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Research Article

Positive Solutions to Boundary Value Problems of Nonlinear Fractional Differential Equations

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We study the existence of positive solutions for the boundary value problem of nonlinear fractional differential equations $D_{0^+}^{\alpha}u(t) + \lambda f(u(t)) = 0$, $0 < t < 1$, $u(0) = u(1) = u'(0) = 0$, where $2 < \alpha \leq 3$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, λ is a positive parameter, and $f : (0, +\infty) \to (0, +\infty)$ is continuous. By the properties of the Green function and Guo-Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established. As an application, some examples are presented to illustrate the main results.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications; see [1–4]. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations on terms of special functions.

Recently, there are some papers dealing with the existence of solutions (or positive solutions) of nonlinear initial fractional differential equations by the use of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, Adomian decomposition method, etc.); see [5–11]. In fact, there has the same requirements for boundary conditions. However, there exist some papers considered the boundary value problems of fractional differential equations; see [12–19].

Yu and Jiang [19] examined the existence of positive solutions for the following problem:

$$D_{0^+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,$$
$$u(0)=u(1)=u'(0)=0, \tag{1.1}$$

where $2<\alpha\leq 3$ is a real number, $f\in C([0,1]\times[0,+\infty),(0,+\infty))$, and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional differentiation. By using the properties of the Green function, they obtained some existence criteria for one or two positive solutions for singular and nonsingular boundary value problems by means of the Krasnosel'skii fixed point theorem and a mixed monotone method.

To the best of our knowledge, there is very little known about the existence of positive solutions for the following problem:

$$D_{0^+}^{\alpha}u(t)+\lambda f(u(t))=0,\quad 0<t<1,$$
$$u(0)=u(1)=u'(0)=0, \tag{1.2}$$

where $2<\alpha\leq 3$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, λ is a positive parameter and $f:(0,+\infty)\to(0,+\infty)$ is continuous.

On one hand, the boundary value problem in [19] is the particular case of problem (1.2) as the case of $\lambda=1$. On the other hand, as Yu and Jiang discussed in [19], we also give some existence results by the fixed point theorem on a cone in this paper. Moreover, the purpose of this paper is to derive a λ-interval such that, for any λ lying in this interval, the problem (1.2) has existence and multiplicity on positive solutions.

In this paper, by analogy with boundary value problems for differential equations of integer order, we firstly give the corresponding Green function named by fractional Green's function and some properties of the Green function. Consequently, the problem (1.2) is reduced to an equivalent Fredholm integral equation. Finally, by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established. As an application, some examples are presented to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate analysis of problem (1.2). These materials can be found in the recent literature; see [19–21].

Definition 2.1 (see [20]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $f:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0^+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{(n)}\int_0^t\frac{f(s)}{(t-s)^{\alpha-n+1}}ds, \tag{2.1}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 (see [20]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.2}$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the following statement.

Lemma 2.3 (see [20]). *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0 \tag{2.3}$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$, $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, as unique solutions, where N is the smallest integer greater than or equal to α.

Lemma 2.4 (see [20]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \tag{2.4}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

In the following, we present the Green function of fractional differential equation boundary value problem.

Lemma 2.5 (see [19]). *Let $h \in C[0,1]$ and $2 < \alpha \leq 3$. The unique solution of problem*

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = 0 \end{gathered} \tag{2.5}$$

is

$$u(t) = \int_0^1 G(t,s) h(s) ds, \tag{2.6}$$

where

$$G(t,s)=\begin{cases}\dfrac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\[2mm] \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1.\end{cases} \tag{2.7}$$

Here $G(t,s)$ *is called the Green function of boundary value problem* (2.5).

The following properties of the Green function play important roles in this paper.

Lemma 2.6 (see [19]). *The function* $G(t,s)$ *defined by* (2.7) *satisfies the following conditions:*

(1) $G(t,s)=G(1-s,1-t)$, *for* $t,s\in(0,1)$;

(2) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}\le\Gamma(\alpha)G(t,s)\le(\alpha-1)s(1-s)^{\alpha-1}$, *for* $t,s\in(0,1)$;

(3) $G(t,s)>0$, *for* $t,s\in(0,1)$;

(4) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}\le\Gamma(\alpha)G(t,s)\le(\alpha-1)(1-t)t^{\alpha-1}$, *for* $t,s\in(0,1)$.

The following lemma is fundamental in the proofs of our main results.

Lemma 2.7 (see [21]). *Let* X *be a Banach space, and let* $P\subset X$ *be a cone in* X. *Assume* $\Omega_1,\ \Omega_2$ *are open subsets of* X *with* $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, *and let* $S:P\to P$ *be a completely continuous operator such that, either*

$(A1)$ $\|Sw\|\le\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\ge\|w\|$, $w\in P\cap\partial\Omega_2$ *or*

$(A2)$ $\|Sw\|\ge\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\le\|w\|$, $w\in P\cap\partial\Omega_2$.

Then S *has a fixed point in* $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

For convenience, we set $q(t)=t^{\alpha-1}(1-t)$, $k(s)=s(1-s)^{\alpha-1}$; then

$$q(t)k(s)\le\Gamma(\alpha)G(t,s)\le(\alpha-1)k(s). \tag{2.8}$$

3. Main Results

In this section, we establish the existence of positive solutions for boundary value problem (1.2).

Let Banach space $E=C[0,1]$ be endowed with the norm $\|u\|=\max_{0\le t\le 1}|u(t)|$. Define the cone $P\subset E$ by

$$P=\left\{u\in E:u(t)\ge\frac{q(t)}{\alpha-1}\|u\|,\ t\in[0,1]\right\}. \tag{3.1}$$

Suppose that u is a solution of boundary value problem (1.2). Then

$$u(t)=\lambda\int_0^1 G(t,s)f(u(s))ds,\quad t\in[0,1]. \tag{3.2}$$

We define an operator $A_\lambda : P \to E$ as follows:

$$(A_\lambda u)(t) = \lambda \int_0^1 G(t,s) f(u(s))ds, \quad t \in [0,1]. \tag{3.3}$$

By Lemma 2.6, we have

$$\begin{aligned} \|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) f(u(s))ds, \\ (A_\lambda u)(t) &\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(t)k(s) f(u(s))ds \\ &\geq \frac{q(t)}{\alpha-1}\|A_\lambda u\|. \end{aligned} \tag{3.4}$$

Thus, $A_\lambda(P) \subset P$.

Then we have the following lemma.

Lemma 3.1. *$A_\lambda : P \to P$ is completely continuous.*

Proof. The operator $A_\lambda : P \to P$ is continuous in view of continuity of $G(t,s)$ and $f(u(t))$. By means of the Arzela-Ascoli theorem, $A_\lambda : P \to P$ is completely continuous.

For convenience, we denote

$$\begin{gathered} F_0 = \lim_{u\to 0^+} \sup \frac{f(u)}{u}, \qquad F_\infty = \lim_{u\to +\infty} \sup \frac{f(u)}{u}, \\ f_0 = \lim_{u\to 0^+} \inf \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to +\infty} \inf \frac{f(u)}{u}, \\ C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s)ds, \\ C_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{(\alpha-1)} q(s)k(s)ds, \\ C_3 = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{(\alpha-1)} k(s)ds. \end{gathered} \tag{3.5}$$

□

Theorem 3.2. *If there exists $l \in (0,1)$ such that $q(l) f_\infty C_2 > F_0 C_1$ holds, then for each*

$$\lambda \in \left((q(l) f_\infty C_2)^{-1}, (F_0 C_1)^{-1} \right), \tag{3.6}$$

the boundary value problem (1.2) *has at least one positive solution. Here we impose $(q(l) f_\infty C_2)^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0 C_1)^{-1} = +\infty$ if $F_0 = 0$.*

Proof. Let λ satisfy (3.6) and $\varepsilon > 0$ be such that

$$\left(q(l)\left(f_\infty - \varepsilon\right)C_2\right)^{-1} \leq \lambda \leq \left((F_0 + \varepsilon)C_1\right)^{-1}. \tag{3.7}$$

By the definition of F_0, we see that there exists $r_1 > 0$ such that

$$f(u) \leq (F_0 + \varepsilon)u, \quad \text{for } 0 < u \leq r_1. \tag{3.8}$$

So if $u \in P$ with $\|u\| = r_1$, then by (3.7) and (3.8), we have

$$\begin{aligned}
\|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s)f(u(s))ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s)(F_0 + \varepsilon)r_1 ds \\
&= \lambda(F_0 + \varepsilon)r_1 C_1 \\
&\leq r_1 = \|u\|.
\end{aligned} \tag{3.9}$$

Hence, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then

$$\|A_\lambda u\| \leq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_1. \tag{3.10}$$

Let $r_3 > 0$ be such that

$$f(u) \geq \left(f_\infty - \varepsilon\right)u, \quad \text{for } u \geq r_3. \tag{3.11}$$

If $u \in P$ with $\|u\| = r_2 = \max\{2r_1, r_3\}$, then by (3.7) and (3.11), we have

$$\begin{aligned}
\|A_\lambda u\| &\geq A_\lambda u(l) \\
&= \lambda \int_0^1 G(l,s)f(u(s))ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l)k(s)f(u(s))ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l)k(s)\left(f_\infty - \varepsilon\right)u(s)ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 \frac{q(l)}{\alpha - 1} q(s)k(s)\left(f_\infty - \varepsilon\right)\|u\|ds \\
&= \lambda q(l)C_2\left(f_\infty - \varepsilon\right)\|u\| \geq \|u\|.
\end{aligned} \tag{3.12}$$

Thus, if we set $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then

$$\|A_\lambda u\| \geq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.13}$$

Now, from (3.10), (3.13), and Lemma 2.7, we guarantee that A_λ has a fixed-point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$, and clearly u is a positive solution of (1.2). The proof is complete. □

Theorem 3.3. *If there exists $l \in (0,1)$ such that $q(l)C_2 f_0 > F_\infty C_1$ holds, then for each*

$$\lambda \in \left(\left(q(l) f_0 C_2\right)^{-1}, \left(F_\infty C_1\right)^{-1} \right), \tag{3.14}$$

the boundary value problem (1.2) *has at least one positive solution. Here we impose $\left(q(l) f_0 C_2\right)^{-1} = 0$ if $f_0 = +\infty$ and $\left(F_\infty C_1\right)^{-1} = +\infty$ if $F_\infty = 0$.*

Proof. Let λ satisfy (3.14) and $\varepsilon > 0$ be such that

$$\left(q(l)\left(f_0 - \varepsilon\right)C_2\right)^{-1} \leq \lambda \leq \left(\left(F_\infty + \varepsilon\right)C_1\right)^{-1}. \tag{3.15}$$

From the definition of f_0, we see that there exists $r_1 > 0$ such that

$$f(u) \geq \left(f_0 - \varepsilon\right)u, \quad \text{for } 0 < u \leq r_1. \tag{3.16}$$

Further, if $u \in P$ with $\|u\| = r_1$, then similar to the second part of Theorem 3.2, we can obtain that $\|A_\lambda u\| \geq \|u\|$. Thus, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then

$$\|A_\lambda u\| \geq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.17}$$

Next, we may choose $R_1 > 0$ such that

$$f(u) \leq \left(F_\infty + \varepsilon\right)u, \quad \text{for } u \geq R_1. \tag{3.18}$$

We consider two cases.

Case 1. Suppose f is bounded. Then there exists some $M > 0$, such that

$$f(u) \leq M, \quad \text{for } u \in (0, +\infty). \tag{3.19}$$

We define $r_3 = \max\{2r_1, \lambda M C_1\}$, and $u \in P$ with $\|u\| = r_3$, then

$$\begin{aligned} \|A_\lambda u\| &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) f(u(s)) ds \\ &\le \frac{\lambda M}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) ds \\ &\le \lambda M C_1 \\ &\le r_3 \le \|u\|. \end{aligned} \tag{3.20}$$

Hence,

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in P_{r_3} = \{u \in P : \|u\| \le r_3\}. \tag{3.21}$$

Case 2. Suppose f is unbounded. Then there exists some $r_4 > \max\{2r_1, R_1\}$, such that

$$f(u) \le f(r_4), \quad \text{for } 0 < u \le r_4. \tag{3.22}$$

Let $u \in P$ with $\|u\| = r_4$. Then by (3.15) and (3.18), we have

$$\begin{aligned} \|A_\lambda u\| &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) f(u(s)) ds \\ &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) (F_\infty + \varepsilon) \|u\| ds \\ &\le \lambda C_1 (F_\infty + \varepsilon) \|u\| \\ &\le \|u\|. \end{aligned} \tag{3.23}$$

Thus, (3.21) is also true.

In both Cases 1 and 2, if we set $\Omega_2 = \{u \in E : \|u\| < r_2 = \max\{r_3, r_4\}\}$, then

$$\|A_\lambda u\| \le \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.24}$$

Now that we obtain (3.17) and (3.24), it follows from Lemma 2.7 that A_λ has a fixed-point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \le \|u\| \le r_2$. It is clear u is a positive solution of (1.2). The proof is complete. □

Theorem 3.4. *Suppose there exist $l \in (0,1)$, $r_2 > r_1 > 0$ such that $q(l) > (\alpha-1) r_1/r_2$, and f satisfy*

$$\min_{(q(l)/(\alpha-1)) r_1 \le u \le r_1} f(u) \ge \frac{r_1}{\lambda(\alpha-1) q(l) C_3}, \qquad \max_{0 \le u \le r_2} f(u) \le \frac{r_2}{\lambda C_1}. \tag{3.25}$$

Then the boundary value problem (1.2) *has a positive solution $u \in P$ with $r_1 \le \|u\| \le r_2$.*

Proof. Choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$; then for $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|A_\lambda u\| &\geq A_\lambda u(l) \\
&= \lambda \int_0^1 G(l,s) f(u(s))ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l)k(s) f(u(s))ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l)k(s) \min_{(q(l)/(\alpha-1))r_1 \leq u \leq r_1} f(u(s))ds \\
&\geq \lambda(\alpha-1)q(l)C_3 \frac{r_1}{\lambda(\alpha-1)q(l)C_3} \\
&= r_1 = \|u\|.
\end{aligned} \tag{3.26}$$

On the other hand, choose $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then for $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) f(u(s))ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) \max_{0 \leq u \leq r_2} f(u(s))ds \\
&\leq \lambda C_1 \frac{r_2}{\lambda C_1} \\
&= r_2 = \|u\|.
\end{aligned} \tag{3.27}$$

Thus, by Lemma 2.7, the boundary value problem (1.2) has a positive solution $u \in P$ with $r_1 \leq \|u\| \leq r_2$. The proof is complete. □

For the reminder of the paper, we will need the following condition.

(H) $(\min_{u \in [(q(l)/(\alpha-1))r,\, r]} f(u))/r > 0$, where $l \in (0,1)$.

Denote

$$\lambda_1 = \sup_{r>0} \frac{r}{C_1 \max_{0 \leq u \leq r} f(u)}, \tag{3.28}$$

$$\lambda_2 = \inf_{r>0} \frac{r}{C_3 \min_{(q(l)/(\alpha-1))r \leq u \leq r} f(u)}. \tag{3.29}$$

In view of the continuity of $f(u)$ and (H), we have $0 < \lambda_1 \leq +\infty$ and $0 \leq \lambda_2 < +\infty$.

Theorem 3.5. *Assume (H) holds. If $f_0 = +\infty$ and $f_\infty = +\infty$, then the boundary value problem* (1.2) *has at least two positive solutions for each $\lambda \in (0, \lambda_1)$.*

Proof. Define

$$a(r) = \frac{r}{C_1 \max_{0\le u\le r} f(u)}. \tag{3.30}$$

By the continuity of $f(u)$, $f_0 = +\infty$ and $f_\infty = +\infty$, we have that $a(r) : (0,+\infty) \to (0,+\infty)$ is continuous and

$$\lim_{r\to 0} a(r) = \lim_{r\to+\infty} a(r) = 0. \tag{3.31}$$

By (3.28), there exists $r_0 \in (0,+\infty)$, such that

$$a(r_0) = \sup_{r>0} a(r) = \lambda_1; \tag{3.32}$$

then for $\lambda \in (0,\lambda_1)$, there exist constants c_1, c_2 $(0 < c_1 < r_0 < c_2 < +\infty)$ with

$$a(c_1) = a(c_2) = \lambda. \tag{3.33}$$

Thus,

$$f(u) \le \frac{c_1}{\lambda C_1}, \quad \text{for } u \in [0, c_1], \tag{3.34}$$

$$f(u) \le \frac{c_2}{\lambda C_1}, \quad \text{for } u \in [0, c_2]. \tag{3.35}$$

On the other hand, applying the conditions $f_0 = +\infty$ and $f_\infty = +\infty$, there exist constants d_1, d_2 $(0 < d_1 < c_1 < r_0 < c_2 < d_2 < +\infty)$ with

$$\frac{f(u)}{u} \ge \frac{1}{q^2(l)\lambda C_3}, \quad \text{for } u \in (0, d_1) \cup \left(\frac{q(l)}{\alpha-1} d_2, +\infty\right). \tag{3.36}$$

Then

$$\min_{(q(l)/(\alpha-1))d_1 \le u \le d_1} f(u) \ge \frac{d_1}{\lambda(\alpha-1)q(l)C_3}, \tag{3.37}$$

$$\min_{(q(l)/(\alpha-1))d_2 \le u \le d_2} f(u) \ge \frac{d_2}{\lambda(\alpha-1)q(l)C_3}. \tag{3.38}$$

By (3.34) and (3.37), (3.35) and (3.38), combining with Theorem 3.4 and Lemma 2.7, we can complete the proof. □

Corollary 3.6. *Assume (H) holds. If $f_0 = +\infty$ or $f_\infty = +\infty$, then the boundary value problem* (1.2) *has at least one positive solution for each $\lambda \in (0, \lambda_1)$.*

Theorem 3.7. *Assume* (H) *holds. If* $f_0 = 0$ *and* $f_\infty = 0$, *then for each* $\lambda \in (\lambda_2, +\infty)$, *the boundary value problem* (1.2) *has at least two positive solutions.*

Proof. Define

$$b(r) = \frac{r}{C_3 \min_{(q(l)/(\alpha-1))r \le u \le r} f(u)}. \tag{3.39}$$

By the continuity of $f(u)$, $f_0 = 0$ and $f_\infty = 0$, we easily see that $b(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r \to 0} b(r) = \lim_{r \to +\infty} b(r) = +\infty. \tag{3.40}$$

By (3.29), there exists $r_0 \in (0, +\infty)$, such that

$$b(r_0) = \inf_{r>0} b(r) = \lambda_2. \tag{3.41}$$

For $\lambda \in (\lambda_2, +\infty)$, there exist constants d_1, d_2 $(0 < d_1 < r_0 < d_2 < +\infty)$ with

$$b(d_1) = b(d_2) = \lambda. \tag{3.42}$$

Therefore,

$$\begin{aligned} f(u) &\ge \frac{d_1}{\lambda(\alpha-1)q(l)C_3}, \quad \text{for } u \in \left[\frac{q(l)}{\alpha-1}d_1, d_1\right], \\ f(u) &\ge \frac{d_2}{\lambda(\alpha-1)q(l)C_3}, \quad \text{for } u \in \left[\frac{q(l)}{\alpha-1}d_2, d_2\right]. \end{aligned} \tag{3.43}$$

On the other hand, using $f_0 = 0$, we know that there exists a constant c_1 $(0 < c_1 < d_1)$ with

$$\frac{f(u)}{u} \le \frac{1}{\lambda C_1}, \quad \text{for } u \in (0, c_1), \tag{3.44}$$

$$\max_{0 \le u \le c_1} f(u) \le \frac{c_1}{\lambda C_1}. \tag{3.45}$$

In view of $f_\infty = 0$, there exists a constant $c_2 \in (d_2, +\infty)$ such that

$$\frac{f(u)}{u} \le \frac{1}{\lambda C_1}, \quad \text{for } u \in (c_2, +\infty). \tag{3.46}$$

Let

$$M = \max_{0 \le u \le c_2} f(u), \quad c_2 \ge \lambda C_1 M. \tag{3.47}$$

It is easily seen that

$$\max_{0\le u\le c_2} f(u) \le \frac{c_2}{\lambda C_1}. \tag{3.48}$$

By (3.45) and (3.48), combining with Theorem 3.4 and Lemma 2.7, the proof is complete. □

Corollary 3.8. *Assume* (H) *holds. If* $f_0 = 0$ *or* $f_\infty = 0$*, then for each* $\lambda \in (\lambda_2, +\infty)$*, the boundary value problem* (1.2) *has at least one positive solution.*

By the above theorems, we can obtain the following results.

Corollary 3.9. *Assume* (H) *holds. If* $f_0 = +\infty$*,* $f_\infty = d$*, or* $f_\infty = +\infty$*,* $f_0 = d$*, then for any* $\lambda \in (0, (dC_1)^{-1})$*, the boundary value problem* (1.2) *has at least one positive solution.*

Corollary 3.10. *Assume* (H) *holds. If* $f_0 = 0, f_\infty = d$*, or if* $f_\infty = 0$*,* $f_0 = d$*, then for any* $\lambda \in ((q(l)dC_2)^{-1}, +\infty)$*, the boundary value problem* (1.2) *has at least one positive solution.*

Remark 3.11. For the integer derivative case $\alpha = 3$, Theorems 3.2–3.7 also hold; we can find the corresponding existence results in [22].

4. Nonexistence

In this section, we give some sufficient conditions for the nonexistence of positive solution to the problem (1.2).

Theorem 4.1. *Assume* (H) *holds. If* $F_0 < +\infty$ *and* $F_\infty < \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$*, the boundary value problem* (1.2) *has no positive solution.*

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist positive numbers m_1, m_2, r_1, and r_2, such that $r_1 < r_2$ and

$$\begin{aligned} f(u) &\le m_1 u, \quad \text{for } u \in [0, r_1], \\ f(u) &\le m_2 u, \quad \text{for } u \in [r_2, +\infty). \end{aligned} \tag{4.1}$$

Let $m = \max\{m_1, m_2, \max_{r_1\le u\le r_2}\{f(u)/u\}\}$. Then we have

$$f(u) \le mu, \quad \text{for } u \in [0, +\infty). \tag{4.2}$$

Assume $v(t)$ is a positive solution of (1.2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := (mC_1)^{-1}$. Since $A_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|A_\lambda v\| \le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) f(v(s))ds \le \frac{m\lambda}{\Gamma(\alpha)} \|v\| \int_0^1 (\alpha-1)k(s)ds < \|v\|, \tag{4.3}$$

which is a contradiction. Therefore, (1.2) has no positive solution. The proof is complete. □

Theorem 4.2. *Assume* (H) *holds. If* $f_0 > 0$ *and* $f_\infty > 0$, *then there exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$, *the boundary value problem* (1.2) *has no positive solution.*

Proof. By $f_0 > 0$ and $f_\infty > 0$, we know that there exist positive numbers $n_1,\ n_2,\ r_1$, and r_2, such that $r_1 < r_2$ and

$$\begin{aligned} f(u) &\geq n_1 u, \quad \text{for } u \in [0, r_1], \\ f(u) &\geq n_2 u, \quad \text{for } u \in [r_2, +\infty). \end{aligned} \tag{4.4}$$

Let $n = \min\{n_1, n_2, \min_{r_1 \leq u \leq r_2}\{f(u)/u\}\} > 0$. Then we get

$$f(u) \geq nu, \quad \text{for } u \in [0, +\infty). \tag{4.5}$$

Assume $v(t)$ is a positive solution of (1.2). We will show that this leads to a contradiction for $\lambda > \lambda_0 := (q(l)nC_2)^{-1}$. Since $A_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|A_\lambda v\| \geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l)k(s)f(v(s))ds > \|v\|, \tag{4.6}$$

which is a contradiction. Thus, (1.2) has no positive solution. The proof is complete. □

5. Examples

In this section, we will present some examples to illustrate the main results.

Example 5.1. Consider the boundary value problem

$$\begin{gathered} D_{0^+}^{5/2} u(t) + \lambda u^a = 0, \quad 0 < t < 1,\ a > 1, \\ u(0) = u(1) = u'(0) = 0. \end{gathered} \tag{5.1}$$

Since $\alpha = 5/2$, we have

$$\begin{aligned} C_1 &= \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s)ds = \frac{1}{\Gamma(5/2)} \int_0^1 \frac{3}{2} s(1-s)^{3/2} ds = 0.1290, \\ C_2 &= \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{(\alpha - 1)} q(s)k(s)ds = \frac{1}{\Gamma(5/2)} \int_0^1 \frac{2}{3} s^{5/2}(1-s)^{5/2} ds = 0.0077. \end{aligned} \tag{5.2}$$

Let $f(u) = u^a,\ a > 1$. Then we have $F_0 = 0$, $f_\infty = +\infty$. Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_\infty > F_0 C_1$ holds. Thus, by Theorem 3.2, the boundary value problem (5.1) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 5.2. Discuss the boundary value problem

$$D_{0^+}^{5/2}u(t) + \lambda u^b = 0, \quad 0 < t < 1,\ 0 < b < 1,$$
$$u(0) = u(1) = u'(0) = 0. \tag{5.3}$$

Since $\alpha = 5/2$, we have $C_1 = 0.1290$ and $C_2 = 0.0077$. Let $f(u) = u^b,\ 0 < b < 1$. Then we have $F_\infty = 0,\ f_0 = +\infty$. Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_0 > F_\infty C_1$ holds. Thus, by Theorem 3.3, the boundary value problem (5.3) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 5.3. Consider the boundary value problem

$$D_{0^+}^{5/2}u(t) + \lambda \frac{(200u^2 + u)(2 + \sin u)}{u + 1} = 0, \quad 0 < t < 1,\ a > 1,$$
$$u(0) = u(1) = u'(0) = 0. \tag{5.4}$$

Since $\alpha = 5/2$, we have $C_1 = 0.129$ and $C_2 = 0.0077$. Let $f(u) = (200u^2 + u)(2 + \sin u)/(u + 1)$. Then we have $F_0 = f_0 = 2$, $F_\infty = 600$, $f_\infty = 200$, and $2u < f(u) < 600u$.

(i) Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_\infty > F_0 C_1$ holds. Thus, by Theorem 3.2, the boundary value problem (5.4) has a positive solution for each $\lambda \in (3.6937, 3.8759)$.

(ii) By Theorem 4.1, the boundary value problem (5.4) has no positive solution for all $\lambda \in (0, 0.0129)$.

(iii) By Theorem 4.2, the boundary value problem (5.4) has no positive solution for all $\lambda \in (369.369, +\infty)$.

Example 5.4. Consider the boundary value problem

$$D_{0^+}^{5/2}u(t) + \lambda \frac{(u^2 + u)(2 + \sin u)}{150u + 1} = 0, \quad 0 < t < 1,\ a > 1,$$
$$u(0) = u(1) = u'(0) = 0. \tag{5.5}$$

Since $\alpha = 5/2$, we have $C_1 = 0.129$ and $C_2 = 0.0077$. Let $f(u) = (u^2 + u)(2 + \sin u)/(150u + 1)$. Then we have $F_0 = f_0 = 2$, $F_\infty = 1/50$, $f_\infty = 1/150$, and $u/150 < f(u) < 2u$.

(i) Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_0 > F_\infty C_1$ holds. Thus, by Theorem 3.3, the boundary value problem (5.5) has a positive solution for each $\lambda \in (369.369, 387.5968)$.

(ii) By Theorem 4.1, the boundary value problem (5.5) has no positive solution for all $\lambda \in (0, 3.8759)$.

(iii) By Theorem 4.2, the boundary value problem (5.5) has no positive solution for all $\lambda \in (110810.6911, +\infty)$.

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References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equation*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.

[2] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.

[3] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[4] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integral and Derivative: Theory and Applications*, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[5] Q. Li and S. Sun, "On the existence of positive solutions for initial value problem to a class of fractional differential equation," in *Proceedings of the 7th Conference on Biological Dynamic System and Stability of Differential Equation*, pp. 886–889, World Academic Press, Chongqing, China, 2010.

[6] Q. Li, S. Sun, M. Zhang, and Y. Zhao, "On the existence and uniqueness of solutions for initial value problem of fractional differential equations," *Journal of University of Jinan*, vol. 24, pp. 312–315, 2010.

[7] Q. Li, S. Sun, Z. Han, and Y. Zhao, "On the existence and uniqueness of solutions for initial value problem of nonlinear fractional differential equations," in *Proceedings of the 6th IEEE/ASME International Conference on Mechatronic and Embedded Systems and Applications (MESA '10)*, pp. 452–457, Qingdao, China, 2010.

[8] D. Delbosco and L. Rodino, "Existence and uniqueness for a nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 204, no. 2, pp. 609–625, 1996.

[9] S. Zhang, "The existence of a positive solution for a nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 804–812, 2000.

[10] S. Zhang, "Existence of positive solution for some class of nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 1, pp. 136–148, 2003.

[11] H. Jafari and V. Daftardar-Gejji, "Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 700–706, 2006.

[12] S. Zhang, "Positive solutions for boundary-value problems of nonlinear fractional differential equations," *Electronic Journal of Differential Equations*, vol. 36, pp. 1–12, 2006.

[13] T. Qiu and Z. Bai, "Existence of positive solutions for singular fractional differential equations," *Electronic Journal of Differential Equations*, vol. 146, pp. 1–9, 2008.

[14] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[15] M. Zhang, S. Sun, Y. Zhao, and D. Yang, "Existence of positive solutions for boundary value problems of fractional differential equations," *Journal of University of Jinan*, vol. 24, pp. 205–208, 2010.

[16] Y. Zhao and S. Sun, "On the existence of positive solutions for boundary value problems of nonlinear fractional differential equations," in *Proceedings of the 7th Conference on Biological Dynamic System and Stability of Differential Equation*, pp. 682–685, World Academic Press, Chongqing, China, 2010.

[17] Y. Zhao, S. Sun, Z. Han, and M. Zhang, "Existence on positive solutions for boundary value problems of singular nonlinear fractional differential equations," in *Proceedings of the 6th IEEE/ASME International Conference on Mechatronic and Embedded Systems and Applications*, pp. 480–485, Qingdao, China, 2010.

[18] Y. Zhao, S. Sun, Z. Han, and Q. Li, "The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 2086–2097, 2011.

[19] Y. Yu and D. Jiang, "Multiple Positive Solutions for the Boundary Value Problem of A Nonlinear Fractional Differential Equation," Northeast Normal University, 2009.

[20] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.
[21] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, The Netherlands, 1964.
[22] W. Ge, *Boundary Value Problem of Nonlinear Ordinary Differential Equations*, Science Press, Beijing, China, 2007.