$$

$$+ (X^*(t)A^*(t) + U^*(t)B^*(t))K(t)X(t), \tag{3.18}$$

and extracting the optimal control $U(t)$ we obtain differential equation for $K(t)$ identical with (3.15). The equality $K(t) = K^*(t)$ follows from the positive definite matrices $Q(t)$, $L(t)$ for $t < T$. Therefore, from (3.17) we get $K(t) = 0$; moreover, from (3.10) it follows that $C = K(0)$.

Applying the formulas (3.15), (3.16) to the system (3.8) with minimal functionals (3.9), the expression for the optimal control can be found in the form

$$U_s(t) = -\Psi_s^{-1}(t)L_s^{-1}(t)B_s^*(t)K_s(t)X_s(t), \quad s = 1, 2, \ldots, n, \tag{3.19}$$

where symmetric matrices $K_s(t)$ satisfy the matrix system of differential equations

$$\frac{dK_s(t)}{dt} = -\Psi_s(t) - Q_s(t) - A_s^*(t)K_s(t) - \sum_{k=1}^{n} q_{ks} C_{ks}^* C_k C_{ks}$$
$$+ K_s(t)B_s(t)\Psi_s^{-1}(t)L_s^{-1}(t)B_s(t)K_s(t) \quad s = 1, 2, \ldots, n. \tag{3.20}$$

The systems (3.9), (3.20) define the necessary condition such that the solutions of the systems (1.4) will be optimal. In addition to this, the system (3.8) defines the matrices $S_k(t)$, $k = 1, 2, \ldots, n$, of the optimal control in the form

$$S_k(t) = -\Psi_k^{-1}(t)L_k^{-1}(t)B_k^*(t)K_k(t), \quad k = 1, 2, \ldots, n. \tag{3.21}$$

We define matrices C_s from the system equations (3.20) in the view of

$$C_s = K_s(0), \quad s = 1, 2, \ldots, n. \tag{3.22}$$

In regards to

$$R_s(t) = -\Psi_s^{-1}(t)K_s(t), \quad \Psi_s(0) = 1, \quad C_s = R_s(0), \quad s = 1, 2, \ldots, n, \tag{3.23}$$

it can makes the system (3.20) simpler. Then the system (3.20) takes the form (3.3), and formula (3.2) defines the optimal control.

Remark 3.2. If the control system (1.1) is deterministic, then $q_{ks}(t) \equiv 0$, $\Psi_s(t) \equiv 0$, $k, s = 1, 2, \ldots, n$ and the system (3.3) is identical to the system of the Riccati type equations (3.15).

4. Particular Cases

The optimal control $U(t)$ for the system (1.1) has some special properties, and the equations determining it are different from those given in the previous section in case the coefficients of the control system (1.1) have special properties or intensities $q_{sk}(t)$ satisfy some relations or some other special conditions are satisfied. Some of these cases will be formulated as corollaries.

Corollary 4.1. *Let the control system* (1.1) *with piecewise constant coefficients have the form*

$$\frac{dX(t)}{dt} = A(\xi(t))X(t) + B(\xi(t))U(t). \tag{4.1}$$

Then the quadratics functional

$$V = \int_0^\infty \langle X^*(t)Q(\xi(t))X(t) + U^*(t)L(\xi(t))U(t)\rangle dt \tag{4.2}$$

determines the optimal control in the form

$$U(t) = S(t,\xi(t))X(t), \tag{4.3}$$

where

$$S(t,\xi(t)) = S_k(t - t_j), \tag{4.4}$$

and the matrices $S_k(t)$ satisfy the equations

$$S_k(t) = -L^{-1}B_k^* R_k(t), \quad k = 1,2,\ldots,n \tag{4.5}$$

if $t_j \leq t < t_{j+1}$, $\xi(t) = \theta_k$.

The matrices $R_k(t)$, $k = 1,2,\ldots,n$ are the solutions of the systems of the Riccati-type equations:

$$
\begin{aligned}
\frac{dR_k(t)}{dt} = {}&-Q_k - A_k^* R_k(t) - R_k(t)A_k \\
&+ R_k(t)B_k L_k^{-1} B_k^* R_k(t) - \frac{\Psi'_k(t)}{\Psi_k(t)} R_k(t) \\
&- \sum_{s=1}^n \frac{q_{sk}(t)}{\Psi_k(t)} C_{sk}^* R_s(0) C_{sk}, \quad k = 1,\ldots,n.
\end{aligned}
\tag{4.6}
$$

Remark 4.2. In the corollary we mention piecewise constant coefficients of the control system (4.1). The coefficients of the functional (4.2) will be piecewise as well, but the optimal control is nonstationary.

Corollary 4.3. *Assume that*

$$\frac{\Psi'_k(t)}{\Psi_k(t)} = \text{const}, \quad \frac{q_{sk}(t)}{\Psi_k(t)} = \text{const}, \quad k,s = 1,2,\ldots,n. \tag{4.7}$$

Then the optimal control $U(t)$ will be piecewise constant.

Taking into consideration that the optimal control is piecewise constant, we find out that the matrices $R_k(t)$, $k = 1,2,\ldots,n$ in (4.5) are constant, which implies the form of the system (4.6) is changed to the form

$$Q_k + A_k^* R_k + R_k A_k - R_k B_k L_k^{-1} B_k^* R_k + \frac{\Psi'_k(t)}{\Psi_k(t)} R_k(t) + \sum_{s=1}^n \frac{q_{sk}(t)}{\Psi_k(t)} C_{sk}^* R_k C_{sk} = 0, \quad k = 1,\ldots,n. \tag{4.8}$$

The system (4.8) has constant solutions R_k, $k = 1, 2, \ldots, n$, if conditions (4.7) hold. Moreover, if the random process $\xi(t)$ is a Markov process then the conditions (4.7) have the form

$$\frac{\Psi'_k(t)}{\Psi_k(t)} = a_{kk} = \text{const}, \quad \frac{q_{sk}(t)}{\Psi_k(t)} = a_{sk} = \text{const}, \quad k, s = 1, 2, \ldots, n, \ k \neq s, \tag{4.9}$$

and the system (4.8) transforms to the form

$$Q_k + A_k^* R_k + R_k A_k - R_k B_k L_k^{-1} B_k^* R_k + \sum_{s=1}^{n} a_{sk} C_{sk}^* R_s C_{sk} = 0, \quad k = 1, \ldots, n \tag{4.10}$$

for which the optimal control is

$$U(t) = S(\xi(t))X(t), \qquad S(\theta_k) \equiv S_k, \quad S_k = -L_k^{-1} B_k^* R_k, \quad k = 1, 2, \ldots, n. \tag{4.11}$$

Corollary 4.4. *Let the state θ_s of the semi-Markov process $\xi(t)$ be no longer than $T_s > 0$. Then the system* (3.8) *has the form*

$$v_k(x) \equiv x^* C_k x$$

$$= \int_0^{T_s} \left(X_k^*(t) \left(\Psi_k(t) Q_k(t) + \sum_{s=1}^{n} q_{sk}(t) C_{sk}^* C_s C_{sk} \right) X_k(t) + U_k^*(t) \Psi_k(t) L_k(t) U_k(t) \right) dt,$$

$$k = 1, 2, \ldots, n. \tag{4.12}$$

Because

$$K_s(T_s) = \Psi_s(t) R_s(t), \quad s = 1, 2, \ldots, n, \tag{4.13}$$

then

$$K_s(T_s) = 0, \quad s = 1, 2, \ldots, n. \tag{4.14}$$

In this case, the search for the matrix $K_s(t)$, $s = 1, 2, \ldots, n$ in concrete tasks is reduced to integration of the matrix system of differential equations (3.15) on the interval $[0, T_s]$ with initial conditions (4.14). In view of $\Psi_s(T_s) = 0$, $s = 1, 2, \ldots, n$, we can expect that every equation (3.15) has a singular point $t = T_s$. If $\Psi_s(t)$ has simple zero at the point $t = T_s$, then the system (4.6) meets the necessary condition

$$\Psi_s(T_s) R_s(T_s) + \sum_{k=1}^{n} q_{sk}(T_s) C_{ks}^* R_s(0) C_{ks} = 0, \quad s = 1, \ldots, n \tag{4.15}$$

for boundary of matrix $R_s(t)$ in the singular points.

Acknowledgments

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