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Research Article

The Fixed Point Theory for Some Generalized Nonexpansive Mappings

Enrique Llorens Fuster[1] and Elena Moreno Gálvez[2]

[1] *Departamento de Análisis Matemático, Facultad de Matemáticas, Universitat de Valencia, 46100 Burjassot, Valéncia, Spain*
[2] *Departamento de Matemáticas, Ciencias Naturales y Ciencias Sociales Aplicadas a la Educación, Universidad Católica de Valencia San Vicente Mártir, 46100 Godella, Valéncia, Spain*

Correspondence should be addressed to Enrique Llorens Fuster, enrique.llorens@uv.es

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We study some aspects of the fixed point theory for a class of generalized nonexpansive mappings, which among others contain the class of generalized nonexpansive mappings recently defined by Suzuki in 2008.

1. Introduction

Nonexpansive mappings are those which have Lipschitz's constant equal to 1. A Banach space X is said to have the fixed point property for nonexpansive mappings (FPP in short) provided that every nonexpansive self-mapping of every nonempty, closed, convex, bounded subset C of X has a fixed point.

Since 1965 considerable effort has been aimed to study the fixed point theory for nonexpansive mappings in the setting of both reflexive and nonreflexive Banach spaces. It turns out that property (FPP) closely depends upon geometric characteristics of the Banach space under consideration. Even when C is a weakly compact convex subset of X, a nonexpansive self-mapping of C needs not have fixed points. Nevertheless, if the norm of X has suitable geometric properties (as e.g., uniform convexity, among many others), every nonexpansive self-mapping of every weakly compact convex subset of X has a fixed point. In this case, X is said to have the weak fixed point property, WFPP in short.

Although nonexpansive mappings are perhaps one of the most important topic in the so-called metric fixed point theory, one can find in the literature considerable amount of research about more general classes of mappings than the nonexpansive ones. Among many others, mappings $T : C \to X$ such that for every $x, y \in C$,

$$\|Tx - Ty\| \le a_1\|x - y\| + a_2\|x - Tx\| + a_3\|y - Ty\| + a_4\|x - Ty\| + a_5\|y - Tx\|, \quad (1.1)$$

where $a_1, \dots, a_5$ are nonnegative constants with $\sum_{i=1}^{5} a_i = 1$ were studied since the late sixties of the last century. In many instances (see [1, 2]), this class of mappings are called *generalized nonexpansive* mappings. Despite its perhaps questionable usefulness, this type of condition appears to be quite natural from a geometric point of view. Of course, one easily verifies that the above condition (1.1) is, equivalent to the existence of nonnegative constants a, b, c with $a + 2b + 2c \le 1$ such that for all $x, y \in C$

$$\|Tx - Ty\| \le a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|). \quad (1.2)$$

Particular cases of mappings satisfying condition (1.1) have been studied by various authors independently [2–5].

On the other hand, in a recent paper, Suzuki [6] defined a class of generalized nonexpansive mappings as follows.

Let C be a nonempty subset of a Banach space X. We say that a mapping $T : C \to X$ satisfies *condition* (C) on C if for all $x, y \in C$,

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \text{ implies } \|Tx - Ty\| \le \|x - y\|. \quad (1.3)$$

Of course, every nonexpansive mapping $T : C \to X$ satisfies condition (C) on C, but in [6] some examples are given of noncontinuous mappings satisfying condition (C).

The aim of this paper is to study a class of mappings which properly contains those satisfying either condition (1.3) or (1.2) in many cases. We will show that one of the most important geometric conditions which implies the (WFPP) for nonexpansive mappings, namely the so-called normal structure, also allows us to derive fixed point results for our class of mappings. In particular, for mappings satisfying Suzuki's condition (C), this result is more general than the ones included in [6, 7].

2. Preliminaries

We will assume throughout this paper that $(X, \|\cdot\|)$ is a Banach space and C is a nonempty, closed, convex, bounded subset of X. For a given mapping $T : C \to X$, the (possibly empty) set of all fixed points of T will be denoted by $\mathrm{Fix}(T)$. In the same way, a sequence (x_n) in C is called an *almost fixed point sequence for* T (a.f.p.s. in short) provided that $x_n - T(x_n) \to 0_X$. It is well known that every nonexpansive mapping $T : C \to C$ has a.f.p. sequences. The same holds if $T : C \to C$ satisfies Suzuki's condition (C) on C, (see [6, Lemma 6]).

We now recall further concepts which will be useful in the forthcoming sections.

We begin with some classes of mappings. Definitions (1) and (2) are given in [8], and the first one is a generalization of condition (C) given by Suzuki in [6].

(1) For $\lambda \in (0,1)$, we say that a mapping $T : C \to X$ satisfies *condition* (C_λ) on C if for all $x, y \in C$ with $\lambda\|x - Tx\| \leq \|x - y\|$ one has that $\|Tx - Ty\| \leq \|x - y\|$. Of course, the original Suzuki condition (C) is just $(C_{1/2})$.

(2) For $\mu \geq 1$, a mapping $T : C \to X$ is said to satisfy *condition* (E_μ) *on* C if, for all $x, y \in C$,

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|. \tag{2.1}$$

We say that T satisfies condition (E) on C if T satisfies (E_μ) on C for some $\mu \geq 1$. In [6] is shown that if a mapping satisfies Suzuki's condition $(C_{1/2})$, then it satisfies condition (E_3).

(3) A mapping $T : C \to X$ is said to be *quasi-nonexpansive on* C provided that it has at least one fixed point $p \in C$ and for every $p_0 \in \mathrm{Fix}(T)$, and for all $x \in C$,

$$\|T(x) - p_0\| \leq \|x - p_0\|. \tag{2.2}$$

This concept is essentially due to Díaz and Metcalf [9] and Dotson [10]. The mapping $T : [-1,1] \to [-1,1]$ defined by

$$T(x) = \begin{cases} \dfrac{x}{2}\sin\dfrac{1}{x} & x \neq 0, \\ 0 & x = 0 \end{cases} \tag{2.3}$$

is quasi-nonexpansive on $[-1,1]$ but not nonexpansive.

(4) Finally, a mapping $T : C \to C$ is said to be *asymptotically regular* on C if, for each $x \in C$, it is the case that $\lim_n\|T^n(x) - T^{n+1}(x)\| = 0$.

Let (x_n) be a bounded sequence on X. One defines the *asymptotic radius of* (x_n) *at* $x \in X$ as the number

$$r(x, (x_n)) = \limsup_n\|x - x_n\|. \tag{2.4}$$

In the same way, the *asymptotic radius of* (x_n) *in* C is the number

$$r(C, (x_n)) = \inf\left\{\limsup_n\|x_n - x\| : x \in C\right\} = \inf\{r(x, (x_n)) : x \in C\}, \tag{2.5}$$

and the *asymptotic center* of (x_n) in C as the (possibly empty) set

$$A(C, (x_n)) = \left\{x \in C : \limsup_n\|x_n - x\| = r(C, (x_n))\right\}. \tag{2.6}$$

It is well known, (e.g., see [11]), that $A(C,(x_n)) \neq \emptyset$ whenever C is weakly compact and that if C is convex, then $A(C,(x_n))$ is convex.

Finally, we recall also some geometric properties of normed spaces that will appear in the remainder of this paper.

(1) A normed space $(X, \|\cdot\|)$ is said to satisfy the *Opial condition* if for any sequence (x_n) in X such that $x_n \rightharpoonup x_0$ it happens that for all $y \in X$, $y \neq x_0$,

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - y\|. \tag{2.7}$$

It can be readily established, on the extraction of appropriate subsequences, that the lower limits can be replaced with upper limits in the above definition.

(2) A Banach space $(X, \|\cdot\|)$ is said to have *normal structure* if for each bounded, convex, subset C of X with $\operatorname{diam}(C) > 0$ there exists a *nondiametral* point $p \in C$, that is a point $p \in C$ such that

$$\sup\{\|p - x\| : x \in C\} < diam(C). \tag{2.8}$$

This property was introduced in 1948 by Brodskii and Milman. Since 1965, it has been widely studied due to its relevance in fixed point theory for nonexpansive mappings. For more information see, for instance [11].

3. Condition (L)

Next we introduce a class of nonlinear mappings.

Definition 3.1. A mapping $T : C \to C$ satisfies condition (L), (or it is an (L)-type mapping), on C provided that it fulfills the following two conditions.

(1) If a set $D \subset C$ is nonempty, closed, convex and T-invariant, (i.e., $T(D) \subset D$), then there exists an a.f.p.s. for T in D.

(2) For any a.f.p.s. (x_n) of T in C and each $x \in C$

$$\limsup_{n\to\infty} \|x_n - T(x)\| \leq \limsup_{n\to\infty} \|x_n - x\|. \tag{3.1}$$

From now on, if not specified, a mapping is said to satisfy condition (L), whenever it satisfies it on its domain.

Assumption (1) of this definition is automatically satisfied by several classes of nonlinear mappings. For instance, it is a well-known property of nonexpansive mappings. Thus, if a mapping $T : C \to C$ is nonexpansive with respect to any equivalent renorming of X, then T satisfies (1). Asymptotically regular mappings automatically satisfy (1) too.

We point out that if $T : C \to C$ satisfies condition (L) and the set $\mathrm{Fix}(T)$ is nonempty, then T is quasi-nonexpansive. Indeed, if $p \in \mathrm{Fix}(T) \subset C$, then the sequence (x_n) with $x_n \equiv p$ is, of course, an a.f.p.s. for T, and from assumption (2),

$$\|p - T(x)\| = \limsup_{n\to\infty}\|x_n - T(x)\| \le \limsup_{n\to\infty}\|x_n - x\| = \|p - x\|. \tag{3.2}$$

This means, among other things, that we have some information about the set $\mathrm{Fix}(T)$ when this set is nonempty and T satisfies condition (L). Indeed, according to Theorem 1 of Dotson's paper [10], if $(X, \|\cdot\|)$ is strictly convex, then $\mathrm{Fix}(T)$ is closed and convex, and T is continuous on $\mathrm{Fix}(T)$.

However, quasi-nonexpansive mappings are not relevant concerning the existence of fixed points (because this existence of fixed points is assumed by definition). Nevertheless, as we will see below, there are fixed point free mappings satisfying condition (L) and, moreover it is possible to give some fixed point results for such (L)-type mappings.

Next, we will see that there are quasi-nonexpansive mappings which fail to be (L)-type mappings.

Example 3.2. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, and consider the compact convex set $C = [-1,1] \times [0,1]$. Let $T : C \to C$ be the mapping given by

$$T(x,y) = (xy, y). \tag{3.3}$$

It is straightforward to check that

$$\mathrm{Fix}(T) = \{(0,y) : y \in [0,1]\} \cup \{(x,1) : x \in [-1,1]\}. \tag{3.4}$$

This set is nonconvex, but this does not contradict the above-mentioned Dotson result, because $(\mathbb{R}^2, \|\cdot\|_\infty)$ is not strictly convex.

First, we will see that T is quasi-nonexpansive. For $(0,y) \in \mathrm{Fix}(T)$ and $(x,y) \in C$,

$$\|(0,y) - T(x,y)\|_\infty = |xy| \le |x| = \|(0,y) - (x,y)\|_\infty. \tag{3.5}$$

On the other hand, if $(x,1) \in \mathrm{Fix}(T)$ and $(x,y) \in C$,

$$\|(x,1) - T(x,y)\|_\infty = \max\{|x - xy|, |1-y|\} = |1-y| = \|(x,1) - (x,y)\|_\infty. \tag{3.6}$$

Thus, for every $p \in \mathrm{Fix}(T)$, and $(x,y) \in C$, $\|p - T(x,y)\|_\infty \le \|p - (x,y)\|_\infty$, that is, T is quasi-nonexpansive.

Moreover, taking $p = (1, 3/4)$ and $q = (3/4, 1/2)$, it is easy to check that T is not nonexpansive on C.

Finally, let us see that T fails to satisfy condition (L) on C. Define the sequence

$$p_n = \begin{cases} (1,1) & n = 1,3,5,\ldots, \\ (0,0) & n = 2,4,6,\ldots. \end{cases} \tag{3.7}$$

Of course $p_n - T(p_n) = 0_X$, and then (p_n) is an a.f.p.s. for T on C. But, for $x = (1/2, 1/2)$, one has

$$\limsup_{n} \|T(x) - p_n\|_\infty = \max\left\{ \left\| \left(\frac{1}{4}, \frac{1}{2} \right) \right\|_\infty, \left\| \left(\frac{1}{4}, \frac{1}{2} \right) - (1,1) \right\|_\infty \right\} = \frac{3}{4}, \tag{3.8}$$

while

$$\limsup_{n} \|x - p_n\|_\infty = \max\left\{ \left\| \left(\frac{1}{2}, \frac{1}{2} \right) \right\|_\infty, \left\| \left(\frac{1}{2}, \frac{1}{2} \right) - (1,1) \right\|_\infty \right\} = \frac{1}{2}. \tag{3.9}$$

Next, we will give some examples of mappings satisfying condition (L). Let $T : C \to C$ be a mapping.

Proposition 3.3. *If T is nonexpansive, then it satisfies condition (L).*

Proof. Let $T : C \to C$ be a nonexpansive mapping. It is well-known that if D is a closed convex T-invariant subset of C, then T has a.f.p. sequences in D. Moreover, since T is nonexpansive, for every a.f.p.s. (x_n) for T and every $x \in C$,

$$\limsup_{n\to\infty} \|x_n - Tx\| \le \limsup_{n\to\infty} (\|x_n - Tx_n\| + \|Tx_n - Tx\|) \le \limsup_{n\to\infty} \|x_n - x\|. \tag{3.10}$$

Then, T satisfies condition (L). □

As a direct consequence, the well known examples of fixed point free nonexpansive self-mappings of weakly compact convex subsets, are instances of (L)-type fixed point free self-mappings of such class of sets.

Proposition 3.4. *If $T : C \to C$ satisfies Suzuki's condition (C), then it satisfies condition (L).*

Proof. Recall that if $T : C \to C$ is a type (C) mapping, and D is a closed, convex, T-invariant subset of C, then there exist a.f.p. sequences for T in D (see [6, Lemma 6]). Moreover, in [6, Lemma 7], it was shown that, for every $x, y \in C$,

$$\|x - Ty\| \le 3\|Tx - x\| + \|x - y\|. \tag{3.11}$$

Hence, if (x_n) is an a.f.p.s. for T and $x \in C$,

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - Tx\| &\le \limsup_{n\to\infty} (3\|Tx_n - x_n\| + \|x_n - x\|) \\ &\le 3\limsup_{n\to\infty} \|Tx_n - x_n\| + \limsup_{n\to\infty} \|x_n - x\| \\ &= \limsup_{n\to\infty} \|x_n - x\|. \end{aligned} \tag{3.12}$$

Hence, such mappings satisfy condition (L). □

The above proof can be easily adapted for mappings which satisfy condition (E).

Proposition 3.5. *If $T : C \to C$ satisfies condition (E_μ) for some $\mu \geq 0$. Then, T satisfies condition (L) provided that it satisfies assumption (1) of Definition 3.1.*

Proof. Replace 3 with μ in the above proof. □

We will study further relationships between the class of mappings (C_λ) and those which satisfy condition (L) in Theorem 4.7 in the next section.

In [6, 8] are given some examples of noncontinuous mappings satisfying conditions (C_λ) and (E_μ).

Proposition 3.6. *Let T be a generalized nonexpansive selfmap of C. If any of the following conditions holds, then T satisfies condition (L):*

(1) $a + 2b + 2c < 1$,

(2) $a + 2b + 2c = 1$ *and* $b > 0,\ c > 0,\ a \geq 0$,

(3) $a + 2b + 2c = 1$ *and* $b > 0,\ c = 0,\ a > 0$,

(4) $a + 2b + 2c = 1$ *and* $b = 0,\ c > 0,\ a \geq 0$,

(5) $a + 2b + 2c = 1$ *and* $b = 0,\ c = 0,\ a > 0$, *which implies that* $a = 1$.

Proof. We first need to verify that there exist almost fixed point sequences for T in any nonempty, closed, convex, and T-invariant subset D of C. We need to split the proof in cases according to the above list.

(1) If $a + 2b + 2c < 1$, it is proved in [12, Theorem 4], that a generalized nonexpansive mapping with coefficients a, b, c satisfying $a+2b+2c < 1$ from a nonempty, closed, bounded, convex set K into itself has a fixed point $x^* \in C$. Then, if such $K \subset C$ is T-invariant, then the sequence (x_n) given by $x_n \equiv x^*$, is obviously an a.f.p.s. for T in K.

(2) If $a + 2b + 2c = 1$ and $b > 0,\ c > 0,\ a \geq 0$, in the proof of Theorem 1 of [5], it is seen that for each $x_0 \in C$, the orbit $(T^n(x_0))$ is an a.f.p.s. Thus, T has a.f.p. sequences on each T invariant closed convex subset K of C.

(3) If $a + 2b + 2c = 1$ and $b > 0,\ c = 0,\ a > 0$, in the proof of Theorem 1.1 in [13], it is shown that $\inf\{\|T(x) - x\| : x \in C\} = 0$. Thus, T has again a.f.p. sequences on each T-invariant closed convex subset of C.

(4) If $a+2b+2c = 1$ and $b = 0,\ c > 0,\ a \geq 0$, then T is asymptotically regular on C, that is, any orbit of T is an a.f.p.s. (see [1, pages 83–85]).

(5) If $a+2b+2c = 1$ and $b = 0,\ c = 0,\ a > 0$, then $a = 1$ and therefore, T is nonexpansive. Thus, T has a.f.p. sequences on each T-invariant, closed, convex subset of C.

Next, we will prove the second condition, that is, given an a.f.p.s. (x_n) for T on C, for each $x \in C$ the following inequality holds:

$$\limsup_{n\to\infty}\|x_n - Tx\| \leq \limsup_{n\to\infty}\|x_n - x\|. \tag{3.13}$$

It is well known that (see the proof of Lemma 3.1 of [2]) if $T : C \to C$ is a generalized nonexpansive mapping and $x, y \in C$, then

$$\|x - Tx\| \leq \frac{1+a}{1-b-c}\|y - x\| + \frac{1+b+c}{1-b-c}\|Ty - y\|. \tag{3.14}$$

Let (x_n) be an a.f.p.s. on C and $x \in C$. Then,

$$\begin{aligned}
\limsup_{n\to\infty}\|x_n - Tx\| &\le \limsup_{n\to\infty}\|x_n - Tx_n\| + \limsup_{n\to\infty}\|Tx_n - Tx\| \\
&\le a\limsup_{n\to\infty}\|x_n - x\| + b\left(\limsup_{n\to\infty}\|x_n - Tx_n\| + \|x - Tx\|\right) \\
&\quad + c\left(\limsup_{n\to\infty}\|x_n - Tx\| + \limsup_{n\to\infty}\|x - Tx_n\|\right) \\
&\le a\limsup_{n\to\infty}\|x_n - x\| \\
&\quad + b\left(\frac{1+a}{1-b-c}\limsup_{n\to\infty}\|x_n - x\| + \frac{1+b+c}{1-b-c}\limsup_{n\to\infty}\|x_n - Tx_n\|\right) \\
&\quad + c\limsup_{n\to\infty}\|x_n - Tx\| + c\left(\limsup_{n\to\infty}\|x - x_n\| + \limsup_{n\to\infty}\|x_n - Tx_n\|\right) \\
&= \left(a + b\frac{1+a}{1-b-c} + c\right)\limsup_{n\to\infty}\|x_n - x\| + c\limsup_{n\to\infty}\|x_n - Tx\|.
\end{aligned} \tag{3.15}$$

Thus,

$$(1-c)\limsup_{n\to\infty}\|x_n - Tx\| \le \left(a + b\frac{1+a}{1-b-c} + c\right)\limsup_{n\to\infty}\|x_n - x\|, \tag{3.16}$$

and this leads to

$$\limsup_{n\to\infty}\|x_n - Tx\| \le \frac{a + b((1+a)/(1-b-c)) + c}{1-c}\limsup_{n\to\infty}\|x_n - x\|. \tag{3.17}$$

As $a + 2b + 2c \le 1$, then $1 + a \le 2 - 2b - 2c$ and hence $(1+a)/(1-b-c) \le 2$. Thus,

$$a + b\frac{1+a}{1-b-c} + c \le a + 2b + c \le 1 - c. \tag{3.18}$$

Consequently,

$$\limsup_{n\to\infty}\|x_n - Tx\| \le \frac{a + b((1+a)/(1-b-c)) + c}{1-c}\limsup_{n\to\infty}\|x_n - x\| \le \limsup_{n\to\infty}\|x_n - x\|, \tag{3.19}$$

as desired. □

The inclusions which follow from Propositions 3.3, 3.6, and 3.4 are strict, as the following easy example shows.

Example 3.7. Let $T : [0, 2/3] \to [0, 2/3]$ given by $T(x) = x^2$. Let us see that T satisfies condition (L) on $[0, 2/3]$, but it fails to be generalized nonexpansive and to satisfy Suzuki's condition (C).

(a) *T satisfies condition (L) on* $[0, 2/3]$.

Let us observe that if (x_n) is an a.f.p.s. for T, then since T is continuous, for any convergent subsequence (x_{n_k}) one has that $x_{n_k} \to 0$. Thus, (x_n) is convergent to 0.

The only invariant subsets of $[0, 2/3]$ are the intervals $[0, a]$ with $0 \le a \le 2/3$, and it is clear that in $[0, a]$ T has a.f.p. sequences, namely all those which are convergent to 0.

On the other hand, if (x_n) is an a.f.p.s. for T, and $x \in [0, 2/3]$, then

$$\limsup_n |x_n - T(x)| = x^2 \le x = \limsup_n |x_n - x|. \tag{3.20}$$

(b) *T fails Suzuki's condition (C) on* $[0, 2/3]$.

Take $x = 2/3$ and $y = 1/2$. One has

$$\frac{1}{2}|x - T(x)| = \frac{1}{9} \le \frac{1}{6} = |x - y|, \tag{3.21}$$

while

$$|T(x) - T(y)| = \frac{7}{36} > \frac{1}{6} = |x - y|. \tag{3.22}$$

(c) *T fails to be generalized nonexpansive on* $[0, 2/3]$.

Suppose for a contradiction that there exist positive constants a, b, c with $a+2b+2c \le 1$ such that for every $x, y \in [0, 2/3]$,

$$|T(x) - T(y)| \le a|x - y| + b(|x - T(x)| + |y - T(y)|) + c(|x - T(y)| + |y - T(x)|). \tag{3.23}$$

Then, if we take $x = 2/3$ and $y = 1/3$, we obtain

$$\frac{1}{3} \le \frac{a}{3} + b\frac{4}{9} + c\frac{4}{9}, \tag{3.24}$$

which implies that

$$1 \le a + \frac{4}{3}b + \frac{4}{3}c = \frac{3a + 4b + 4c}{3} \le \frac{a + 2}{3}. \tag{3.25}$$

Therefore, $1 \le a$ which implies that $a = 1$ and $b = c = 0$. But, in this case, T would be nonexpansive, which is impossible because nonexpansive mappings satisfy Suzuki's condition.

Remark 3.8. If $a + 2b + 2c = 1$ and $b > 0$, $c = 0$, $a = 0$ (which implies that $b = 1/2$), then the above proposition also holds whenever the space $(X, \|\cdot\|)$ satisfies suitable geometrical conditions.

(1) From [14], the following fact is well known: every weakly compact convex subset of X has close-to-normal structure if and only if every map of the above type on a weakly compact convex subset of X has a unique fixed point, and hence it has a.f.p. sequences in every T-invariant, closed, convex subset of C.

(2) If X has uniformly normal structure (see [15]), from [4, Theorem 1], we know that T has a unique fixed point and hence it has a.f.p. sequences. The same conclusion is true if T is Lipschitzian on C, without the assumption of uniformly normal structure for X (see [4, Theorem 4]).

Remark 3.9. Condition (1) and (2) in the definition of (L) type mappings are independent as it is shown in the following two examples.

Example 3.10 (see [16]). Let (e_i) be the standard orthonormal basis of ℓ_2 and let

$$K = \left\{ (a_i) \in \ell_2 : \sum_{i=1}^{+\infty} a_i^2 \leq 1 \text{ and } a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0 \right\}. \tag{3.26}$$

For each $x = \sum_{i=1}^{+\infty} a_i e_i \in K$, let

$$g(x) = \max(a_1, 1 - \|x\|_2)\ e_1 + \sum_{i=2}^{+\infty} a_{i-1} e_i, \tag{3.27}$$

where $\|x\|_2 = (\sum_{n=1}^{\infty} |a_n|^2)^{1/2}$. Define the mapping $f : K \to K$ by

$$f(x) = \frac{g(x)}{\|g(x)\|_2}. \tag{3.28}$$

In [16], it is shown that f is fixed point free and asymptotically regular, that is, for every $x \in K$, $f^n(x) - f^{n+1}(x) \to 0_{\ell_2}$, and hence any orbit of f is an a.f.p.s. for f. Therefore, if M is a nonempty invariant subset of K, then for every $x \in M$, $(f^n(x))$ is an a.f.p.s. in M. Thus, f satisfies condition (1) of Definition 3.1.

Consider now the orbit of the mapping f which starts at the point 0_{ℓ_2}. We claim that $f^n(0_{\ell_2}) = \sum_{i=1}^{n} (e_i/\sqrt{n})$

$$f(0_{\ell_2}) = e_1 = \sum_{i=1}^{1} \frac{e_i}{\sqrt{1}}. \tag{3.29}$$

Suppose that for a positive integer n our claim holds. Then,

$$\begin{aligned}
f^{n+1}(0_{\ell_2}) &= f\left(\sum_{i=1}^{n}\frac{e_i}{\sqrt{n}}\right) \\
&= \frac{\max(1/\sqrt{n},1-1)e_1+\sum_{i=2}^{n+1}(e_i/\sqrt{n})}{\left\|\max(1/\sqrt{n},1-1)e_1+\sum_{i=2}^{n+1}(e_i/\sqrt{n})\right\|_2} \\
&= \frac{\sum_{i=1}^{n+1}(e_i/\sqrt{n})}{\sqrt{(n+1)/n}} \\
&= \sum_{i=1}^{n+1}\frac{e_i}{\sqrt{n+1}}.
\end{aligned} \tag{3.30}$$

Since f is asymptotically regular in K, $(x_n) := (f^n(0_{\ell_2}))$ is an a.f.p.s. for f. Considering this a.f.p.s. and the point 0_{ℓ_2}, we will see that f fails condition (2). Indeed,

$$\begin{aligned}
\limsup_{n\to\infty}\|x_n - f(0_{\ell_2})\|_2 &= \limsup_{n\to\infty}\left\|\sum_{i=1}^{n}\frac{e_i}{\sqrt{n}} - e_1\right\|_2 \\
&= \limsup_{n\to\infty}\sqrt{\left(\frac{1}{\sqrt{n}}-1\right)^2+(n-1)\left(\frac{1}{\sqrt{n}}\right)^2} \\
&= \sqrt{2} \\
&> \limsup_{n\to\infty}\|x_n - 0_{\ell_2}\|_2 \\
&= 1.
\end{aligned} \tag{3.31}$$

Example 3.11. Let $T : [0,1] \to [0,1]$ be the mapping given by $T(x) = \sqrt{x}$. It is easy to see that the only closed convex, T-invariant subsets of $C = [0,1]$ are $\{0\}$ and the intervals $[a,1]$ whenever $0 \le a \le 1$. Since 0 is a fixed point for T, then in $\{0\}$, we have the (trivial) a.f.p.s. (x_n) given by $x_n \equiv 0$. In the same way, since 1 is a fixed point for T, then in $[a,1]$, we have the (trivial) a.f.p.s. (x_n) given by $x_n \equiv 1$. Thus, T fulfills condition (1) of Definition 3.1.

On the other hand, for the sequence (x_n) given by $x_n \equiv 0$ one has that, if $x \in (0,1)$,

$$\limsup_{n}|x_n - T(x)| = \sqrt{x} > x = \limsup_{n}|x_n - x|. \tag{3.32}$$

Thus, T fails to satisfy condition (2) of the definition of (L)-type mappings.

4. Fixed Point Theorems

Remark 4.1. Let C be a nonempty closed and convex subset of a Banach space X, and $T : C \to C$ a mapping which satisfies condition (L). If (x_n) is an a.f.p.s. for T, then, for every $x \in A(C,(x_n))$,

$$\limsup_{n\to\infty}\|x_n - Tx\| \le \limsup_{n\to\infty}\|x_n - x\| = r(C,(x_n)), \tag{4.1}$$

that is, asymptotic centers of a.f.p. sequences are invariant under mappings satisfying condition (L).

Theorem 4.2. *Let C be a nonempty compact convex subset of a Banach space X and $T : C \to C$ a mapping satisfying condition (L). Then, T has a fixed point.*

Proof. Since C is nonempty, closed, bounded and convex, and T-invariant, there exists an a.f.p.s. for T, say (x_n), in C. Since C is compact, there exists a subsequence (x_{n_j}) of (x_n) such that (x_{n_j}) converges to some $z \in C$. By assumption (2) of Definition 3.1,

$$\limsup_{j\to\infty}\left\|x_{n_j} - Tz\right\| \le \limsup_{j\to\infty}\left\|x_{n_j} - z\right\| = 0 \tag{4.2}$$

and by unicity of the limit, $Tz = z$. □

Corollary 4.3. *Suppose that the asymptotic center in C of each sequence in C is nonempty and compact. Then, T has a fixed point.*

Proof. Since T satisfies condition (L), there exists an a.f.p.s. for T in C, say (x_n). Let A be the asymptotic center of (x_n) relative to C. By our assumption, A is nonempty and compact and, by Remark 4.1, A is T-invariant. Then, from Theorem 4.2, T has a fixed point on A. □

Notice that whenever the asymptotic center in C of a bounded sequence (x_n) consists of just one point, this point has to be a fixed point for mappings that leave asymptotic centers invariant, which is the case for the mappings satisfying condition (L) studied here.

Some geometrical regularity conditions of the space X or the set C force the asymptotic centers to be "nice". For example, it is known that in uniformly convex spaces asymptotic centers are singletons. Even more, it is also known that the asymptotic centers of bounded sequences wit respect to weakly compact convex sets are compact on k-uniformly convex Banach spaces. (see [17, page 77].)

However, Banach spaces which satisfy either Opial condition or uniform convexity have normal structure and then they fall into the scope of the following theorem.

Let K be weakly compact convex set and T an arbitrary self-map of K. The standard Zorn's Lemma argument gives that K contains a closed, convex, T-invariant, minimal subset M.

Theorem 4.4. *Let X be a Banach space with normal structure. Let K be a nonempty, weakly compact and convex subset of X. Let $T : K \to K$ be a mapping satisfying condition (L). Then, T has a fixed point.*

Proof. Let C be a minimal subset of K. Since T satisfies condition (L), there exists an a.f.p.s. (x_n) for T in C. This sequence is either constant, and hence it consists of a fixed point of T, or it is nonconstant. In this case, since X has normal structure, from Corollary 1 of [5], the real function $g : C \to [0,\infty)$ given by

$$g(x) := \limsup_n \|x - x_n\| \tag{4.3}$$

is not constant in $\operatorname{conv}\{x_n : n = 1,2,\ldots\} \subset C$. Then, g takes at least two values. If r is an intermediate value, then the set

$$M := \{x \in C : g(x) \le r\} \tag{4.4}$$

is nonempty, convex, and closed and $M \ne C$. From condition (L), M is also T-invariant which contradicts the minimality of C. □

Remark 4.5. In the above proof, we have shown that if C is a minimal subset of K, then the functions $g_{(x_n)}(x) := \limsup_n \|x - x_n\|$ are constant on C. It is unclear if this constant value coincides with the diameter of C, as in the nonexpansive case. In the affirmative, we would obtain a Karlovitz-like result for (L) type mappings.

Theorem 4.6. *Let X be a Banach space which satisfies the Opial condition. Let $T : C \to C$ be a mapping satisfying condition (L). Then, if (x_n) is an a.f.p.s. for T such that it converges weakly to $z \in C$, then z is a fixed point of T.*

Proof. Since (x_n) is an a.f.p.s. for T, $z \in C$ and T satisfies condition (L),

$$\limsup_{n\to\infty} \|x_n - Tz\| \le \limsup_{n\to\infty} \|x_n - z\|. \tag{4.5}$$

Given that $x_n \rightharpoonup z$, if $z \ne Tz$, from the Opial condition, we obtain

$$\limsup_{n\to\infty} \|x_n - z\| < \limsup_{n\to\infty} \|x_n - Tz\|, \tag{4.6}$$

which is a contradiction. □

We finish with a result which establishes an alternative for mappings satisfying some condition (C_λ).

Theorem 4.7. *Let C be a closed, convex, bounded subset of a Banach space $(X, \|\cdot\|)$ and let $T : C \to C$ be a continuous mapping satisfying condition (C_λ) on C for some $\lambda \in (0,1)$. Then, at least one of the following statements is true:*

(1) *T has a fixed point,*

(2) *T satisfies condition (L).*

Proof. It is shown in [8] that mappings satisfying some condition (C_λ) have a.f.p. sequences on each closed bounded convex T-invariant subset D of C. Let (x_n) be an a.f.p.s. for T in C.

If for some $x \in C$, there is a subsequence (x_{n_j}) of (x_n) converging to $x \in C$, since (x_{n_j}) is an a.f.p.s., then the sequence (Tx_{n_j}) has the same limit as (x_{n_j}), and therefore by the continuity of T, $Tx = x$, and statement 1 holds.

Suppose now that for every $x \in C$, the sequence (x_n) does not have any subsequence converging to x. We claim that there exists some $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\lambda \|x_n - Tx_n\| \leq \|x_n - x\|. \tag{4.7}$$

Otherwise, for every $n \in \mathbb{N}$, there would exist $j_n \geq n$ such that

$$\lambda \left\|x_{j_n} - Tx_{j_n}\right\| > \left\|x_{j_n} - x\right\|. \tag{4.8}$$

Then, since (x_n) does not have any subsequence converging to x,

$$0 = \liminf_{n \to \infty} \lambda \left\|x_{j_n} - Tx_{j_n}\right\| \geq \liminf_{n \to \infty} \left\|x_{j_n} - x\right\| > 0, \tag{4.9}$$

a contradiction which proves our claim.

From (4.7), bearing in mind that T satisfies condition (C_λ), we have

$$\|Tx_n - Tx\| \leq \|x_n - x\| \tag{4.10}$$

and then

$$\limsup_{n \to \infty} \|x_n - Tx\| \leq \limsup_{n \to \infty} (\|x_n - Tx_n\| + \|Tx_n - Tx\|) \leq \limsup_{n \to \infty} \|x_n - x\|, \tag{4.11}$$

that is, T satisfies condition (L). □

For continuous mappings, a more general result than Theorem 8 of [8] can be obtained by combining the above result with Theorem 4.4.

Corollary 4.8. *Let X be a Banach space with normal structure and K a convex weakly compact subset of X. Let $T : K \to K$ be a continuous mapping satisfying some condition (C_λ) on K. Then, T has a fixed point.*

Proof. If T does not have a fixed point in K, then from the above theorem, it satisfies condition (L). Consequently, from Theorem 4.4, we get a contradiction. □

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