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Research Article

Existence and Asymptotic Behavior of Boundary Blow-Up Solutions for Weighted $p(x)$-Laplacian Equations with Exponential Nonlinearities

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This paper investigates the following $p(x)$-Laplacian equations with exponential nonlinearities: $-\Delta_{p(x)}u+\rho(x)e^{f(x,u)}=0$ in Ω, $u(x)\to+\infty$ as $d(x,\partial\Omega)\to 0$, where $-\Delta_{p(x)}u=-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, $\rho(x)\in C(\Omega)$. The asymptotic behavior of boundary blow-up solutions is discussed, and the existence of boundary blow-up solutions is given.

1. Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. On the background of this class of problems, we refer to [1–3]. Many results have been obtained on this kind of problems, for example, [4–18]. On the regularity of weak solutions for differential equations with nonstandard $p(x)$-growth conditions, we refer to [4, 5, 8]. On the existence of solutions for $p(x)$-Laplacian equation Dirichlet problems in bounded domain, we refer to [7, 9, 15, 18]. In this paper, we consider the following $p(x)$-Laplacian equations with exponential nonlinearities

$$\begin{aligned} -\Delta_{p(x)}u+\rho(x)e^{f(x,u)}=0, \quad &\text{in } \Omega,\\ u(x)\longrightarrow+\infty, \quad &\text{as } d(x,\partial\Omega)\longrightarrow 0, \end{aligned} \tag{P}$$

where $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ and $\Omega = B(0,R) \subset \mathbb{R}^N$ is a bounded radial domain ($B(0,R) = \{x \in \mathbb{R}^N \mid |x| < R\}$). Our aim is to give the asymptotic behavior and the existence of boundary blow-up solutions for problem (P).

Throughout the paper, we assume that $p(x)$, $\rho(x)$, and $f(x,u)$ satisfy the following.

(H_1) $p(x) \in C^1(\overline{\Omega})$ is radial and satisfies

$$1 < p^- \le p^+ < +\infty, \quad \text{where } p^- = \inf_{\Omega} p(x), \quad p^+ = \sup_{\Omega} p(x). \tag{1.1}$$

(H_2) $f(x,u)$ is radial with respect to x, $f(x,\cdot)$ is increasing, and $f(x,0) = 0$ for any $x \in \Omega$.

(H_3) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\left|f(x,t)\right| \le C_1 + C_2|t|^{\gamma(x)}, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \tag{1.2}$$

where C_1, C_2 are positive constants and $0 \le \gamma \in C(\overline{\Omega})$.

(H_4) $\rho(x) \in C(\Omega)$ is a radial nonnegative function, and there exists a constant $\sigma \in [R/2, R)$ such that

$$\rho_0(R-r)^{-\beta(r)} \le \rho(r) \le \rho_1(R-r)^{-\beta_1(r)} \quad \text{for } r \in [\sigma, R) \text{ uniformly}, \tag{1.3}$$

where ρ_0 and ρ_1 are positive constants and $\beta(r)$ and $\beta_1(r)$ are Lipschitz continuous on $[\sigma, R]$, which satisfy $\beta(r) \le \beta_1(r) < p(r)$ for any $r \in [\sigma, R]$.

The operator $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. Specifically, if $p(x) \equiv p$ (a constant), (P) is the well-known p-Laplacian problem. If $f(x,u)$ can be represented as $h(x)f(u)$, on the boundary blow-up solutions for the following p-Laplacian equations (p is a constant):

$$-\Delta_p u + h(x)f(u) = 0, \quad \text{in } \Omega, \tag{1.4}$$

we refer to [19–26], and the following generalized Keller-Osserman condition is crucial

$$\int_1^{\infty} \frac{1}{(F(t))^{1/p}} dt < +\infty, \quad \text{where } F(t) = \int_0^t f(s)ds, \tag{1.5}$$

but the typical form of $p(x)$-Laplacian equation is

$$-\Delta_{p(x)}u + |u|^{q(x)-2}u = 0, \quad \text{in } \Omega, \tag{1.6}$$

and there are some differences between the results of (1.4) and (1.6) (see [16]).

On the boundary blow-up solutions for the following p-Laplacian equations with exponential nonlinearities (p is a constant):

$$-\Delta_p u + e^{h(x)f(u)} = 0, \quad \text{in } \Omega, \tag{1.7}$$

we refer to [20–22], but the results on the boundary blow-up solutions for $p(x)$-Laplacian equations are rare (see [16]).

In [16], the present author discussed the existence and asymptotic behavior of boundary blow-up solutions for the following $p(x)$-Laplacian equations:

$$-\Delta_{p(x)}u + f(x,u) = 0, \quad \text{in } \Omega, \qquad u(x) \longrightarrow +\infty, \quad \text{as } d(x,\partial\Omega) \longrightarrow 0, \tag{1.8}$$

on the condition that $f(x,\cdot)$ satisfies polynomial growth condition.

If $p(x)$ is a function, the typical form of (P) is the following:

$$-\Delta_{p(x)}u + \rho(x)e^{|u|^{q(x)-2}u} = 0, \tag{1.9}$$

and the method to construct subsolution and supersolution in [16] cannot give the exact asymptotic behavior of solutions for (P). Our results partially generalized the results of [20–22].

Because of the nonhomogeneity of $p(x)$-Laplacian, $p(x)$-Laplacian problems are more complicated than those of p-Laplacian ones (see [10]); another difficulty of this paper is that $f(x,u)$ cannot be represented as $h(x)f(u)$.

2. Preliminary

In order to deal with $p(x)$-Laplacian problems, we need some theories on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian, which we will use later (see [6, 11]). Let

$$L^{p(x)}(\Omega) = \left\{ u \mid u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}dx < \infty \right\}. \tag{2.1}$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 \mid \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \tag{2.2}$$

The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space. We call it generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive, and uniform convex Banach space (see [6, Theorems 1.10, 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \right\}, \tag{2.3}$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \tag{2.4}$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive, and uniform convex Banach spaces (see [6, Theorem 2.1]).

If $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$, u is called a blow-up solution of (P) when it satisfies

$$\int_Q |\nabla u|^{p(x)-2} \nabla u \nabla q \, dx + \int_Q \rho(x) f(x,u) q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(Q), \tag{2.5}$$

for any domain $Q \Subset \Omega$, and $\max(k-u,0) \in W_0^{1,p(x)}(\Omega)$ for every positive integer k.

Let $W_{0,\mathrm{loc}}^{1,p(x)}(\Omega) = \{u \mid \text{there is an open domain } Q \Subset \Omega \text{ such that } u \in W_0^{1,p(x)}(Q)\}$, and define $A : W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega) \to (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$ as

$$\langle Au, \varphi \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + \rho(x) e^{f(x,u)} \varphi \right) dx, \quad \forall u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega), \ \forall \varphi \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega). \tag{2.6}$$

Lemma 2.1 (see [9, Theorem 3.1]). *Let $h \in W^{1,p(x)}(\Omega) \cap C(\Omega)$, and $X = h + W_0^{1,p(x)}(\Omega) \cap C(\Omega)$. Then, $A : X \to (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$ is strictly monotone.*

Letting $g \in (W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, if $\langle g, \varphi \rangle \geq 0$,for all $\varphi \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega)$ with $\varphi \geq 0$ a.e. in Ω, then denote $g \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$; correspondingly, if $-g \geq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then denote $g \leq 0$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$.

Definition 2.2. Let $u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$. If $Au \geq 0$ $(Au \leq 0)$ in $(W_{0,\mathrm{loc}}^{1,p(x)}(\Omega))^*$, then u is called a weak supersolution (weak subsolution) of (P).

Copying the proof of [14], we have the following.

Lemma 2.3 (comparison principle). *Let $u, v \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C(\Omega)$ satisfy*

$$Au - Av \geq 0, \quad in \left(W_0^{1,p(x)}(\Omega) \right)^*. \tag{2.7}$$

Let $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ (i.e., $u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in Ω.

Lemma 2.4 (see [8, Theorem 1.1]). *Under the conditions (H_1) and (H_3), if $u \in W^{1,p(x)}(\Omega)$ is a bounded weak solution of $-\Delta_{p(x)} u + \rho(x) e^{f(x,u)} = 0$ in Ω, then $u \in C_{\mathrm{loc}}^{1,\vartheta}(\Omega)$, where $\vartheta \in (0,1)$ is a constant.*

3. Asymptotic Behavior of Boundary Blow-Up Solutions

If u is a radial solution for (P), then (P) can be transformed into

$$\begin{gathered}\left(r^{N-1}\left|u'\right|^{p(r)-2}u'\right)' = r^{N-1}\rho(r)e^{f(r,u)}, \quad r \in (0,R), \\ u(0) = u_0, \quad u'(0) = 0, \quad u'(r) \geq 0, \quad \text{for } 0 < r < R.\end{gathered} \tag{3.1}$$

It means that $u(r)$ is increasing.

Theorem 3.1. *If $f(r,u)$ satisfies*

$$f(r,u) \geq \alpha u^s \quad (\text{as } u \longrightarrow +\infty) \text{ for } r \in [\sigma, R) \text{ uniformly}, \tag{3.2}$$

where σ is defined in (H_4) and α and s are positive constants, then there exists a supersolution $\Phi_1(x)$ which satisfies $\Phi_1(x) \to +\infty$ (as $d(x,\partial\Omega) \to 0$), such that for every solution u of problem (P), *one has $u(x) \leq \Phi_1(x)$.*

Proof. Define the function $g(r,a,\lambda)$ on $[0, R_\lambda)$ as

$$g(r,a,\lambda) = \begin{cases} \left(a \ln \dfrac{1}{(R-r)^{1-\theta} - \lambda}\right)^{1/s} + k, \quad R_0 \leq r < R_\lambda, \\ k - \displaystyle\int_r^{R_0} \left[\dfrac{a^{1/s}(1-\theta)(R-R_0)^{-\theta}}{s\left((R-R_0)^{1-\theta} - \lambda\right)}\left(\ln \dfrac{1}{(R-R_0)^{1-\theta} - \lambda}\right)^{(1/s)-1}\right]^{(p(R_0)-1)/(p(t)-1)} \\ \qquad \times \left[\dfrac{(R_0)^{N-1}}{t^{N-1}} \sin \varepsilon(t-\sigma)\right]^{1/(p(t)-1)} dt \\ \qquad + \left(a \ln \dfrac{1}{(R-R_0)^{1-\theta} - \lambda}\right)^{1/s}, \quad \sigma < r < R_0, \\ k - \displaystyle\int_\sigma^{R_0} \left[\dfrac{a^{1/s}(1-\theta)(R-R_0)^{-\theta}}{s\left((R-R_0)^{1-\theta} - \lambda\right)}\left(\ln \dfrac{1}{(R-R_0)^{1-\theta} - \lambda}\right)^{(1/s)-1}\right]^{(p(R_0)-1)/(p(t)-1)} \\ \qquad \times \left[\dfrac{(R_0)^{N-1}}{t^{N-1}} \sin \varepsilon(t-\sigma)\right]^{1/(p(t)-1)} dt \\ \qquad + \left(a \ln \dfrac{1}{(R-R_0)^{1-\theta} - \lambda}\right)^{1/s}, \quad r \leq \sigma, \end{cases} \tag{3.3}$$

where $\theta < \beta(R)/p(R)$, $a > (1/\alpha)\sup_{|x|\geq R_0} p(x)$ are constants, $R_0 \in (\sigma, R)$, $R - R_0$ is small enough, parameter $\lambda \in [0, (R-R_0)^{1-\theta}/2]$, R_λ satisfies $(R-R_\lambda)^{1-\theta} - \lambda = 0$, $\varepsilon = \pi/2(R_0 - \sigma)$

$$
\begin{aligned}
k = {} & \left[\frac{2p^{+}((1+s)/s + 1/(1-\theta)) + |\beta|^{+}/(1-\theta)}{\alpha} \ln \frac{2}{(R-R_0)^{(1-\theta)}}\right]^{1/s} \\
& + \int_{\sigma}^{R_0} \left[\frac{2a^{1/s}(1-\theta)}{s(R-R_0)} \left(\ln \frac{2}{(R-R_0)^{1-\theta}}\right)^{(1/s)-1}\right]^{(p(R_0)-1)/(p(t)-1)} \\
& \times \left[\frac{(R_0)^{N-1}}{t^{N-1}} \sin \varepsilon(t-\sigma)\right]^{1/(p(t)-1)} dt.
\end{aligned} \tag{3.4}
$$

Obviously, for any positive constant a, we have $g(r, a, \lambda) \in C^1[0, R_\lambda)$.

When $R_0 < r < R_\lambda < R$, we have

$$
\begin{aligned}
g' = g'(r, a, \lambda) &= \frac{a^{1/s}}{s} \left(\ln \frac{1}{(R-r)^{1-\theta} - \lambda}\right)^{(1/s)-1} \frac{(1-\theta)(R-r)^{-\theta}}{(R-r)^{1-\theta} - \lambda}, \\
|g'|^{p(r)-2} g' &= \left[\frac{(1-\theta)a^{1/s}}{s}\right]^{p(r)-1} \left(\ln \frac{1}{(R-r)^{1-\theta} - \lambda}\right)^{((1/s)-1)(p(r)-1)} \frac{(R-r)^{-\theta(p(r)-1)}}{\left[(R-r)^{1-\theta} - \lambda\right]^{p(r)-1}}, \\
\left(r^{N-1}|g'|^{p(r)-2} g'\right)' &= r^{N-1} \left[\frac{(1-\theta)a^{1/s}}{s}\right]^{p(r)-1} \left(\ln \frac{1}{(R-r)^{1-\theta} - \lambda}\right)^{((1/s)-1)(p(r)-1)} \\
& \quad \times \frac{(p(r)-1)(R-r)^{-\theta p(r)}}{\left[(R-r)^{1-\theta} - \lambda\right]^{p(r)}} [(1-\theta) + \Pi(r)],
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
\Pi(r) = {} & \frac{\left\{r^{N-1}\left[(1-\theta)a^{1/s}/s\right]^{p(r)-1}\right\}'}{(p(r)-1) r^{N-1} \left[(1-\theta)a^{1/s}/s\right]^{p(r)-1}} \frac{(R-r)^{1-\theta} - \lambda}{(R-r)^{1-\theta}} (R-r) + \frac{((1/s)-1)(1-\theta)}{\left(\ln\left(1/\left((R-r)^{1-\theta} - \lambda\right)\right)\right)} \\
& + \frac{(R-r)^{1-\theta} - \lambda}{(R-r)^{1-\theta}} (R-r) \frac{((1/s)-1)p'(r)}{(p(r)-1)} \ln\left[\ln \frac{1}{(R-r)^{1-\theta} - \lambda}\right] \\
& + \frac{\theta p'(r)}{(p(r)-1)} \frac{(R-r)^{1-\theta} - \lambda}{(R-r)^{1-\theta}} (R-r) \ln \frac{1}{(R-r)} \\
& + \theta \frac{(R-r)^{1-\theta} - \lambda}{(R-r)^{1-\theta}} + \frac{-p'(r)}{p(r)-1} (R-r) \frac{(R-r)^{1-\theta} - \lambda}{(R-r)^{1-\theta}} \ln\left[(R-r)^{1-\theta} - \lambda\right].
\end{aligned} \tag{3.6}
$$

If $(R - R_0)$ is small enough, it is easy to see that

$$|\Pi(r)| \le \ln \frac{1}{(R-r)^{1-\theta} - \lambda}, \quad \text{for } \lambda \in \left[0, \frac{(R-R_0)^{1-\theta}}{2}\right] \text{uniformly}, \tag{3.7}$$

and then

$$\begin{aligned} \left(r^{N-1}\left|g'\right|^{p(r)-2} g'\right)' \le{}& r^{N-1}\left[\frac{(1-\theta)a^{1/s}}{s}\right]^{p(r)-1}\left(\ln \frac{1}{(R-r)^{1-\theta}-\lambda}\right)^{((1/s)-1)(p(r)-1)+1} \\ & \times \frac{(p(r)-1)(R-r)^{-\theta p(r)}}{\left[(R-r)^{1-\theta}-\lambda\right]^{p(r)}}, \quad \forall r \in (R_0, R_\lambda). \end{aligned} \tag{3.8}$$

Thus, when $0 < R - R_0$ is small enough, from (3.5) and (3.8), for $\lambda \in [0, (R-R_0)^{1-\theta}/2]$ uniformly, we have

$$\begin{aligned} &\left(r^{N-1}\left|g'\right|^{p(r)-2} g'\right)' \\ &\quad \le 2r^{N-1}\left[\frac{(1-\theta)a^{1/s}}{s}\right]^{p(r)-1}\left(\ln \frac{1}{(R-r)^{1-\theta}-\lambda}\right)^{((1/s)-1)(p(r)-1)+1} \frac{(p(r)-1)(R-r)^{-\theta p(r)}}{\left[(R-r)^{1-\theta}-\lambda\right]^{p(r)}} \\ &\quad \le r^{N-1}\rho(r)\left(\frac{1}{(R-r)^{1-\theta}-\lambda}\right)^{\alpha a} = r^{N-1}\rho(r)e^{\alpha g^s} \le r^{N-1}\rho(r)e^{f(r,g)}, \quad \forall r \in (R_0, R_\lambda). \end{aligned} \tag{3.9}$$

Thus, when $0 < R - R_0$ is small enough, the following inequality is valid for $\lambda \in [0, (R-R_0)^{1-\theta}/2]$ uniformly:

$$\left(r^{N-1}\left|g'\right|^{p(r)-2} g'\right)' \le r^{N-1}\rho(r) f(r, g), \quad \forall r \in (R_0, R_\lambda). \tag{3.10}$$

Obviously, if $R-R_0$ is small enough, then $g \geq [((2p^+((s+1)/s+1/(1-\theta))+|\beta|^+/(1-\theta))/\alpha)\ln(2/(R-R_0)^{1-\theta})]^{1/s}$ is large enough. Since $\lambda \in [0,(R-R_0)^{1-\theta}/2]$,

$$\begin{aligned}
&\left(r^{N-1}|g'|^{p(r)-2}g'\right)' \\
&\quad = \varepsilon(R_0)^{N-1}\left[\frac{a^{1/s}(1-\theta)(R-R_0)^{-\theta}}{s\left((R-R_0)^{1-\theta}-\lambda\right)}\left(\ln\frac{1}{(R-R_0)^{1-\theta}-\lambda}\right)^{(1/s)-1}\right]^{(p(R_0)-1)}\cos(\varepsilon(r-\sigma)) \\
&\quad \leq \varepsilon(R_0)^{N-1}\left[\frac{a^{1/s}(1-\theta)(R-R_0)^{-\theta}}{s(1/2)(R-R_0)^{1-\theta}}\left(\ln\frac{2}{(R-R_0)^{1-\theta}}\right)^{(1/s)+1}\right]^{(p(R_0)-1)} \\
&\quad \leq \varepsilon(R_0)^{N-1}\left[\frac{2a^{1/s}(1-\theta)}{s(R-R_0)}\left(\frac{2}{(R-R_0)^{1-\theta}}\right)^{(1/s)+1}\right]^{(p(R_0)-1)} \\
&\quad \leq \varepsilon(R_0)^{N-1}\left[\frac{2a^{1/s}(1-\theta)}{s}\left(\frac{2}{R-R_0}\right)^{((s+1)/s)(1-\theta)+1}\right]^{p^+} \\
&\quad \leq r^{N-1}\rho(r)e^{\alpha g^s} \leq r^{N-1}\rho(r)e^{f(r,g)}, \quad \sigma < r < R_0.
\end{aligned} \tag{3.11}$$

Thus,

$$\left(r^{N-1}|g'|^{p(r)-2}g'\right)' \leq r^{N-1}\rho(r)e^{f(r,g)}, \quad \sigma < r < R_0. \tag{3.12}$$

Obviously,

$$\left(r^{N-1}|g'|^{p(r)-2}g'\right)' = 0 \leq r^{N-1}\rho(r)e^{f(r,g)}, \quad 0 \leq r < \sigma. \tag{3.13}$$

Since $g(x,a,\lambda)=g(|x|,a,\lambda)$ is a C^1 function on $B(0,R_\lambda)$, if $0<R-R_0$ is small enough (R_0 depends on R, p, s, α), from (3.10), (3.12), and (3.13), for any $\lambda \in [0,(R-R_0)^{1-\theta}/2]$, we can see that $g(|x|,a,\lambda)$ is a supersolution for (P) on $B(0,R_\lambda)$, and then $g(|x|,a,0)$ is a supersolution for (P).

Defining the function $g_m(|x|,a-\epsilon)=g(r,a-\epsilon,1/m)$ on $[0,R_{1/m})$, where $a-\epsilon > (1/\alpha)\sup_{|x|\geq R_0}p(x)$, then $g_m(|x|,a-\epsilon)$ is a supersolution for (P) on $B(0,R-(1/m))$. If u is a solution for (P), according to the comparison principle, we get that $g_m(|x|,a-\epsilon)\geq u(x)$ for any $x\in B(0,R_{1/m})$. For any $x\in B(0,R)\setminus B(0,R_0)$, we have $g_m(|x|,a-\epsilon)\geq g_{m+1}(|x|,a-\epsilon)$, when m is large enough. Thus

$$u(x) \leq \lim_{m\longrightarrow+\infty} g_m(|x|,a-\epsilon), \quad \forall x \in B(0,R)\setminus B(0,R_0). \tag{3.14}$$

When $d(x, \partial\Omega) > 0$ is small enough, we have

$$\lim_{m \longrightarrow +\infty} g_m(|x|, a-\epsilon) < \left(a \ln \frac{1}{(R-r)^{1-\theta}} \right)^{1/s} + k \leq g(|x|, a, 0). \tag{3.15}$$

According to the comparison principle, we get that $g(|x|, a, 0) \geq u(x)$, for all $x \in B(0, R)$; then $\Phi_1(x) = \Phi_1(|x|) = g(|x|, a, 0)$ is a radial upper control function of all of the solutions for (P), and $\Phi_1(x) = \Phi_1(|x|)$ is a radial supersolution for (P). The proof is completed. □

Theorem 3.2. *If $f(r, u)$ satisfies*

$$\begin{gathered} f(r,u) \longrightarrow -\infty \quad (\text{as } u \longrightarrow -\infty) \text{ for } r \in [\sigma, R) \text{ uniformly}, \\ f(r,u) \leq \delta u^s \quad (\text{as } u \longrightarrow +\infty) \text{ for } r \in [\sigma, R) \text{ uniformly}, \end{gathered} \tag{3.16}$$

where σ is defined in (H_4) and δ and s are positive constants, then there exists a subsolution $\Phi_2(x)$ which satisfies $\Phi_2(x) \to +\infty$ (as $d(x, \partial\Omega) \to 0$), such that for every solution $u(x)$ for problem (P), *one has $u(x) \geq \Phi_2(x)$.*

Proof. We will prove this theorem in the following two cases.

(i) $\beta_1(R) > 0$.

(ii) $\beta_1(R) \leq 0$.

Case 1 ($\beta_1(R) > 0$). Let z_1 be a radial solution of

$$-\Delta_{p(x)} z_1(x) = -\mu, \quad \text{in } \Omega_1 = B(0, \sigma), \quad z_1 = 0, \quad \text{on } \partial\Omega_1, \tag{3.17}$$

where $\mu > 2(\max_{r \in [0, R_0]} \rho(r) + 1)^{2(p^+ - 1)/(p^- - 1)}$ is a positive constant. We denote $z_1 = z_1(r) = z_1(|x|)$. Then, z_1 satisfies

$$\begin{gathered} -\left(r^{N-1} |z_1'|^{p(r)-2} z_1' \right)' = -r^{N-1}\mu, \quad z_1(\sigma) = 0, \qquad z_1'(0) = 0, \\ z_1' = \left| \frac{r\mu}{N} \right|^{1/(p(r)-1)}, z_1 = -\int_r^{\sigma} \left| \frac{r\mu}{N} \right|^{1/(p(r)-1)} dr. \end{gathered} \tag{3.18}$$

Denote $h_b(r, \lambda)$ on $[\sigma, R_0]$ as

$$\begin{aligned} h_b(r,\lambda) = \int_r^{R_0} \Bigg\{ & \frac{(R_0)^{N-1}}{t^{N-1}} \frac{t-\sigma}{R_0-\sigma} \left[\frac{b(1-\theta)(R-R_0)^{-\theta}}{s\left((R-R_0)^{1-\theta}+\lambda\right)} \left(b \ln \frac{1}{(R-R_0)^{1-\theta}+\lambda} \right)^{(1/s)-1} \right]^{p(R_0)-1} \\ & + \frac{(\sigma)^{N-1}}{t^{N-1}} \frac{R_0 - t}{R_0 - \sigma} \left| \frac{\sigma\mu}{N} \right| \Bigg\}^{1/(p(t)-1)} dt. \end{aligned} \tag{3.19}$$

It is easy to see that

$$
\begin{aligned}
-h_b'(\sigma,\lambda) &= z_1'(\sigma) = \left|\frac{\sigma\mu}{N}\right|^{1/(p(\sigma)-1)}, \\
-h_b'(R_0,\lambda) &= \frac{b(1-\theta)(R-R_0)^{-\theta}}{s\left((R-R_0)^{1-\theta}+\lambda\right)}\left(b\ln\frac{1}{(R-R_0)^{1-\theta}+\lambda}\right)^{(1/s)-1}.
\end{aligned}
\tag{3.20}
$$

Define the function $v(r,b,\lambda)$ on $[0,R)$ as

$$
v(r,b,\lambda)=\begin{cases}
\left(b\ln\dfrac{1}{(R-r)^{1-\theta}+\lambda}\right)^{1/s}-k^*, & R_0\le r<R,\\
\left(b\ln\dfrac{1}{(R-R_0)^{1-\theta}+\lambda}\right)^{1/s}-k^*-h_b(r,\lambda), & \sigma<r<R_0,\\
-\displaystyle\int_r^{\sigma}\left|\frac{r\mu}{N}\right|^{1/(p(r)-1)}dr+\left(b\ln\dfrac{1}{(R-R_0)^{1-\theta}+\lambda}\right)^{1/s}-k^*-h_b(\sigma,\lambda), & r\le\sigma,
\end{cases}
\tag{3.21}
$$

where $\theta\in(\beta_1(R)/p(R),1)$, $b\in(0,(1/\delta)\inf_{|x|\ge R_0}p(x))$ are constants, $R_0\in(\sigma,R)$,$R-R_0$ is small enough, parameter $\lambda\in[0,(R-R_0)^{1-\theta}/2]$, and

$$
k^*=M+\left(b\ln\frac{1}{(R-R_0)^{1-\theta}}\right)^{1/s},
\tag{3.22}
$$

where M satisfies

$$
(\sigma)^{N-1}\frac{1}{R_0-\sigma}\ge r^{N-1}\rho(r)e^{f(r,y)},\quad \forall y\le -M,\ \forall r\in[0,R_0].
\tag{3.23}
$$

Obviously, for any positive constant b, $v(r,b,\lambda)\in C^1[0,R)$.
By computation, when $r\in(R_0,R)$, we have

$$
\begin{aligned}
v' = v'(r,b,\lambda) &= \frac{b^{1/s}}{s}\left(\ln\frac{1}{(R-r)^{1-\theta}+\lambda}\right)^{1/s-1}\frac{(1-\theta)(R-r)^{-\theta}}{(R-r)^{1-\theta}+\lambda},\\
\left|v'\right|^{p(r)-2}v' &= \left[\frac{(1-\theta)b^{1/s}}{s}\right]^{p(r)-1}\left(\ln\frac{1}{(R-r)^{1-\theta}+\lambda}\right)^{(1/s-1)(p(r)-1)}\frac{(R-r)^{-\theta(p(r)-1)}}{\left[(R-r)^{1-\theta}+\lambda\right]^{p(r)-1}},
\end{aligned}
$$

$$\left(r^{N-1}\left|v'\right|^{p(r)-2}v'\right)' = r^{N-1}\left[\frac{(1-\theta)b^{1/s}}{s}\right]^{p(r)-1}\left(\ln\frac{1}{(R-r)^{1-\theta}+\lambda}\right)^{(1/s-1)(p(r)-1)}$$
$$\times\frac{\left(p(r)-1\right)(R-r)^{-\theta(p(r)-1)-1}}{\left[(R-r)^{1-\theta}+\lambda\right]^{p(r)-1}}(\theta+\Lambda(r)), \tag{3.24}$$

where

$$\Lambda(r) = \frac{\left\{r^{N-1}\left[(1-\theta)b^{1/s}/s\right]^{p(r)-1}\right\}'}{\left(p(r)-1\right)r^{N-1}\left[(1-\theta)b^{1/s}/s\right]^{p(r)-1}}(R-r) + \frac{(1/s-1)(1-\theta)}{\left(\ln\left(1/\left((R-r)^{1-\theta}+\lambda\right)\right)\right)\left[(R-r)^{1-\theta}+\lambda\right]}$$
$$\times(R-r)^{1-\theta} + \frac{(1/s-1)p'(r)}{(p(r)-1)}(R-r)\ln\left[\ln\frac{1}{(R-r)^{1-\theta}+\lambda}\right] + \frac{\theta p'(r)}{(p(r)-1)}(R-r)\ln\frac{1}{(R-r)}$$
$$+\frac{(1-\theta)}{\left[(R-r)^{1-\theta}+\lambda\right]}(R-r)^{1-\theta} + \frac{-p'(r)}{p(r)-1}(R-r)\ln\left[(R-r)^{1-\theta}+\lambda\right]. \tag{3.25}$$

By computation, when $R-R_0$ is small enough, for $\lambda\in[0,(R-R_0)^{1-\theta}/2]$ uniformly, we have

$$\begin{aligned}
&\left(r^{N-1}\left|v'\right|^{p(r)-2}v'\right)' \\
&\quad\geq r^{N-1}\left[\frac{(1-\theta)b^{1/s}}{s}\right]^{p(r)-1}\left(\ln\frac{1}{(R-r)^{1-\theta}+\lambda}\right)^{(1/s-1)(p(r)-1)} \\
&\qquad\times\frac{\left(p(r)-1\right)(R-r)^{-\theta(p(r)-1)-1}}{\left[(R-r)^{1-\theta}+\lambda\right]^{p(r)-1}}\theta\left(1-\frac{1}{2}\right) \\
&\quad\geq \frac{\theta}{2}r^{N-1}\left[\frac{(1-\theta)b^{1/s}}{s}\right]^{p(r)-1}\left(\ln\frac{1}{(R-r)^{1-\theta}+\lambda}\right)^{(1/s-1)(p(r)-1)} \\
&\qquad\times\frac{\left(p(r)-1\right)(R-r)^{-\theta(p(r)-1)-1}}{\left[(R-r)^{1-\theta}+\lambda\right]^{p(r)}}(R-r)^{1-\theta} \\
&\quad\geq \frac{\theta}{2}r^{N-1}\left[\frac{(1-\theta)b^{1/s}}{s}\right]^{p(r)-1}\left(\ln\frac{1}{(R-r)^{1-\theta}+\lambda}\right)^{(1/s-1)(p(r)-1)}\frac{\left(p(r)-1\right)(R-r)^{-\theta p(r)}}{\left[(R-r)^{1-\theta}+\lambda\right]^{p(r)}} \\
&\quad\geq r^{N-1}\rho_1(R-r)^{-\beta_1(r)}e^{\delta v^s} \\
&\quad\geq r^{N-1}\rho(r)e^{f(r,v)}, \quad \forall r\in(R_0,R).
\end{aligned} \tag{3.26}$$

Then, for $\lambda \in [0, (R - R_0)^{1-\theta}/2]$ uniformly, we have

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)' \geq r^{N-1}\rho(r)e^{f(r,v)}, \quad \forall r \in (R_0, R). \tag{3.27}$$

When $R - R_0$ is small enough, for all $r \in (\sigma, R_0)$, since $v \leq -M$, it is easy to see that

$$\begin{aligned}
\left(r^{N-1}|v'|^{p(r)-2}v'\right)' &\geq \left(r^{N-1}|h'|^{p(r)-2}h'\right)' \\
&= (R_0)^{N-1}\frac{1}{R_0 - \sigma}\left[\frac{b(1-\theta)(R-R_0)^{-\theta}}{s\left((R-R_0)^{1-\theta}+\lambda\right)}\left(b\ln\frac{1}{(R-R_0)^{1-\theta}+\lambda}\right)^{1/s-1}\right]^{p(R_0)-1} \\
&\quad - (\sigma)^{N-1}\frac{1}{R_0-\sigma}\left|\frac{\sigma\mu}{N}\right| \\
&\geq (\sigma)^{N-1}\frac{1}{R_0-\sigma} \\
&\geq r^{N-1}\rho(r)e^{f(r,v)},
\end{aligned} \tag{3.28}$$

Then,

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)' \geq r^{N-1}\rho(r)e^{f(r,v)}, \quad \forall r \in (\sigma, R_0). \tag{3.29}$$

Obviously,

$$\left(r^{N-1}|v'|^{p(r)-2}v'\right)' = r^{N-1}\mu \geq r^{N-1}\rho(r)e^{f(r,v)}, \quad \forall r \in (0, \sigma). \tag{3.30}$$

Combining (3.27), (3.29), and (3.30), when $R - R_0$ is large enough, for any $\lambda \in [0, (R - R_0)^{1-\theta}/2]$, one can see that $v(r, a, \lambda)$ is a subsolution for (P).

Define the function $v_m(r, b+\epsilon)$ on $B(0, R)$ as

$$v_m(r, b+\epsilon) = v_m\left(r, b+\epsilon, \frac{1}{m}\right), \tag{3.31}$$

where ϵ is a small enough positive constant such that $(b+\epsilon) < (1/\delta)\inf_{|x|\geq R_0} p(x)$.

For any $m = 1, 2, \ldots$, we can see that $v_m(r, b+\epsilon) \in C^1([0, R))$ is a subsolution for (P) on $B(R_0, R)$. According to the comparison principle, we get that $v_m(r, b+\epsilon) \leq u(x)$ for any $x \in B(0, R)$. For any $x \in B(0, R) \setminus B(0, R_0)$, we have $v_m(|x|, b+\epsilon) \leq v_{m+1}(|x|, b+\epsilon)$. Thus

$$u(x) \geq \lim_{m\longrightarrow+\infty} v_m(|x|, b+\epsilon), \quad \forall x \in B(0, R) \setminus B(0, R_0). \tag{3.32}$$

When $d(x, \partial\Omega)$ is small enough, we have $\lim_{m\longrightarrow+\infty} v_m(|x|, b+\epsilon) > v(|x|, b, 0)$.

According to the comparison principle, we get that $v(|x|, b, 0) \leq u(x)$, $\forall x \in B(0, R)$; then $\Phi_2(x) = \Phi_2(|x|) = v(|x|, b, 0)$ is a radial lower control function of all of the solutions for (P), and $\Phi_2(x)$ is a radial subsolution for (P).

Case 2 ($\beta_1(R) \leq 0$). Let $\mu > 2(\max_{r\in[0,R_0]}\rho(r) + 1)^{2(p^+-1)/(p^--1)}$ be a positive constant. Denote $\varpi_b(r, \lambda)$ on $[\sigma, R_0]$ as

$$\begin{aligned}\varpi_b(r,\lambda) = \int_r^{R_0} \Bigg\{ &\frac{(R_0)^{N-1}}{t^{N-1}} \frac{t-\sigma}{R_0-\sigma} \left[\frac{b}{s(R+\lambda-R_0)} \left(b\ln(R+\lambda-R_0)^{-1}\right)^{1/s-1}\right]^{p(R_0)-1} \\ &+ \frac{(\sigma)^{N-1}}{t^{N-1}} \frac{R_0-t}{R_0-\sigma} \left|\frac{\sigma\mu}{N}\right| \Bigg\}^{1/(p(t)-1)} dt.\end{aligned} \tag{3.33}$$

It is easy to see that

$$-\varpi_b'(\sigma,\lambda) = z_1'(\sigma) = \left|\frac{\sigma\mu}{N}\right|^{1/(p(\sigma)-1)}, \quad -\varpi_b'(R_0,\lambda) = \frac{b}{s(R+\lambda-R_0)}\left(b\ln(R+\lambda-R_0)^{-1}\right)^{1/s-1}. \tag{3.34}$$

Define the function $\eta(r, b, \lambda)$ on $B(0, R)$ as

$$\eta(r,b,\lambda) = \begin{cases} \left(b\ln(R+\lambda-r)^{-1}\right)^{1/s} - k^*, & R_0 \leq r < R, \\ \left(b\ln(R+\lambda-R_0)^{-1}\right)^{1/s} - k^* - \varpi_b(r,\lambda), & \sigma < r < R_0, \\ -\displaystyle\int_r^{\sigma} \left|\frac{r\mu}{N}\right|^{1/(p(r)-1)} dr + \left(b\ln(R+\lambda-R_0)^{-1}\right)^{1/s} - k^* - \varpi_b(\sigma,\lambda), & r \leq \sigma, \end{cases} \tag{3.35}$$

where $b \in (0, (1/\delta)\inf_{|x|\geq R_0}[p(x) - \beta_1(x)])$ is a constant, $R_0 \in (\sigma, R)$,$R - R_0$ is small enough, parameter $\lambda \in [0, (R - R_0)/2]$, and

$$k^* = M + \left(b\ln\frac{1}{R-R_0}\right)^{1/s}, \tag{3.36}$$

where M is defined in (3.23).

Obviously, for any positive constant b, $\eta(r, b, \lambda) \in C^1[0, R)$.

Similar to the proof of Case 1, when $R - R_0$ is small enough, we have

$$\begin{aligned}
&\left(r^{N-1}|\eta'|^{p(r)-2}\eta'\right)' \\
&\quad \geq r^{N-1}\left(\frac{b^{1/s}}{s}\right)^{p(r)-1}(p(r)-1)(R+\lambda-r)^{-p(r)}\left(\ln(R+\lambda-r)^{-1}\right)^{(1/s-1)(p(r)-1)}\left(1-\frac{1}{2}\right) \\
&\quad \geq r^{N-1}\rho(r)e^{f(r,\eta)}, \quad \forall r \in (R_0, R).
\end{aligned} \tag{3.37}$$

When $R - R_0$ is small enough, for all $r \in (\sigma, R_0)$, from the definition of k^*, it is easy to see that

$$\left(r^{N-1}|\eta'|^{p(r)-2}\eta'\right)' \geq (\sigma)^{N-1}\frac{1}{R_0-\sigma} \geq r^{N-1}\rho(r)e^{f(r,\eta)}. \tag{3.38}$$

Obviously

$$\left(r^{N-1}|\eta'|^{p(r)-2}\eta'\right)' = r^{N-1}\mu \geq r^{N-1}\rho(r)e^{f(r,\eta)}, \quad \forall r \in (0,\sigma). \tag{3.39}$$

Combining (3.37), (3.38), and (3.39), when $R - R_0$ is large enough, for any $\lambda \in [0,(R - R_0)/2]$, one can see that $\eta(r,a,\lambda)$ is a subsolution for (P).

Define the function $\eta_m(r, b+\varepsilon)$ on $B(0,R)$ as

$$\eta_m(r,b+\varepsilon) = \eta\left(r, b+\varepsilon, \frac{1}{m}\right), \tag{3.40}$$

where ε is a small enough positive constant such that $(b+\varepsilon) < (1/\delta)\inf_{|x|\geq R_0} p(x)$.

We can see that $\eta_m(r,b+\varepsilon) \in C^1[0,R)$ is a subsolution for (P) for any $m = 1,2\ldots$. According to the comparison principle, we get that $\eta_m(r,b+\varepsilon) \leq u(x)$ for any $x \in B(0,R)$. For any $x \in B(0,R)\setminus B(0,R_0)$, we have $\eta_m(|x|,b+\varepsilon) \leq \eta_{m+1}(|x|,b+\varepsilon)$. Then,

$$u(x) \geq \lim_{m\to+\infty}\eta_m(|x|,b+\varepsilon), \quad \forall x \in B(0,R)\setminus B(0,R_0). \tag{3.41}$$

When $d(x,\partial\Omega)$ is small enough, we have

$$\lim_{m\longrightarrow+\infty}\eta_m(|x|,b+\varepsilon) > \eta(|x|,b,0). \tag{3.42}$$

According to the comparison principle, we get that $\eta(|x|,b,0) \leq u(x)$, $\forall\, x \in B(0,R)$; then $\Phi_2(x) = \Phi_2(|x|) = \eta(|x|,b,0)$ is a radial lower control function of all of the solutions for (P), and $\Phi_2(x) = \Phi_2(|x|)$ is a radial subsolution for (P). □

Theorem 3.3. *If $f(r,u)$ satisfies*

$$\lim_{u\longrightarrow+\infty}\frac{f(r,u)}{u^s}=\delta \quad (\text{as } u\longrightarrow+\infty) \text{ for } r\in[\sigma,R) \text{ uniformly}, \tag{3.43}$$

where σ is defined in (H$_4$), δ and s are positive constants, $\rho(r)=\rho_0(R-r)^{-\beta(r)}$, where $\beta(R)<p(R)$, then each solution $u(x)$ for (P) *satisfies*

$$\lim_{|x|\longrightarrow R}\frac{u(x)}{\left((p(R)/\delta)\left(\ln 1/(R-|x|)^{1-\theta}\right)\right)^{1/s}}=1, \quad \text{where } \theta=\frac{\beta(R)}{p(R)}. \tag{3.44}$$

Proof. It is easy to be seen from Theorems 3.1 and 3.2 □

4. The Existence of Boundary Blow-Up Solutions

Theorem 4.1. *If $\inf_{x\in\Omega}p(x)>N$ and $f(r,u)$ satisfies*

$$f(r,u)\geq au^s \quad (\text{as } u\rightarrow+\infty) \text{ for } r\in[\sigma,R) \text{ uniformly}, \tag{4.1}$$

where σ is defined in (H$_4$), a and s are positive constants, then (P) *possesses a boundary blow-up solution.*

Proof. In order to deal with the existence of boundary blow-up solutions, let us consider the problem

$$\begin{gathered}-\Delta_{p(x)}u+\rho(r)e^{f(x,u)}=0, \quad \text{in } \Omega_0,\\ u(x)=c, \quad \text{for } x\in\partial\Omega_0,\end{gathered} \tag{4.2}$$

where c is a positive constant and $\Omega_0\Subset\Omega$ is a radial subdomain of Ω. Since $\inf_{x\in\Omega}p(x)>N$, then $W^{1,p(x)}(\Omega_0)\hookrightarrow C^{\alpha}(\overline{\Omega_0})$, where $\alpha\in(0,1)$. The relative functional of (4.2) is

$$\varphi=\int_{\Omega_0}\frac{1}{p(x)}|\nabla u(x)|^{p(x)}dx+\int_{\Omega_0}F(x,u)dx, \tag{4.3}$$

where $F(x,u)=\int_0^u e^{f(x,t)}dt$. Since φ is coercive in $X:=c+W_0^{1,p(x)}(\Omega_0)$, then φ possesses a nontrivial minimum point u. So, problem (4.2) possesses a weak solution u.

Since $au^s\leq f(r,u)\leq C_1+C_2|u|^{\gamma(x)}$, from Theorems 3.1 and 3.2, we get that (P) possesses a supersolution $g^*(x)$ and a subsolution $g_*(x)$, which satisfy $g^*(x)\geq g_*(x)$, when $d(x,\partial\Omega)$ (the distance from x to $\partial\Omega$) is small enough. According to the comparison principle, we get that $g^*(x)\geq g_*(x)$ for any $x\in\Omega$.

Denote $D_j = \{x \mid |x| < 1 - 1/(j+1)R\}$ $(j = 1, 2, \ldots)$. Let us consider the problem

$$-\Delta_{p(x)} u_j + \rho(x) e^{f(x,u_j)} = 0, \quad \text{in } D_j,$$
$$u_j(x) = g_*(x), \quad \text{for } x \in \partial D_j, \tag{4.4}$$

and the relative functional is

$$\varphi = \int_{D_j} \frac{1}{p(x)} \left|\nabla u_j(x)\right|^{p(x)} dx + \int_{D_j} \rho(x) F(x, u_j) dx. \tag{4.5}$$

Let $g_{*j}(x) = g_*(x)|_{D_j}$. Since the functional φ is coercive in $X_j = g_{*j}(x) + W_0^{1,p(x)}(D_j)$, then φ has a nontrivial minimum point u_j. Therefore, problem (4.4) has a weak solution u_j.

According to the comparison principle, we get that $g_*(x) \le u_j(x)$ for any $x \in D_j$ $(j = 1, 2, \ldots)$. Since $u_j(x) = g_*(x)$ for any $x \in \partial D_j$, then $u_j(x) \le u_{j+1}(x)$ for any $x \in \partial D_j$ $(j = 1, 2, \ldots)$. According to the comparison principle, we get that $u_j(x) \le u_{j+1}(x)$ for any $x \in D_j$ $(j = 1, 2, \ldots)$.

Since $g^*(x)$ is a supersolution and $g^*(x) \ge g_*(x)$ for any $x \in \Omega$, so we have $u_j(x) = g_*(x) \le g^*(x)$ for any $x \in \partial D_j$ $(j = 1, 2, \ldots)$. According to the comparison principle, we get that $u_j(x) \le g^*(x)$ for any $x \in D_j$ $(j = 1, 2, \ldots)$.

Since $g^*(x)$ and $g_*(x)$ are locally bounded, from Lemma 2.4, each weak solution of (4.4) is a $C^{1,\alpha}_{\mathrm{loc}}$ function. The $C^{1,\alpha}$ interior regularity result implies that the sequences $\{u_j\}$ and $\{\nabla u_j\}$ are equicontinuous in D_2, and hence we can choose a subsequence, which we denoted by $\{u_j^1\}$, such that $u_j^1 \to w_1$ and $\nabla u_j^1 \to \varpi_1$ uniformly on D_1 for some $w_1 \in C(D_1)$ and $\varpi_1 \in (C(D_1))^N$. In fact, $\varpi_1 = \nabla w_1$ on D_1, and from the interior $C^{1,\alpha}$ estimate, we conclude that $\nabla w_1 \in (C^\alpha(D_1))^N$ for some $0 < \alpha < 1$. Thus, $w_1 \in W^{1,p(x)}(D_1) \cap C^{1,\alpha}(D_1)$. From the $C^{1,\alpha}$ interior regularity result, we see that $|\nabla u_j|^{p-1} |\nabla \varphi| \le C |\nabla \varphi|$ on D_1, and since the function $\xi \to |\xi|^{p-2}\xi$ is continuous on $\mathbb{R}^N$, it follows that $|\nabla u_j^1(x)|^{p-2} \nabla u_j^1(x) \cdot \nabla \varphi(x) \to |\nabla w_1(x)|^{p-2} \nabla w_1(x) \cdot \nabla \varphi(x)$ for $x \in D_1$. Thus, by the dominated convergence theorem, we have

$$\int_{D_1} \left|\nabla u_j^1(x)\right|^{p-2} \nabla u_j^1(x) \cdot \nabla \varphi(x) dx \longrightarrow \int_{D_1} |\nabla w_1(x)|^{p-2} \nabla w_1(x) \cdot \nabla \varphi(x) dx, \quad \forall \varphi \in W_0^{1,p(x)}(D_1). \tag{4.6}$$

Furthermore, since $0 \le f(u_j^1) \le f(u_{j+1}^1)$ and $f(u_j^1(x)) \to f(w_1(x))$ for each $x \in D_1$, by the monotone convergence theorem, we obtain

$$\int_{D_1} \rho e^{f(u_j^1)} q \, dx \longrightarrow \int_{D_1} \rho e^{f(w_1)} q \, dx, \quad \forall q \in W_0^{1,p(x)}(D_1). \tag{4.7}$$

Therefore, it follows that

$$\int_{D_1} |\nabla w_1(x)|^{p-2} \nabla w_1(x) \cdot \nabla q(x) dx + \int_{D_1} \rho e^{f(w_1)} q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(D_1), \tag{4.8}$$

and hence w_1 is a weak solution for $-\Delta_{p(x)} w_1 + \rho e^{f(w_1)} = 0$ on D_1.

Thus, there exists a subsequence of $\{u_j\}$ which we denote it by $\{u_j^1\}$, such that $u_j^1 \to w_1$ in D_1 (as $j \to \infty$), where $w_1 \in W^{1,p(x)}(D_1) \cap C^{1,\alpha_1}(D_1)$ and satisfies

$$\int_{D_1} |\nabla w_1|^{p(x)-2} \nabla w_1 \nabla q \, dx + \int_{D_1} \rho(x) e^{f(x,w_1)} q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(D_1). \tag{4.9}$$

Similarly, we can prove that there exists a subsequence of $\{u_j^1\}$ which we denote by $\{u_j^2\}$, such that $u_j^2 \to w_2$ in D_2 (as $j \to \infty$), where $w_2 \in W^{1,p(x)}(D_2) \cap C^{1,\alpha_2}(D_2)$ satisfies $w_1 = w_2|_{D_1}$ and

$$\int_{D_2} |\nabla w_2|^{p(x)-2} \nabla w_2 \nabla q \, dx + \int_{D_2} \rho(x) e^{f(x,w_2)} q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(D_2). \tag{4.10}$$

Repeating the above steps, we can get a subsequence of $\{u_j^i \mid j = 1,2,\ldots\}$ which we denote by $\{u_j^{i+1} \mid j = 1,2,\ldots\}$ $(i = 1,2,\ldots)$ and satisfies the following.

(1^0) For any fixed i, $\{u_j^{i+1}\}$ is a subsequence of $\{u_j^i\}$.

(2^0) For any fixed i, $u_j^{i+1} \to w_{i+1}$ in D_{i+1} (as $j \to \infty$), where $w_{i+1} \in W^{1,p(x)}(D_{i+1}) \cap C^{1,\alpha_{i+1}}(D_{i+1})$ satisfies $w_i = w_{i+1}|_{D_i}$.

(3^0) For any fixed i, w_i satisfies

$$\int_{D_i} |\nabla w_i|^{p(x)-2} \nabla w_i \nabla q \, dx + \int_{D_i} \rho(x) e^{f(x,w_i)} q \, dx = 0, \quad \forall q \in W_0^{1,p(x)}(D_i). \tag{4.11}$$

Thus, we can conclude that

(i) $\{u_j^j\}$ is a subsequence of $\{u_j\}$,

(ii) there exists a function $w \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) \cap C_{\mathrm{loc}}^{1,\alpha}(\Omega)$ such that $w_i = w|_{D_i}$, and for any $x \in \Omega$, there exists a constant j_x such that when $j \geq j_x$, $u_j^j(x)$ is defined at x, and $\lim_{j\to\infty} u_j^j(x) = w(x)$,

(iii)

$$\int_{\Omega} |\nabla w|^{p(x)-2} \nabla w \nabla q \, dx + \int_{\Omega} \rho(x) e^{f(x,w)} q \, dx = 0, \quad \forall q \in W_{0,\mathrm{loc}}^{1,p(x)}(\Omega). \tag{4.12}$$

Obviously, w is a boundary blow-up solution for (P).

This completes the proof. $\square$

In Theorem 4.1, when $\inf_{x\in\Omega}p(x) > N$, the existence of solutions for (P) is given. In the following, we will consider the existence of solutions for (P) in the general case $1 < \inf_{x\in\Omega}p(x) \leq \sup_{x\in\Omega}p(x) < \infty$. We need to do some preparation. Let us consider

$$\begin{gathered}\left(r^{N-1}\left|u'\right|^{p(r)-2}u'\right)' = r^{N-1}\rho(r)e^{f(r,u)}, \quad r \in (0,R_\lambda),\\ u'(0)=0, \quad u(R_\lambda)=d,\end{gathered} \tag{I}$$

where $R_\lambda \in (0,R)$ and d is a constant.

Lemma 4.2. *If* $\Phi_2(R_\lambda) \leq d \leq \Phi_1(R_\lambda)$, *where* Φ_1 *and* Φ_2 *are defined in Theorems 3.13.2, respectively, then* (4.13) *has a solution* u *satisfying*

$$\Phi_2(r) \leq u(r) \leq \Phi_1(r), \quad \forall r \in [0,R_\lambda]. \tag{4.13}$$

Proof. Denote

$$h(r,u) = \begin{cases} e^{f(r,\Phi_1(r))} + \arctan(u(r)-\Phi_1(r)), & u(r) > \Phi_1(r),\\ e^{f(r,u)}, & \Phi_2(r) \leq u(r) \leq \Phi_1(r),\\ e^{f(r,\Phi_2(r))} + \arctan(u(r)-\Phi_2(r)), & u(r) < \Phi_2(r). \end{cases} \tag{4.14}$$

Let $\rho_E(t) = \rho(|t|)$, and $h_E(t,u) = h(|t|,u)$, for all $t \in [-R_\lambda,R_\lambda]$. Let us consider the even solutions of the following

$$\begin{gathered}\left(|t|^{N-1}\left|u'\right|^{p(|t|)-2}u'\right)' = |t|^{N-1}\rho_E(t)h_E(t,u), \quad t \in (-R_\lambda,R_\lambda),\\ u(-R_\lambda)=d, \quad u(R_\lambda)=d.\end{gathered} \tag{II}$$

It is easy to see that u is an even solution for (4.15) if and only if u is even and

$$u = d - \int_r^{R_\lambda}\left[|t|^{1-N}\int_0^t |s|^{N-1}\rho(s)h(s,u(s))ds\right]^{1/(p(t)-1)}dt, \quad \forall r \in [0,R_\lambda]. \tag{4.15}$$

Denote $\Psi(u,\mu) = \mu d - \mu\int_r^{R_\lambda}[|t|^{1-N}\int_0^t|s|^{N-1}\rho(s)h(s,u(s))ds]^{1/(p(t)-1)}dt$. Similar to the proof of Lemma 2.3 of [18], for any $\mu \in [0,1]$, it is easy to see that $\Psi(u,\mu)$ is compact continuous and bounded from $C_E^1[0,R_\lambda]$ to $C_E^1[0,R_\lambda]$, where $C_E^1[0,R_\lambda] = \{u \in C^1[0,R_\lambda] \mid u \text{ is even}\}$. Thus, $u = \Psi(u,1)$ has a solution u in $C_E^1[0,R_\lambda]$ and satisfies $u'(0) = \lim_{r\to 0^+}u'(r) = 0$. Then, $u(|t|)$ is an even solution for (4.15).

Denote $\Phi_{1,E}(t) = \Phi_1(|t|)$,$\Phi_{2,E}(t) = \Phi_2(|t|)$. From the definitions of Φ_1 and Φ_2, we can see that $\Phi_1'(0) = 0 = \Phi_2'(0)$; therefore, $\Phi_{1,E}(t)$ and $\Phi_{2,E}(t)$ are supersolution and subsolution for (4.15), respectively.

Since $\Phi_2(R_\lambda) \le u(R_\lambda) \le \Phi_1(R_\lambda)$ and $h_E(t,\cdot)$ is increasing, from the comparison principle, we have

$$\Phi_{2,E}(t) \le u(t) \le \Phi_{1,E}(t), \quad \forall t \in [-R_\lambda, R_\lambda]. \tag{4.16}$$

It means that u is a solution for (4.13) and u satisfies

$$\Phi_2(r) \le u(r) \le \Phi_1(r), \quad \forall r \in [0, R_\lambda]. \tag{4.17}$$

Thus u is a radial solution for (P). This completes the proof. □

Theorem 4.3. *If $f(r,u)$ satisfies*

$$f(r,u) \ge au^s \quad (\text{as } u \longrightarrow +\infty) \text{ for } r \in [\sigma, R) \text{ uniformly}, \tag{4.18}$$

where σ is defined in (H_4) and a and s are positive constants, then (P) *possesses a boundary blow-up solution.*

Proof. From Lemma 4.2, we have that (4.4) has a weak solution $u_j(x) = u_j(|x|) = u_j(r)$. Similar to the proof of Theorem 4.1, we can obtain the existence of solutions for (P). □

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