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Research Article

Non-Self-Adjoint Singular Sturm-Liouville Problems with Boundary Conditions Dependent on the Eigenparameter

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Let A denote the operator generated in $L_2(\mathcal{R}_+)$ by the Sturm-Liouville problem: $-y'' + q(x)y = \lambda^2 y$, $x \in \mathcal{R}_+ = [0,\infty)$, $(y'/y)(0) = (\beta_1\lambda + \beta_0)/(\alpha_1\lambda + \alpha_0)$, where q is a complex valued function and $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathcal{C}$, with $\alpha_0\beta_1 - \alpha_1\beta_0 \neq 0$. In this paper, using the uniqueness theorems of analytic functions, we investigate the eigenvalues and the spectral singularities of A. In particular, we obtain the conditions on q under which the operator A has a finite number of the eigenvalues and the spectral singularities.

1. Introduction

Let L denote the non-self-adjoint Sturm-Liouville operator generated in $L_2(\mathcal{R}_+)$ by the differential expression

$$l(y) = -y'' + q(x)y, \quad x \in \mathcal{R}_+ \tag{1.1}$$

and the boundary condition $y(0) = 0$, where q is a complex valued function. The spectral analysis of L with continuous and discrete spectrum was studied by Naĭmark [1]. In this article, the spectrum of L was investigated and shown that it is composed of the eigenvalues, the continuous spectrum and the spectral singularities.The spectral singularities of L are poles of the resolvent which are imbedded in the continuous spectrum and are not the eigenvalues.

If the function q satisfies the Naĭmark condition, that is,

$$\int_0^\infty e^{\varepsilon x}|q(x)|dx < \infty \tag{1.2}$$

for some $\varepsilon > 0$, then L has a finite number of the eigenvalues and spectral singularities with finite multiplicities.

The results of Naĭmark were extended to the Sturm-Liouville operators on the entire real axis by Kemp [2] and to the differential operators with a singularity at the zero point by Gasymov [3]. The spectral analysis of dissipative Sturm-Liouville operators with spectral singularities was considered by Pavlov [4]. A very important development in the spectral analysis of L was made by Lyance [5, 6]. He showed that the spectral singularities play an important role in the spectral theory of L. He also investigated the effect of the spectral singularities in the spectral expansion. The spectral singularities of the non-self-adjoint Sturm-Liouville operator generated in $L_2(\mathcal{R}_+)$ by (1.1) and the boundary condition

$$\int_0^\infty K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0, \tag{1.3}$$

in which $K \in L_2(\mathcal{R}_+)$ is a complex valued function and $\alpha, \beta \in \mathcal{C}$, was studied in detail by Krall [7–9].

Some problems of spectral theory of differential and difference operators with spectral singularities were also investigated in [10–16]. Note that, the boundary conditions used in [1–17] are independent of spectral parameter. In recent years, various problems of the spectral theory of regular Sturm-Liouville problem whose boundary conditions depend on spectral parameter have been examined in [18–22].

Let us consider the boundary value problem

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathcal{R}_+, \tag{1.4}$$

$$\frac{y'}{y}(0) = \frac{\beta_1\lambda + \beta_0}{\alpha_1\lambda + \alpha_0}, \tag{1.5}$$

where q is a complex valued function and $\alpha_0, \alpha_1, \beta_0, \beta_1$ are complex numbers such that $\alpha_0\beta_1 - \alpha_1\beta_0 \neq 0$. By A we will denote the operator generated in $L_2(\mathcal{R}_+)$ by (1.4) and (1.5). In this paper we discuss the discrete spectrum of A and prove that the operator A has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity if

$$\lim_{x\to\infty} q(x) = 0, \qquad \int_0^\infty e^{\varepsilon x^\delta}|q'(x)|dx < \infty \tag{1.6}$$

for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$. We also show that the analogue of the Naĭmark condition for A is the form

$$\lim_{x\to\infty} q(x) = 0, \qquad \int_0^\infty e^{\varepsilon x}|q'(x)|dx < \infty \tag{1.7}$$

for some $\varepsilon > 0$.

2. Jost Solution of (1.4)

We will denote the solution of (1.4) satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C},\ \operatorname{Im}\lambda \geq 0\}, \tag{2.1}$$

by $e(x,\lambda)$. The solution $e(x,\lambda)$ is called the Jost solution of (1.4). Under the condition

$$\int_0^\infty x\left|q(x)\right|dx < \infty, \tag{2.2}$$

the Jost solution has a representation

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt \tag{2.3}$$

for $\lambda \in \overline{\mathbb{C}}_+$, where the kernel $K(x,t)$ satisfies

$$\begin{aligned} K(x,t) = {} & \frac{1}{2}\int_{(x+t)/2}^\infty q(\xi)d\xi + \frac{1}{2}\int_x^{(x+t)/2}\int_{t+x-\xi}^{t+\xi-x} K(\xi,\eta)q(\xi)d\eta\, d\xi \\ & + \frac{1}{2}\int_{(x+t)/2}^\infty \int_\xi^{t+\xi-x} K(\xi,\eta)q(\xi)d\eta\, d\xi. \end{aligned} \tag{2.4}$$

Moreover, $K(x,t)$ is continuously differentiable with respect to its arguments and

$$|K(x,t)| \leq c\int_{(x+t)/2}^\infty \left|q(\xi)\right|d\xi, \tag{2.5}$$

$$|K_x(x,t)|, |K_t(x,t)| \leq \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + c\int_{(x+t)/2}^\infty \left|q(\xi)\right|d\xi, \tag{2.6}$$

where $c > 0$ is a constant [23, Chapter 3].

The solution $e(x,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C},\ \operatorname{Im}\lambda > 0\}$ and continuous on the real axis.

Let $\mathcal{AC}(\mathcal{R}_+)$ denote the class of complex valued absolutely continuous functions in $\mathcal{R}_+$. In the sequel we will need the following.

Lemma 2.1. *If*

$$q \in \mathcal{AC}(\mathcal{R}_+), \quad \lim_{x\to\infty} q(x) = 0, \quad \int_0^\infty x^2 \left|q'(x)\right| dx < \infty, \tag{2.7}$$

then $K_{xt}(x,t) := (\partial^2/\partial t\partial x)K(x,t)$ *exists and*

$$\begin{aligned} K_{xt}(x,t) &= -\frac{1}{8}q'\left(\frac{x+t}{2}\right) - \frac{1}{4}K\left(\frac{x+t}{2}, \frac{x+t}{2}\right)q\left(\frac{x+t}{2}\right) \\ &\quad - \frac{1}{2}\int_x^{(x+t)/2} \left[K_t(\xi, t+x-\xi) + K_t(\xi, t-x+\xi)\right]q(\xi)d\xi \\ &\quad - \frac{1}{2}\int_{(x+t)/2}^\infty K_t(\xi, t-x+\xi)q(\xi)d\xi. \end{aligned} \tag{2.8}$$

The proof of the lemma is the direct consequence of (2.4).

From (2.5)–(2.8) we find that

$$|K_{xt}(0,t)| \le c\left[\left|q\left(\frac{t}{2}\right)\right| + \left|q'\left(\frac{t}{2}\right)\right| + \int_{t/2}^\infty |q(\xi)|d\xi\right], \tag{2.9}$$

where $c > 0$ is a constant.

3. The Green Function and the Continuous Spectrum

Let $\varphi(x,\lambda)$ denote the solution of (1.4) subject to the initial conditions $\varphi(0,\lambda) = \alpha_0 + \alpha_1\lambda$, $\varphi'(0,\lambda) = \beta_0 + \beta_1\lambda$. Therefore $\varphi(x,\lambda)$ is an entire function of λ.

Let us define the following functions:

$$D_\pm(\lambda) = (\alpha_0 + \alpha_1\lambda)e_x(0,\pm\lambda) - (\beta_0 + \beta_1\lambda)e(0,\pm\lambda) \quad \lambda \in \overline{\mathcal{C}}_\pm, \tag{3.1}$$

where $\overline{\mathcal{C}}_\pm = \{\lambda : \lambda \in \mathcal{C}, \pm\operatorname{Im}\lambda \ge 0\}$. It is obvious that the functions D_+ and D_- are analytic in $\mathcal{C}_+$ and $\mathcal{C}_- := \{\lambda : \lambda \in \mathcal{C}, \operatorname{Im}\lambda < 0\}$, respectively and continuous on the real axis.

Let

$$G(x,t;\lambda) = \begin{cases} G_+(x,t;\lambda), & \lambda \in \mathcal{C}_+, \\ G_-(x,t;\lambda), & \lambda \in \mathcal{C}_- \end{cases} \tag{3.2}$$

be the Green function of A (obtained by the standard techniques), where

$$G_\pm(x,t;\lambda) = \begin{cases} -\dfrac{e(x,\pm\lambda)\varphi(t,\lambda)}{D_\pm(\lambda)}, & 0 \le t \le x, \\[2ex] -\dfrac{e(t,\pm\lambda)\varphi(x,\lambda)}{D_\pm(\lambda)}, & x \le t < \infty. \end{cases} \tag{3.3}$$

We will denote the continuous spectrum of A by σ_c. Using (3.1)–(3.3) in a way similar to Theorem 2 [17, page 303], we get the following:

$$\sigma_c = \mathcal{R}. \tag{3.4}$$

4. The Discrete Spectrum of the Operator A

Let us denote the eigenvalues and the spectral singularities of the operator A by σ_d and σ_{ss} respectively. From (2.3) and (3.1)–(3.4) it follows that

$$\begin{aligned} \sigma_d &= \{\lambda : \lambda \in \mathcal{C}_+,\ D_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathcal{C}_-,\ D_-(\lambda) = 0\}, \\ \sigma_{ss} &= \{\lambda : \lambda \in \mathcal{R}^*,\ D_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathcal{R}^*,\ D_-(\lambda) = 0\}, \end{aligned} \tag{4.1}$$

where $\mathcal{R}^* = \mathcal{R} - \{0\}$.

Definition 4.1. The multiplicity of a zero of D_+ (or D_-) in $\overline{\mathcal{C}}_+$ (or $\overline{\mathcal{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of A.

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of A we need to discuss the quantative properties of the zeros of D_+ and D_- in $\overline{\mathcal{C}}_+$ and $\overline{\mathcal{C}}_-$, respectively. For the sake of simplicity we will consider only the zeros of D_+ in $\overline{\mathcal{C}}_+$. A similar procedure may also be employed for zeros of D_- in $\overline{\mathcal{C}}_-$.

Let us define

$$M_1^{\pm} = \{\lambda : \lambda \in \mathcal{C}_{\pm},\ D_{\pm}(\lambda) = 0\}, \qquad M_2^{\pm} = \{\lambda : \lambda \in \mathcal{R},\ D_{\pm}(\lambda) = 0\}. \tag{4.2}$$

So we have, by (4.1), that

$$\sigma_d = M_1^+ \cup M_1^-, \qquad \sigma_{ss} = M_2^+ \cup M_2^- - \{0\}. \tag{4.3}$$

Theorem 4.2. *Under the conditions in* (2.7)*:*

(i) *the discrete spectrum σ_d is a bounded, at most countable set and its limit points lie on the bounded subinterval of the real axis;*

(ii) *the set σ_{ss} is a bounded and its linear Lebesgue measure is zero.*

Proof. From (2.3) and (3.1) we obtain that D_+ is analytic in $\mathcal{C}_+$, continuous on the real axis and has the form

$$D_+(\lambda) = i\alpha_1\lambda^2 + a\lambda + b + \int_0^\infty f(t)e^{i\lambda t}dt, \tag{4.4}$$

where

$$a = i\alpha_0 - \alpha_1 K(0,0) - \beta_1,$$
$$b = -(\alpha_0 + i\beta_1)K(0,0) - \beta_0 + i\alpha_1 K_x(0,0), \tag{4.5}$$
$$f(t) = -\beta_0 K(0,t) - i\beta_1 K_t(0,t) + \alpha_0 K_x(0,t) + i\alpha_1 K_{xt}(0,t).$$

Using (2.5), (2.6), and (2.9) we get that $f \in L_1(\mathcal{R}_+)$. So

$$D_+(\lambda) = i\alpha_1\lambda^2 + a\lambda + b + o(1), \quad \lambda \in \overline{\mathcal{C}}_+, \ |\lambda| \longrightarrow \infty. \tag{4.6}$$

From (4.3), (4.6) and uniqueness theorem for analytic functions [24], we get (i) and (ii). □

Theorem 4.3. *If*

$$q \in \mathcal{AC}(\mathcal{R}_+), \qquad \lim_{x\to\infty} q(x) = 0, \qquad \int_0^\infty x^3 \left|q'(x)\right| dx < \infty, \tag{4.7}$$

then

$$\sum_\nu |l_\nu| \ln \frac{1}{|l_\nu|} < \infty, \tag{4.8}$$

where $|l_\nu|$ *is the lengths of the boundary complementary intervals of* σ_{ss}.

Proof. From (2.5), (2.6), (2.9), (4.4) and (4.7) we see that D_+ is continuously differentiable on $\mathcal{R}$. Since the function D_+ is not identically equal to zero, by Beurling's theorem we obtain (4.8) [25]. □

Theorem 4.4. *Under the conditions*

$$q \in \mathcal{AC}(\mathcal{R}_+), \quad \lim_{x\to\infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x} \left|q'(x)\right| dx < \infty, \ \varepsilon > 0, \tag{4.9}$$

the operator A *has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

Proof. (2.5), (2.7), (2.9), (4.4) and (4.9) imply that the function D_+ has an analytic continuation to the half-plane $\mathrm{Im}\lambda > -\varepsilon/2$. Hence the limit points of its zeros on $\overline{\mathcal{C}}_+$ cannot lie in $\mathcal{R}$. Therefore using Theorem 4.2, we have the finiteness of zeros of D_+ in $\overline{\mathcal{C}}_+$. Similarly we find that the function D_- has a finite number of zeros with finite multiplicity in $\overline{\mathcal{C}}_-$. Then the proof of the theorem is the direct consequence of (4.3).

Note that the conditions in (4.9) are analogous to the Naĭmark condition (1.2) for the operator A.

It is clear that the condition (4.9) guarantees the analytic continuation of D_+ and D_- from the real axis to the lower and the upper half-planes respectively. So the finiteness of the eigenvalues and the spectral singularities of A are obtained as a result of these analytic continuations.

Now let suppose that

$$q \in \mathcal{AC}(\mathcal{R}_+), \quad \lim_{x\to\infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x^\delta} \left|q'(x)\right| dx < \infty, \tag{4.10}$$

for some $\varepsilon > 0$ and $1/2 \leq \delta < 1$, which is weaker than (4.9). It is obvious that under the condition (4.10) the function D_+ is analytic in $\mathcal{C}_+$ and infinitely differentiable on the real axis. But D_+ does not have analytic continuation from the real axis to the lower half-plane. Similarly, D_- does not have analytic continuation from the real axis to the upper half-plane either. Consequently, under the conditions in (4.10) the finiteness of the eigenvalues and the spectral singularities of A cannot be shown in a way similar to Theorem 4.4.

Let us denote the sets of limit points of M_1^+ and M_2^+ by M_3^+ and M_4^+ respectively and the set of all zeros of D_+ with infinite multiplicity in $\overline{\mathcal{C}}_+$ by M_∞^+. Analogously define the sets M_3^-, M_4^- and M_∞^-.

It is clear from the boundary uniqueness theorem of analytic functions that [24]

$$\begin{gathered} M_1^{\pm} \cap M_\infty^{\pm} = \emptyset, \qquad M_3^{\pm} \subset M_2^{\pm}, \qquad M_4^{\pm} \subset M_2^{\pm}, \\ M_\infty^{\pm} \subset M_2^{\pm}, \qquad M_3^{\pm} \subset M_\infty^{\pm}, \qquad M_4^{\pm} \subset M_\infty^{\pm}, \end{gathered} \tag{4.11}$$

and $\mu(M_3^{\pm}) = \mu(M_4^{\pm}) = \mu(M_\infty^{\pm}) = 0$, where μ denote the Lebesgue measure on the real axis. □

Theorem 4.5. *If* (4.10) *holds, then* $M_\infty^+ = M_\infty^- = \emptyset$.

Proof. We will prove that $M_\infty^+ = \emptyset$. The case $M_\infty^- = \emptyset$ is similar. Under the condition (4.10) D_+ is analytic in $\mathcal{C}_+$ all of its derivatives are continuous on the real axis and there exists $N > 0$ such that

$$\begin{gathered} \left|\frac{d^n}{d\lambda^n} D_+(\lambda)\right| \leq B_n, \quad n = 0,1,2,\ldots, \ \lambda \in \overline{\mathcal{C}}_+, \ |\lambda| < 2N, \\ B_0 = 4|\alpha_1|N^2 + 2|a|N + |b| + \int_0^\infty \left|f(t)\right| dt, \\ B_1 = 4|\alpha_1|N + |a| + \int_0^\infty t\left|f(t)\right| dt, \\ B_2 = 2|\alpha_1| + \int_0^\infty t^2\left|f(t)\right| dt, \\ B_n = \int_0^\infty t^n\left|f(t)\right| dt, \quad n \geq 3. \end{gathered} \tag{4.12}$$

From Theorem 4.2, we get that

$$\left|\int_{-\infty}^{-N} \frac{\ln|D_+(\lambda)|}{1+\lambda^2} d\lambda\right| < \infty, \qquad \left|\int_{N}^{\infty} \frac{\ln|D_+(\lambda)|}{1+\lambda^2} d\lambda\right| < \infty. \tag{4.13}$$

Let us define the function

$$T(s) = \inf_n \frac{B_n s^n}{n!}. \tag{4.14}$$

Since the function D_+ is not equal to zero identically, by Pavlov's theorem [4],

$$\int_0^h \ln T(s) d\mu(M_{\infty,s}^+) > -\infty \tag{4.15}$$

holds, where $h > 0$ is a constant and $\mu(M_{\infty,s}^+)$ is the Lebesgue measure of s-neighborhood of M_∞^+. Using (2.5), (2.6), (2.9) and (4.4) we obtain that

$$B_n \leq B d^n n! n^{n(1/\delta - 1)}, \tag{4.16}$$

where B and d are constants depending on ε and δ. Substituting (4.16) in the definition of $T(s)$ we get

$$T(s) \leq B \exp\left\{-\left(\frac{1}{\delta} - 1\right) e^{-1} d^{-\delta/(1-\delta)} s^{-\delta/(1-\delta)}\right\}. \tag{4.17}$$

Now (4.15) and (4.17) imply that

$$\int_0^h s^{-\delta/(1-\delta)} d\mu(M_{\infty,s}^+) < \infty. \tag{4.18}$$

Since $\delta/(1-\delta) \geq 1$, consequently (4.18) holds for arbitrary s if and only if $\mu(M_{\infty,s}^+) = 0$ or $M_\infty^+ = \emptyset$. □

Theorem 4.6. *Under the condition* (4.10) *the operator A has a finite number of the eigenvalues and the spectral singularities and each of them is of a finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the functions D_+ and D_- have finite number of zeros with finite multiplicities in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. We will prove it only for D_+. The case of D_- is similar.

It follows from (4.11) that $M_3^+ = M_4^+ = \emptyset$. So the bounded sets M_1^+ and M_2^+ have no limit points, that is, the D_+ has only a finite number of zeros in $\overline{\mathbb{C}}_+$. Since $M_\infty^+ = \emptyset$ these zeros are of a finite multiplicity. □

Theorem 4.7. *If the condition* (2.7) *is satisfied then the set σ_{ss} is of the first category.*

Proof. From the continuity of D_+ it is clear that the set M_2^+ is closed and is a set of Lebesgue measure zero which is of type F_σ. According to Martin's theorem [26] there is measurable set whose metric density exists and is different from 0 and 1 at every point of M_2^+. So, M_2^+ is of the first category from the theorem due to Goffman [27]. We also have obviously same things for M_2^-. Consequently σ_{ss} is of the first category by (4.3). □

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