Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 393875, 8 pages
doi:10.1155/2011/393875

Research Article

On a Maximal Number of Period Annuli

Yelena Kozmina[1] and Felix Sadyrbaev[2]

[1] *Information Systems and Management Institute, Ludzas Street 91, 1019 Riga, Latvia*
[2] *Institute of Mathematics and Computer Science, University of Latvia Rainis Boulevard 29, 1459 Riga, Latvia*

Correspondence should be addressed to Felix Sadyrbaev, felix@latnet.lv

Received 30 October 2010; Accepted 30 December 2010

Academic Editor: Elena Braverman

We consider equation $x'' + g(x) = 0$, where $g(x)$ is a polynomial, allowing the equation to have multiple period annuli. We detect the maximal number of possible period annuli for polynomials of odd degree and show how the respective optimal polynomials can be constructed.

1. Introduction

Consider equation

$$x'' + g(x) = 0, \tag{1.1}$$

where $g(x)$ is an odd degree polynomial with simple zeros.

The equivalent differential system

$$x' = y, \qquad y' = -g(x) \tag{1.2}$$

has critical points at $(p_i, 0)$, where p_i are zeros of $g(x)$. Recall that a critical point O of (1.2) is *a center* if it has a punctured neighborhood covered with nontrivial cycles.

We will use the following definitions.

Definition 1.1 (see [1]). A *central region* is the largest connected region covered with cycles surrounding O.

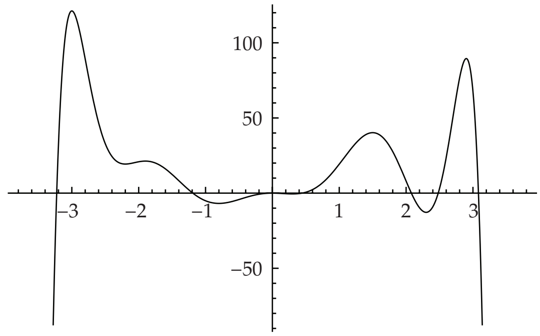

Figure 1

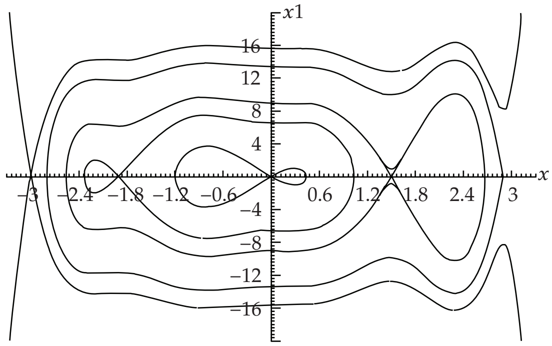

Figure 2: The phase portrait for (1.1), where $G(x)$ is as in Figure 1.

Definition 1.2 (see [1]). A *period annulus* is every connected region covered with nontrivial concentric cycles.

Definition 1.3. We will call a period annulus associated with a central region *a trivial period annulus.* Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type center.

Definition 1.4. Respectively, a period annulus enclosing several (more than one) critical points will be called a *nontrivial period annulus.*

For example, there are four central regions and three nontrivial period annuli in the phase portrait depicted in Figure 2.

Period annuli are the continua of periodic solutions. They can be used for constructing examples of nonlinear equations which have a prescribed number of solutions to the Dirichlet problem

$$x'' + g(x) = 0, \qquad x(0) = 0, \qquad x(1) = 0, \tag{1.3}$$

or a given number of positive solutions [2] to the same problem.

Under certain conditions, period annuli of (1.1) give rise to limit cycles in a dissipative equation

$$x'' + f(x)x' + g(x) = 0. \tag{1.4}$$

The Liénard equation with a quadratical term

$$x'' + f(x)x'^2 + g(x) = 0 \tag{1.5}$$

can be reduced to the form (1.1) by Sabatini's transformation [3]

$$u := \Phi(x) = \int_0^x e^{F(s)} ds, \tag{1.6}$$

where $F(x) = \int_0^x f(s)ds$. Since $du/dx > 0$, this is one-to-one correspondence and the inverse function $x = x(u)$ is well defined.

Lemma 1.5 (see [3, Lemma 1]). *The function $x(t)$ is a solution of* (1.5) *if and only if $u(t) = \Phi(x(t))$ is a solution to*

$$u'' + g(x(u))e^{F(x(u))} = 0. \tag{1.7}$$

Our task in this article is to define the maximal number of nontrivial period annuli for (1.1).

(A) We suppose that $g(x)$ is an odd degree polynomial with simple zeros and with a negative coefficient at the principal term (so $g(-\infty) = +\infty$ and $g(+\infty) = -\infty$). A zero z is called simple if $g(z) = 0$ and $g'(z) \neq 0$.

The graph of a primitive function $G(x) = \int_0^x g(s)ds$ is an even degree polynomial with possible multiple local maxima.

The function $g(x) = -x(x^2 - p^2)(x^2 - q^2)$ is a sample.

We discuss nontrivial period annuli in Section 2. In Section 3, a maximal number of *regular pairs* is detected. Section 4 is devoted to construction of polynomials $g(x)$ which provide the maximal number of *regular pairs* or, equivalently, nontrivial period annuli in (1.1).

2. Nontrivial Period Annuli

The result below provides the criterium for the existence of nontrivial period annuli.

Theorem 2.1 (see [4]). *Suppose that $g(x)$ in* (1.1) *is a polynomial with simple zeros. Assume that M_1 and M_2 ($M_1 < M_2$) are nonneighboring points of maximum of the primitive function $G(x)$. Suppose that any other local maximum of $G(x)$ in the interval (M_1, M_2) is (strictly) less than* $\min\{G(M_1); G(M_2)\}$.

Then, there exists a nontrivial period annulus associated with a pair (M_1, M_2).

It is evident that if $G(x)$ has m pairs of non-neighboring points of maxima then m nontrivial period annuli exist.

Consider, for example, (1.1), where

$$g(x) = -x(x+3)(x+2.2)(x+1.9)(x+0.8)(x-0.3)(x-1.5)(x-2.3)(x-2.9). \tag{2.1}$$

The equivalent system has alternating "saddles" and "centers", and the graph of $G(x)$ is depicted in Figure 1.

There are three pairs of non-neighboring points of maxima and three nontrivial period annuli exist, which are depicted in Figure 2.

3. Polynomials

Consider a polynomial $G(x)$. Points of local maxima x_i and x_j of $G(x)$ are non-neighboring if the interval (x_i, x_j) contains at least one point of local maximum of $G(x)$.

Definition 3.1. Two non-neighboring points of maxima $x_i < x_j$ of $G(x)$ will be called a *regular pair* if $G(x) < \min\{G(x_i), G(x_j)\}$ at any other point of maximum lying in the interval (x_i, x_j).

Theorem 3.2. *Suppose $g(x)$ is a polynomial which satisfies the condition A. Let $G(x)$ be a primitive function for $g(x)$ and n a number of local maxima of $G(x)$.*

Then, the maximal possible number of regular pairs is $n-2$.

Proof. By induction, let $x_1, x_2, \ldots, x_n$ be successive points of maxima of $G(x)$, $x_1 < x_2 < \cdots < x_n$.

(1) Let $n = 3$. The following combinations are possible at three points of maxima:

(a) $G(x_1) \geq G(x_2) \geq G(x_3)$,

(b) $G(x_2) < G(x_1), G(x_2) < G(x_3)$,

(c) $G(x_1) \leq G(x_2) \leq G(x_3)$,

(d) $G(x_2) \geq G(x_1), G(x_2) \geq G(x_3)$.

Only the case (b) provides *a regular pair*. In this case, therefore, the maximal number of *regular pairs* is 1.

(2) Suppose that for any sequence of $n > 3$ ordered points of maxima of $G(x)$ the maximal number of *regular pairs* is $n-2$. Without loss of generality, add to the right one more point of maximum of the function $G(x)$. We get a sequence of $n+1$ consecutive points of maximum $x_1, x_2, \ldots, x_n, x_{n+1}$, $x_1 < x_2 < \ldots < x_n < x_{n+1}$. Let us prove that the maximal number of *regular pairs* is $n-1$. For this, consider the following possible variants.

(a) The couple x_1, x_n is *a regular pair*. If $G(x_1) > G(x_n)$ and $G(x_{n+1}) > G(x_n)$, then, beside the *regular pairs* in the interval $[x_1, x_n]$, only one new *regular pair* can appear, namely, x_1, x_{n+1}. Then, the maximal number of *regular pairs* which can be composed of the points $x_1, x_2, \ldots, x_n, x_{n+1}$, is not greater than $(n-2)+1 = n-1$. If $G(x_1) \leq G(x_n)$ or $G(x_{n+1}) \leq G(x_n)$, then the additional *regular pair* does not appear. In a particular case $G(x_2) < G(x_3) < \cdots < G(x_n) < G(x_{n+1})$ and $G(x_1) > G(x_n)$ the following *regular pairs* exist, namely, x_1 and x_3, x_1 and $x_4, \ldots$, x_1 and x_n, and the new pair x_1 and x_{n+1} appears, totally $n-1$ pairs.

(b) Suppose that x_1, x_n is not *a regular pair*. Let x_i and x_j be a *regular pair*, $1 \leq i < j \leq n$, and there is no other *regular pair* x_p, x_q such that $1 \leq p \leq i < j \leq q \leq n$. Let us mention that if such a pair x_i, x_j does not exist, then the function $G(x)$ does not have *regular pairs* at all and the sequence $\{G(x_k)\}$, $k = 1, \cdots, n$, is monotone. Then, if $G(x_{n+1})$ is greater than any other maximum, there are exactly $(n+1) - 2 = n - 1$ *regular pairs*.

Otherwise, we have two possibilities:

$$\text{either } G(x_i) \geq G(x_p), \quad p = 1, \ldots, i-1,$$
$$\text{or } G(x_j) \geq G(x_q), \quad q = j+1, \ldots, n.$$

In the first case, the interval $[x_1, x_i]$ contains i points of maximum of $G(x)$, $i < n$, and hence the number of *regular pairs* in this interval does not exceed $i - 2$. There are no *regular pairs* x_p, x_k for $1 \leq p < i$, $i < k \leq n+1$. The interval $[x_i, x_{n+1}]$ contains $(n+1) - (i-1)$ points of maximum of $G(x)$, and hence the number of *regular pairs* in this interval does not exceed $(n+1) - (i-1) - 2 = n - i$. Totally, there are no more *regular pairs* than $(i-2) + (n-i) = n - 2$.

In the second case, the number of *regular pairs* in $[x_i, x_j]$ does not exceed $j - (i-1) - 2 = j - i - 1$. In $[x_j, x_{n+1}]$,there are no more than $(n+1) - (j-1) - 2 = n - j$ *regular pairs*. The points x_p, $p = 1, \ldots, i-1$, x_q, $j < q \leq n$ do not form *regular pairs*, by the choice of x_p and x_q. The points x_p, $p = 1, \ldots, i$, together with x_{n+1} (it serves as the $i+1$th point in a collection of points) form not more than $(i+1) - 2 = i - 1$ *regular pairs.* Totally, the number of *regular pairs* is not greater than $(j - i - 1) + (n - j) + (i - 1) = n - 2$. □

4. Existence of Polynomials with Optimal Distribution

Theorem 4.1. *Given number n, a polynomial $g(x)$ can be constructed such that*

(a) *the condition (A) is satisfied,*

(b) *the primitive function $G(x)$ has exactly n points of maximum and the number of regular pairs is exactly $n - 2$.*

Proof of the Theorem. Consider the polynomial

$$G(x) = -\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\left(x + \frac{5}{2}\right)\left(x - \frac{5}{2}\right)\left(x + \frac{7}{2}\right)\left(x - \frac{7}{2}\right). \tag{4.1}$$

It is an even function with the graph depicted in Figure 3.

Consider now the polynomial

$$G_\varepsilon(x) = -\left(x + \frac{1}{2} + \varepsilon\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\left(x + \frac{5}{2}\right)\left(x - \frac{5}{2}\right)\left(x + \frac{7}{2}\right)\left(x - \frac{7}{2}\right), \tag{4.2}$$

where $\varepsilon > 0$ is small enough. The graph of $G_\varepsilon(x)$ with $\varepsilon = 0.2$ is depicted in Figure 4.

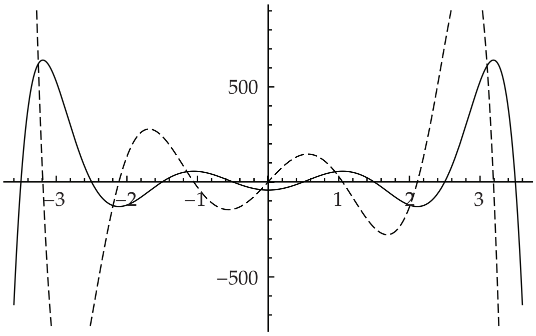

Figure 3: $G(x)$ (solid) and $G'(x) = g(x)$ (dashed).

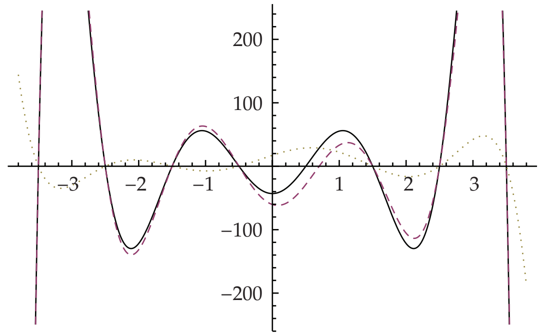

Figure 4: $G(x)$ (solid line), $G_\varepsilon(x)$ (dashed line), and $G(x) - G_\varepsilon(x)$ (dotted line).

Denote the maximal values of $G(x)$ and $G_\varepsilon(x)$ to the right of $x = 0$ m_1^+, m_2^+. Denote the maximal values of $G(x)$ and $G_\varepsilon(x)$ to the left of $x = 0$ m_1^-, m_2^-. One has for $G(x)$ that $m_1^+ = m_1^- < m_2^- = m_2^+$. One has for $G_\varepsilon(x)$ that $m_1^+ < m_1^- < m_2^+ < m_2^-$. Then, there are two regular pairs (resp., m_1^- and m_2^+, m_2^+ and m_2^-).

For arbitrary even n the polynomial

$$G_\varepsilon(x) = -\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\cdots\left(x + \frac{2n-1}{2}\right)\left(x - \frac{2n-1}{2}\right), \tag{4.3}$$

is to be considered where the maximal values $m_1^+, m_2^+, \dots, m_{n/2}^+$ to the right of $x = 0$ form ascending sequence, and, respectively, the maximal values $m_1^-, m_2^-, \dots, m_{n/2}^-$ to the left of $x = 0$ also form ascending sequence. One has that $m_i^+ = m_i^-$ for all i. For a slightly modified polynomial

$$G_\varepsilon(x) = -\left(x + \frac{1}{2} + \varepsilon\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\dots\left(x + \frac{2n-1}{2}\right)\left(x - \frac{2n-1}{2}\right), \tag{4.4}$$

the maximal values are arranged as

$$m_1^+ < m_1^- < m_2^+ < m_2^- < \cdots < m_{n/2}^+ < m_{n/2}^-. \tag{4.5}$$

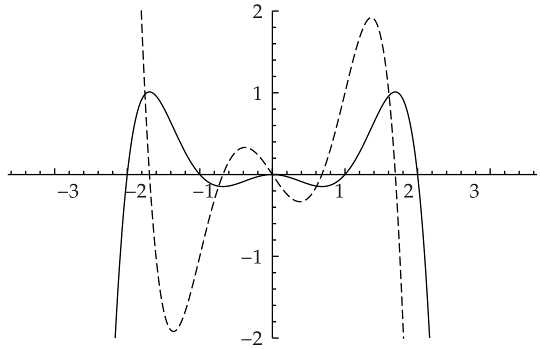

Figure 5: $G(x)$ (solid) and $G'(x) = g(x)$ (dashed).

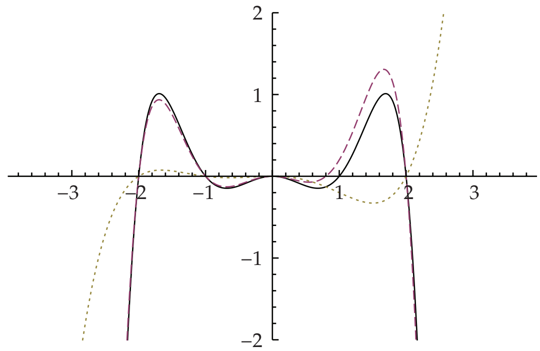

Figure 6: $G(x)$ (solid), $G_\varepsilon(x)$ (dashed), and $G(x) - G_\varepsilon(x)$ (dotted).

Therefore, there exist exactly $n-2$ *regular pairs* and, consequently, $n-2$ nontrivial period annuli in the differential equation (1.1).

If n is odd, then the polynomial

$$G(x) = -x^2(x-1)(x+1)(x-2)(x+2)\cdots(x-(n-1))(x+(n-1)) \tag{4.6}$$

with n local maxima is to be considered. The maxima are descending for $x < 0$ and ascending if $x > 0$. The polynomial with three local maxima is depicted in Figure 5.

The slightly modified polynomial

$$G(x) = -x^2(x-1-\varepsilon)(x+1)(x-2)(x+2)\cdots(x-(n-1))(x+(n-1)) \tag{4.7}$$

has maxima which are not equal and are arranged in an optimal way in order to produce the maximal $(n-2)$ *regular pairs*.

The graph of $G_\varepsilon(x)$ with $\varepsilon = 0.2$ is depicted in Figure 6. □

Acknowledgments

This work has been supported by ERAF project no. 2010/0206/2DP/2.1.1.2.0/10/APIA/VIAA/011 and Latvian Council of Science Grant no. 09.1220.

References

[1] M. Sabatini, "Liénard limit cycles enclosing period annuli, or enclosed by period annuli," *The Rocky Mountain Journal of Mathematics*, vol. 35, no. 1, pp. 253–266, 2005.

[2] S. Atslega and F. Sadyrbaev, "Period annuli and positive solutions of nonlinear boundary value problems," in *Proceedings of the 7th Congress of The International Society for Analysis, Its Applications and Computation (ISAAC '09)*, July 2009, http://www.isaac2009.org/Congress/Welcome.html.

[3] M. Sabatini, "On the period function of $x'' + f(x)x'^2 + g(x) = 0$," *Journal of Differential Equations*, vol. 196, no. 1, pp. 151–168, 2004.

[4] S. Atslega and F. Sadyrbaev, "Multiple solutions of the second order nonlinear Neumann BVP," in *Proceedings of the 6th International Conference on Differential Equations and Dynamical Systems*, pp. 100–103, Watam Press, Baltimore, Md, USA, May 2009.