Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2010, Article ID 868059, 19 pages
doi:10.1155/2010/868059

Research Article

Exact Solutions for Nonclassical Stefan Problems

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Received 17 February 2010; Revised 17 June 2010; Accepted 3 August 2010

Academic Editor: Maurizio Grasselli

We consider one-phase nonclassical unidimensional Stefan problems for a source function F which depends on the heat flux, or the temperature on the fixed face $x = 0$. In the first case, we assume a temperature boundary condition, and in the second case we assume a heat flux boundary condition or a convective boundary condition at the fixed face. Exact solutions of a similarity type are obtained in all cases.

1. Introduction

The one-phase Stefan problem for a semi-infinite material is a free boundary problem for the classical heat equation which requires the determination of the temperature distribution u of the liquid phase (melting problem) or the solid phase (solidification problem) and the evolution of the free boundary $x = s(t)$. Phase change problems appear frequently in industrial processes and other problems of technological interest [1–4].

Nonclassical heat conduction problem for a semi-infinite material was studied in [5–11]. A problem of this type is the following:

$$\begin{aligned}
&\text{(i)} \quad u_t - u_{xx} = -F(W(t), t), \quad x > 0, \ t > 0,\\
&\text{(ii)} \quad u(0,t) = f(t), \quad t > 0,\\
&\text{(iii)} \quad u(x,0) = h(x), \quad x > 0,
\end{aligned} \tag{1.1}$$

where functions $f = f(t)$ and $h = h(x)$ are continuous real functions, and F is a given function of two variables. A particular and interesting case is the following:

$$F(W(t), t) = \frac{\lambda_0}{\sqrt{t}} W(t) \quad (\lambda_0 > 0), \tag{1.2}$$

where $W = W(t)$ represents the heat flux on the boundary $x = 0$, that is $W(t) = u_x(0,t)$. Problems of the types (1.1) and (1.2) can be thought of by modelling of a system of temperature regulation in isotropic mediums [10, 11], with a nonuniform source term which provides a cooling or heating effect depending upon the properties of F related to the course of the heat flux (or the temperature in other cases) at the boundary $x = 0$ [10].

In the particular case of a bounded domain, a class of problems, when the heat source is uniform and belongs to a given multivalued function from $\mathbb{R}$ into itself, was studied in [8] regarding existence, uniqueness, and asymptotic behavior. Moreover, in [5] conditions are given on the nonlinearity of the source term F so as to accelerate the convergence of the solution to the steady-state solution. Other references on the subject are in [7, 12, 13].

Nonclassical free boundary problems of the Stefan type were recently studied in [14–16] from a theoretical point of view by using an equivalent formulation through a system of second kind Volterra integral equations [17–19]. A large bibliography on free boundary problems for the heat equation was given in [20].

In this paper, firstly we consider a free boundary problem which consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ such that the following conditions are satisfied:

$$\rho c u_t - k u_{xx} = -\gamma F(W(t),t), \quad 0 < x < s(t), \; t > 0, \tag{1.3}$$

$$u(0,t) = f > 0, \quad t > 0, \tag{1.4}$$

$$u(s(t),t) = 0, \quad t > 0, \tag{1.5}$$

$$k u_x(s(t),t) = -\rho l \dot{s}(t), \quad t > 0, \tag{1.6}$$

$$s(0) = 0, \tag{1.7}$$

where the thermal coefficients $k, \rho, c, l, \gamma > 0$, the boundary temperature $f > 0$, and the control function F depend on the evolution of the heat flux at the boundary $x = 0$ as follows:

$$W(t) = u_x(0,t), \qquad F(W(t),t) = F(u_x(0,t),t) = \frac{\lambda_0}{\sqrt{t}} u_x(0,t), \tag{1.8}$$

where $\lambda_0 > 0$ is a given constant. The existence and the uniqueness of the solution of a general free boundary problem of the type (1.3)–(1.8) was given recently in [14, 15]. Moreover, we consider other two free boundary problems which consist in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ such that (1.3), (1.5), (1.6), and (1.7) are satisfied, and in these cases the control function F depends on the evolution of the temperature at the boundary $x = 0$ as follows:

$$W(t) = u(0,t), \qquad F(W(t),t) = F(u(0,t),t) = \frac{\lambda_0}{t} u(0,t), \quad \lambda_0 > 0. \tag{1.9}$$

In this case, a heat flux boundary condition

$$k u_x(0,t) = \frac{-q_0}{\sqrt{t}} > 0, \quad t > 0 \tag{1.10}$$

or a convective boundary condition

$$ku_x(0,t) = \frac{q_0}{\sqrt{t}}(u(0,t) - f) > 0, \quad t > 0 \tag{1.11}$$

can be considered at the fixed face $x = 0$ in order to obtain the corresponding explicit solutions.

The plan of this paper is the following. In Section 2, we show an explicit solution of a similarity type for the nonclassical one-phase Stefan problem (1.3)–(1.7) for a control function F given by (1.8).

In Sections 3 and 4, we obtain sufficient conditions on data in order to have a similarity type solution to the problems (1.3), (1.5), (1.6), and (1.7), where the control function F is given by (1.9) (instead of (1.8)) and we take into account the heat flux condition (1.10) or the convective condition (1.11) at the fixed face $x = 0$, respectively.

The restrictions on data we have obtained for these two free boundary problems with a heat flux boundary condition (1.10) or a convective boundary condition (1.11) at the fixed face $x = 0$ can be interpreted in the same way as we have obtained in the classical Stefan problem with the same boundary conditions in [21, 22] in order to have an instantaneous phase-change problem (see, e.g., sufficient condition $\lambda_0 < \rho c/2\gamma$ in Theorems 3.2 and 4.1).

2. Explicit Solution to a One-Phase Stefan Problem for a Nonclassical Heat Equation with Control Function of the Type $F(u_x(0,t),t) = (\lambda_0/\sqrt{t})u_x(0,t)$ and a Temperature Condition at the Fixed Boundary

We consider the following free boundary problem for a semi-infinite material given by the following conditions:

$$\begin{gathered}
\rho c u_t - k u_{xx} = -\gamma F(u_x(0,t),t), \quad 0 < x < s(t),\ t > 0, \\
u(0,t) = f > 0, \quad t > 0, \\
u(s(t),t) = 0, \quad t > 0, \\
k u_x(s(t),t) = -\rho l \dot{s}(t), \quad t > 0, \\
s(0) = 0,
\end{gathered} \tag{2.1}$$

where the thermal coefficients k, ρ, c, l, γ are positive and the control function F, which depends on the evolution of the heat flux at the extremum $x = 0$, is given by (1.8).

In order to obtain an explicit solution of a similarity type, we define

$$\Phi(\eta) = u(x,t), \quad \eta = \frac{x}{2a\sqrt{t}}, \tag{2.2}$$

where $a^2 = k/\rho c$ is the diffusion coefficient of the phase change material. The problem (2.1) and (1.8) become

$$\Phi''(\eta) + 2\eta\Phi'(\eta) = 2\lambda\Phi'(0), \quad 0 < \eta < \eta_0, \tag{2.3}$$

$$\Phi(0) = f, \tag{2.4}$$

$$\Phi(\eta_0) = 0, \tag{2.5}$$

$$\Phi'(\eta_0) = -\frac{2l}{c}\eta_0, \tag{2.6}$$

where the dimensionless parameter λ is defined by

$$\lambda = \frac{\gamma\lambda_0}{\rho c a} > 0, \tag{2.7}$$

and the free boundary $s(t)$ must be of the type

$$s(t) = 2a\eta_0\sqrt{t}, \tag{2.8}$$

where η_0 is an unknown parameter to be determined later. The general solution of the differential equation (2.3) is given by

$$\Phi(\eta) = C_2 + C_1\left[\frac{\sqrt{\pi}}{2}\operatorname{erf}(\eta) + 2\lambda\int_0^{\eta} f_1(z)dz\right], \tag{2.9}$$

where C_1 and C_2 are arbitrary constants, and

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-z^2\right)dz, \qquad f_1(x) = \exp\left(-x^2\right)\int_0^x \exp\left(r^2\right)dr \tag{2.10}$$

are the error function and the Dawson's integral (see [23, page 298] and [24, page 43]), respectively.

After some elementary computations, from (2.3), (2.4), and (2.5) we obtain

$$\Phi(\eta) = f\left[1 - \frac{E(\eta,\lambda)}{E(\eta_0,\lambda)}\right], \quad 0 < \eta < \eta_0, \tag{2.11}$$

where

$$E(x,\lambda) = \operatorname{erf}(x) + \frac{4\lambda}{\sqrt{\pi}}\int_0^x f_1(r)dr. \tag{2.12}$$

Taking into account condition (2.6), the unknown parameter $\eta_0 = \eta_0(\lambda, \mathrm{Ste})$ must be the solution of the following equation:

$$\frac{\mathrm{Ste}}{\sqrt{\pi}}\left[\exp\left(-x^2\right) + 2\lambda f_1(x)\right] = x\left[\mathrm{erf}(x) + \frac{4\lambda}{\sqrt{\pi}}\int_0^x f_1(z)dz\right], \quad x > 0, \tag{2.13}$$

where $\mathrm{Ste} = fc/l > 0$ is the Stefan's number. Equation (2.13) is equivalent to the following one:

$$W_1(x) = 2\lambda W_2(x), \quad x > 0, \tag{2.14}$$

where the real functions W_1 and W_2 are defined by

$$W_1(x) = \mathrm{Ste}\,\exp\left(-x^2\right) - \sqrt{\pi}x\,\mathrm{erf}(x), \tag{2.15}$$

$$W_2(x) = 2x\int_0^x f_1(r)dr - \mathrm{Ste}\,f_1(x). \tag{2.16}$$

Remark 2.1. If $\lambda = 0$ (i.e., $\lambda_0 = 0$), then the problem (2.1) and (1.8) represented the classical Lamé-Clapeyron problem [25]. In this case, there exists a unique solution η_{00} of (2.17) (equivalent to (2.13)) given by

$$F_0(x) = \frac{\mathrm{Ste}}{\sqrt{\pi}}, \quad x > 0, \tag{2.17}$$

where

$$F_0(x) = x\,\mathrm{erf}(x)\exp\left(x^2\right), \tag{2.18}$$

and the explicit solution is given by [2, 23]:

$$u(x,t) = f\left[1 - \frac{\mathrm{erf}(\eta)}{\mathrm{erf}(\eta_{00})}\right], \quad 0 < \eta = \frac{x}{2a\sqrt{t}} < \eta_{00},$$
$$s(t) = 2a\eta_{00}\sqrt{t}. \tag{2.19}$$

In order to solve (2.14), we will study firstly the behavior of function f_1. We obtain some preliminary properties.

Lemma 2.2. *The Dawson's integral satisfies the following properties:*

(i) $f_1(0) = 0$,

(ii) $f_1(+\infty) = 0$,

(iii)

$$f_1'(x) = 1 - 2xf_1(x) = \begin{cases} > 0 & \text{if } 0 < x < x_1, \\ = 0 & \text{if } x = x_1, \\ < 0 & \text{if } x > x_1, \end{cases} \tag{2.20}$$

where $x_1 \simeq 0.924$, $f_1(x_1) \simeq 0.541$,

(iv)

$$f_1''(x) = -2\left[1 + f_1(x)\left(1 - 2x^2\right)\right] = \begin{cases} < 0 & \text{if } 0 < x < x_2, \\ = 0 & \text{if } x = x_2, \\ > 0 & \text{if } x > x_2, \end{cases} \tag{2.21}$$

where $x_2 \simeq 1.502$, $f_1(x_2) \simeq 0.428$,

(v) $\lim_{x\to+\infty} 2xf_1(x) = 1$.

Proof. The properties (i)–(iv) have been proved in [23, page 298] (see also [24, pages 42–45])

(v) By the L'Hopital Theorem, we have

$$\begin{aligned} \lim_{x\to+\infty} 2xf_1(x) &= \lim_{x\to+\infty} \frac{2x\int_0^x \exp(r^2)dr}{\exp(x^2)} = \lim_{x\to+\infty} \frac{\int_0^x \exp(r^2)dr + x\exp(x^2)}{x\exp(x^2)} \\ &= \lim_{x\to+\infty}\left(1 + \frac{\int_0^x \exp(r^2)dr}{x\exp(x^2)}\right) = \lim_{x\to+\infty}\left(1 + \frac{f_1(x)}{x}\right) = 1, \end{aligned} \tag{2.22}$$

then (v) holds. □

Next, we define the following auxiliary functions:

$$\begin{gathered} \varphi_1(x) = \int_0^x f_1(r)dr, \qquad \varphi_2(x) = x\varphi_1(x) = x\int_0^x f_1(r)dr, \\ \varphi_3(x) = xf_1(x), \qquad \varphi_4(x) = x(2xf_1(x) - 1) = -xf_1'(x), \\ \varphi_5(x) = f_1(x) - xf_1'(x), \qquad \varphi_6(x) = \mathrm{Ste} - 2(1 + \mathrm{Ste})xf_1(x). \end{gathered} \tag{2.23}$$

We have the following results.

Lemma 2.3.

(a) *Function* φ_1 *satisfies the following properties:*

(i) $\varphi_1(0) = 0$,
(ii) $\varphi_1'(x) = f_1(x)$,
(iii) $\varphi_1'(0^+) = 0$,
(iv) $\varphi_1(+\infty) = +\infty$,
(v)

$$\varphi_1''(x) = f_1'(x) = 1 - 2xf_1(x) = \begin{cases} > 0 & \text{if } 0 < x < x_1, \\ = 0 & \text{if } x = x_1, \\ < 0 & \text{if } x > x_1, \end{cases} \tag{2.24}$$

(vi) $\lim_{x\to+\infty}(\varphi_1(x)/\log(x)) = 1/2$,
(vii) $\lim_{x\to+\infty}\varphi_1(x)f_1'(x) = 0$.

(b) *Function* φ_4 *satisfies the following properties:*

(i) $\varphi_4(0^+) = 0^-$,
(ii) $\varphi_4'(x) = -1 + 4xf_1(x) - 2x^2(2xf_1(x) - 1)$,
(iii) $\varphi_4(+\infty) = 0^+$,
(iv) $\varphi_4'(0^+) = -1$,
(v) $\varphi_4'(+\infty) = 0^+$,
(vi) $\varphi_4(x) = 0 \Leftrightarrow x = x_1$ (*the maximum point of* f_1),
(vii) $\varphi_4'(x_1) = 1$.

(c) *Function* φ_3 *satisfies the following properties:*

(i) $\varphi_3(0^+) = 0$,
(ii) $\varphi_3(+\infty) = 1/2$,
(iii) $\varphi_3'(x) = f_1(x) + x(1 - 2xf_1(x))$,
(iv) $\varphi_3'(0^+) = 0$,
(v) $\varphi_3'(+\infty) = 0$,
(vi) $\varphi_3(x_1) = x_1f_1(x_1) \simeq 0.4999$,
(vii) $\varphi_3(x_2) = x_2f_1(x_2) \simeq 0.64$.

(d) *Function* φ_2 *satisfies the following properties:*

(i) $\varphi_2(0^+) = 0$,
(ii) $\varphi_2(+\infty) = +\infty$,
(iii) $\varphi_2'(x) = \varphi_1(x) + xf_1(x) > 0$, *for all* $x > 0$,
(iv) $\varphi_2'(0^+) = 0$,
(v) $\varphi_2'(+\infty) = +\infty$,
(vi) $\varphi_2''(x) = 2f_1(x) - x(2xf_1(x) - 1)$,
(vii) $\varphi_2''(+\infty) = 0$,
(viii) $\varphi_2''(0^+) = 0$.

(e) *Function φ_5 satisfies the following properties:*

(i) $\varphi_5(0^+) = 0$,
(ii) $\varphi_5(+\infty) = 0^+$,
(iii)

$$\varphi_5'(x) = -x f_1''(x) = \begin{cases} > 0 & \text{if } 0 < x < x_2, \\ = 0 & \text{if } x = x_2, \\ < 0 & \text{if } x > x_2, \end{cases} \tag{2.25}$$

(iv) $\varphi_5(x) > 0$, *for all* $x > 0$.

(f) *Function φ_6 satisfies the following properties:*

(i) $\varphi_6(0^+) = Ste > 0$,
(ii) $\varphi_6(+\infty) = -1$,
(iii) $\varphi_6'(x) = -2(1 + Ste)\varphi_3'(x)$,
(iv) $\varphi_6'(0^+) = 0$,
(v) $\varphi_6'(+\infty) = 0$,
(vi) $\varphi_6(x_1) = x_1 f_1(x_1) \simeq 0.4999$,
(vii) $\varphi_6(x_2) = x_2 f_1(x_2) \simeq 0.64$.

Proof. (a) Taking into account properties of f_1, we have

$$\varphi_1'(x) = f_1(x) > 0, \quad \forall x > 0, \qquad \varphi_1'(0) = f_1(0) = 0, \tag{2.26}$$

and (v) holds. If we consider Lemma 2.2(v), we get $\varphi_1(+\infty) = +\infty$ and we have

$$\lim_{x \to +\infty} \frac{\varphi_1(x)}{\log(x)} = \lim_{x \to +\infty} x f_1(x) = \frac{1}{2}, \tag{2.27}$$

then (iv) and (vi) hold.

To prove (vii), we consider

$$\varphi_1(x) f_1'(x) = \left(\int_0^x f_1(r) dr \right) f_1'(x) = f_1(c) x f_1'(x), \tag{2.28}$$

where $c = c(x) \in (0, x)$. Then $\lim_{x \to +\infty} \varphi_1(x) f_1'(x) = 0$ because $\lim_{x \to +\infty} x f_1'(x) = 0$ and f_1 is a bounded function.

(b) From the definition of φ_4, we obtain (i) and (ii). To prove (iii), we have

$$\begin{aligned} \varphi_4(+\infty) &= \lim_{x \to +\infty} x\left(2x f_1(x) - 1\right) = \lim_{x \to +\infty} \frac{2x f_1(x) - 1}{1/x} \\ &= \lim_{x \to +\infty} 2 \frac{\left[f_1(x) + x\left(1 - 2x f_1(x)\right)\right]}{1/x^2} = 2 \lim_{x \to +\infty} \left[x^2 f_1(x) + x^2 x\left(1 - 2x f_1(x)\right)\right], \end{aligned} \tag{2.29}$$

then

$$\lim_{x\to+\infty} x\left(2xf_1(x)-1\right) = 2\lim_{x\to+\infty}\left[x^2x\left(2xf_1(x)-1\right)-x^2f_1(x)\right]. \tag{2.30}$$

If we suppose that

$$\lim_{x\to+\infty} x\left(2xf_1(x)-1\right) = L > 0, \tag{2.31}$$

we get

$$L = 2\lim_{x\to+\infty}\left[x^2x\left(2xf_1(x)-1\right)-x^2f_1(x)\right] = +\infty, \tag{2.32}$$

which is a contradiction. If we suppose that

$$\lim_{x\to+\infty} x\left(2xf_1(x)-1\right) = +\infty, \tag{2.33}$$

then

$$\varphi_4'(+\infty) = \lim_{x\to+\infty} -1+4xf_1(x)-2x^2\left(2xf_1(x)-1\right) = -\infty, \tag{2.34}$$

which is also a contradiction. Therefore, $\lim_{x\to+\infty}x(2xf_1(x)-1)=0$ and (iii) hold.

Taking into account (ii), we have $\varphi_4'(x) = -1+4xf_1(x)-2x^2(2xf_1(x)-1)$, then $\varphi_4'(0) = -1$ and if we consider (iii) we have $\varphi_4'(+\infty) = 0^+$. From properties of f_1, we have

$$\varphi_4(x) = 0 \Longleftrightarrow 2xf_1(x)-1=0 \Longleftrightarrow f_1'(x)=0 \Longleftrightarrow x = x_1, \tag{2.35}$$

and (vi) holds. Taking into account $f_1'(x) = 1-2xf_1(x) = 0$, we get $\varphi_4'(x_1) = 1$.

(c) From Lemmas 2.2 and 2.3(b) we get (i)–(vii).

(d) We have $\varphi_2(x) = x\varphi_1(x) = x\int_0^x f_1(r)dr$, then from (a) and (b)(iii) we get (i)–(vi).

(e) As we have $\varphi_5(x) = f_1(x)-xf_1'(x) = f_1(x)+\varphi_4(x)$, then by using the properties of f_1 and (b) we obtain the properties of φ_5.

(f) We have $\varphi_6(x) = \text{Ste}-2(1+\text{Ste})xf_1(x) = \text{Ste}-2(1+\text{Ste})\varphi_3(x)$, and from the properties of φ_3,we obtain (i)–(v). □

Corollary 2.4. *One has*

(i) $\lim_{x\to+\infty}x^2[2xf_1(x)-1] = 1/2$,

(ii) $\lim_{x\to+\infty}x[x^2(2xf_1(x)-1)-xf_1(x)] = 0$.

Now, we are in conditions to enunciate properties of functions W_1 and W_2 in order to study after (2.14).

Lemma 2.5. *The functions $W_1(x)$ and $W_2(x)$, defined by* (2.15) *and* (2.16), *respectively, satisfy the following properties.*

(a) *Properties of function* W_1*:*

(i) $W_1(0) = \text{Ste}$,
(ii) $W_1(+\infty) = -\infty$,
(iii) $\lim_{x\to+\infty}(W_1(x)/x) = -\sqrt{\pi}$,
(iv) $\lim_{x\to+\infty}(W_1(x) + \sqrt{\pi}x) = 0$,
(v) $W_1'(x) < 0$, *for all* $x > 0$,
(vi) $W_1(\eta_{00}) = 0$, *where* η_{00} *is the unique solution of* (2.17),
(vii)

$$W_1''(x) = \begin{cases} < 0 & \textit{if } 0 < x < x_0, \\ = 0 & \textit{if } x = x_0, \\ < 0 & \textit{if } x > x_0, \end{cases} \tag{2.36}$$

where

$$x_0 = \sqrt{\frac{3 + 2\,\text{Ste}}{4(1 + \text{Ste})}}, \tag{2.37}$$

(viii) $W_1''(0^+) = -2(3 + 2\,\text{Ste}) < 0$.

(b) *Properties of function* W_2 :

(i) $W_2(0) = 0$,
(ii) $W_2(+\infty) = +\infty$,
(iii) *there exists a unique* $x_4 > 0$ *such that* $W_2(x_4) = 0$,
(iv) $W_2'(x) = 2\int_0^x f_1(r)dr + 2xf_1(x)(1 + \text{Ste}) - \textit{Ste}$,
(v) *there exists a unique* $x_3 > 0$ *such that* $W_2'(x_3) = 0$ *and* $W_2(x_3) < 0$,
(vi) $W_2'(0^+) = -\text{Ste} < 0$,
(vii) $W_2'(+\infty) = +\infty$,
(viii) $W_2''(x) = 2(1 + \text{Ste})x + 2f_1(x)[2 + \text{Ste} - 2(1 + \text{Ste})x^2]$,
(ix) $W_2''(0^+) = 0$,
(x) $W_2(\eta_{00}) < 0$.

Proof. (a) Taking into account the definition of the function W_1, we get (i) and (ii).
(iii) We have

$$\lim_{x\to+\infty}\frac{W_1(x)}{x} = \lim_{x\to+\infty}\left[\text{Ste}\frac{\exp(-x^2)}{x} - \sqrt{\pi}\,\text{erf}(x)\right] = -\sqrt{\pi}. \tag{2.38}$$

(iv) We have

$$\begin{aligned}\lim_{x\to+\infty}\left(W_1(x)+\sqrt{\pi}x\right) &= \lim_{x\to+\infty}\left(\mathrm{Ste}\,\exp\left(-x^2\right)-\sqrt{\pi}x\,\mathrm{erf}(x)+\sqrt{\pi}x\right)\\ &= \lim_{x\to+\infty}\left(\mathrm{Ste}\,\exp\left(-x^2\right)+\sqrt{\pi}x\,\mathrm{erf}\,c(x)\right)\\ &= \lim_{x\to+\infty}\left(\mathrm{Ste}\,\exp\left(-x^2\right)+Q(x)\exp\left(-x^2\right)\right)\\ &= \lim_{x\to+\infty}\exp\left(-x^2\right)(\mathrm{Ste}+Q(x))=0,\end{aligned} \tag{2.39}$$

where Q is the function defined by

$$Q(x)=\sqrt{\pi}x\exp\left(x^2\right)\mathrm{erf}\,c(x),\quad \mathrm{erf}\,c(x)=1-\mathrm{erf}(x), \tag{2.40}$$

which satisfies the following properties:

$$Q(0)=0,\qquad Q(+\infty)=1,\qquad Q'(x)>0,\quad \forall x>0. \tag{2.41}$$

(v) We have

$$W_1'(x)=-\sqrt{\pi}\,\mathrm{erf}(x)-2x\exp\left(-x^2\right)[\mathrm{Ste}+1]<0,\quad \forall x>0. \tag{2.42}$$

(vi) Taking into account (i), (iii), and (v), we get that there exists a unique zero of W_1 which is given by η_{00},the unique solution of (2.17).

(vii) We have

$$W_1''(x)=-2\exp\left(-x^2\right)\left[3+2\mathrm{Ste}-4(1+\mathrm{Ste})x^2\right], \tag{2.43}$$

then

$$W_1''(x)=0\Longleftrightarrow 4(1+\mathrm{Ste})x^2=3+2\mathrm{Ste}\Longleftrightarrow x=x_0=\sqrt{\frac{3+2\mathrm{Ste}}{4(1+\mathrm{Ste})}}. \tag{2.44}$$

Since $\mathrm{sign}(W_1''(x))=\mathrm{sign}(4(1+\mathrm{Ste})x^2-3-2\mathrm{Ste})$,then we obtain (vii).

(b) Taking into account Lemmas 2.2 and 2.3, we have (i) and (ii).

We can write

$$W_2'(x)=2\int_0^x f_1(r)dr+2xf_1(x)(1+\mathrm{Ste})-\mathrm{Ste}=2\varphi_1(x)-\varphi_6(x), \tag{2.45}$$

then $W_2'(0^+)=-\mathrm{Ste}$, $W_2'(+\infty)=+\infty$ and $W_2''(x)=2\varphi_1'(x)-\varphi_6'(x)$ satisfies $W_2''(0^+)=0$. Then (iv), (vi), (vii), (viii), and (ix) hold.

We have

$$W_2(x) = 0 \Longleftrightarrow 2\varphi_2(x) = \text{Ste}\, f_1(x), \tag{2.46}$$

then taking into account the properties of φ_2 and f_1, we get that there exists a unique $x_4 > 0$ such that

$$W_2(x) = 0, \quad x > 0. \tag{2.47}$$

Moreover, we have

$$W_2(x) = \begin{cases} = 0 & \text{if } x = 0, \\ < 0 & \text{if } 0 < x < x_4, \\ = 0 & \text{if } x = x_4, \\ > 0 & \text{if } x > x_4. \end{cases} \tag{2.48}$$

In the same way, we have

$$W_2'(x) = 0 \Longleftrightarrow 2\varphi_1(x) = \varphi_6(x). \tag{2.49}$$

Then, if we consider the properties of the functions φ_1 and φ_2, we have that there exists a unique x_3 such that $W_2'(x_3) = 0$. Moreover, $W_2(x_3) = -2x_3^2 f_1(x_3) - \text{Ste}\ \varphi_5(x_3) < 0$ and then (v) holds.

To prove (x), we take into account that

$$\begin{aligned} W_2(x) &= 2x\int_0^x f_1(r)\,dr - \text{Ste} f_1(x) \\ &= \sqrt{\pi} x \operatorname{erf}(x) F(x) - \sqrt{\pi} x \int_0^x \operatorname{erf}(r) \exp\left(r^2\right) dr - \text{Ste}\ \exp\left(-x^2\right) F(x) \\ &= \sqrt{\pi} \exp\left(-x^2\right) \left[F_0(x) - \frac{\text{Ste}}{\sqrt{\pi}}\right] F(x) - \sqrt{\pi} x \int_0^x \operatorname{erf}(r) \exp\left(r^2\right) dr, \end{aligned} \tag{2.50}$$

where $F(x) = \int_0^x \exp(r^2)dr$ and F_0 was defined in (2.18). Then by using (2.17), we have

$$W_2(\eta_{00}) = -\sqrt{\pi}\eta_{00} \int_0^{\eta_{00}} \operatorname{erf}(r) \exp\left(r^2\right) dr < 0. \tag{2.51}$$

□

Lemma 2.6. *For each $\lambda > 0$, there exists a unique solution η_0 of* (2.14). *This solution $\eta_0 = \eta_0(\lambda)$ satisfies the following properties:*

$$\begin{aligned} &\text{(i)}\quad \eta_0(0^+) = \eta_{00},\\ &\text{(ii)}\quad \eta_0(+\infty) = x_4,\\ &\text{(iii)}\quad \eta_0 = \eta_0(\lambda) \text{ is an increasing function on } \lambda, \end{aligned} \tag{2.52}$$

where η_{00} and x_4 are the unique solution of (2.17) *and* (2.47), *respectively.*

Proof. Taking into account Lemma 2.5, we get that there exists a unique solution η_0 of (2.14). Let $0 < \lambda_1 < \lambda_2$ be given, taking into account properties of function W_2, we obtain that the real functions Z_1 and Z_2 defined by

$$Z_1(x) = 2\lambda_1 W_2(x), \qquad Z_2(x) = 2\lambda_2 W_2(x) \tag{2.53}$$

satisfy the following properties:

$$\begin{aligned} Z_2(x) &< Z_1(x) \quad \text{if } 0 < x < x_4,\\ Z_2(x) &= Z_1(x) \quad \text{if } x = x_4,\\ Z_2(x) &> Z_1(x) \quad \text{if } x > x_4. \end{aligned} \tag{2.54}$$

Then $\eta_0(\lambda_1) < \eta_0(\lambda_2)$, where $\eta_0(\lambda_i)$ is the solution of equation $Z_i(x) = W_1(x), i = 1,2$. Therefore, $\eta_0 = \eta_0(\lambda)$ is an increasing function on λ. Moreover, we obtain $\eta_{00} < \eta_0(\lambda) < x_4$ because $W_2(\eta_{00}) < 0$. □

Then, we have proved the following result.

Theorem 2.7. *For each $\lambda > 0$, the free boundary problem* (2.1), *where F is defined by* (1.8), *has a unique similarity solution of the type*

$$\begin{gathered} u(x,t,\lambda) = f\left[1 - \frac{E(n,\lambda)}{E(\eta_0(\lambda),\lambda)}\right], \quad 0 < \eta = \frac{x}{2a\sqrt{t}} < \eta_0(\lambda),\\ s(t,\lambda) = 2a\eta_0(\lambda)\sqrt{t}, \end{gathered} \tag{2.55}$$

where

$$E(\eta,\lambda) = \operatorname{erf}(\eta) + \frac{4\lambda}{\sqrt{\pi}} \int_0^{\eta} f_1(r)\,dr \tag{2.56}$$

and $\eta_0 = \eta_0(\lambda)$ is the unique solution of (2.14) *with $\eta_{00} < \eta_0(\lambda) < x_4$.*

3. Explicit Solution to a One-Phase Stefan Problem for a Nonclassical Heat Equation with Control Function of the Type $F(u(0,t),t) = (\lambda_0/t)u(0,t)$ and a Heat Flux Condition at the Fixed Face

In this section, the free boundary problem consists in determining the temperature $u = u(x,t)$ and the free boundary $x = s(t)$ with a control function F which depends on the evolution of the temperature at the extremum $x = 0$ given by the following conditions:

$$\begin{gathered}
\rho c u_t - k u_{xx} = -\gamma F(u(0,t),t), \quad 0 < x < s(t), \ t > 0, \\
k u_x(0,t) = \frac{-q_0}{\sqrt{t}} > 0, \quad t > 0, \\
u(s(t),t) = 0, \quad t > 0, \\
k u_x(s(t),t) = -\rho l \dot{s}(t), \quad t > 0, \\
s(0) = 0,
\end{gathered} \tag{3.1}$$

where the coefficient $q_0 > 0$ characterizes the heat flux on the $x = 0$ [21] and the control function F is given by (1.9).

In order to obtain an explicit solution of a similarity type, we define the same transformation given by (2.2). The problem (3.1) and (1.9) are equivalent to the following one:

$$\Phi''(\eta) + 2\eta\Phi'(\eta) = \Lambda\Phi(0), \quad 0 < \eta < \mu_0, \tag{3.2}$$

$$\Phi'(0) = -q_0^*, \tag{3.3}$$

$$\Phi(\mu_0) = 0, \tag{3.4}$$

$$\Phi'(\mu_0) = -\frac{2l}{c}\mu_0, \tag{3.5}$$

where the dimensionless parameters Λ and q_0^* are defined by

$$\Lambda = \frac{4\gamma\lambda_0}{\rho c} > 0, \qquad q_0^* = \frac{2aq_0}{k}, \tag{3.6}$$

$$s(t) = 2a\mu_0\sqrt{t} \tag{3.7}$$

is the free boundary, where μ_0 is an unknown parameter to be determined.

From (3.2), (3.3), and (3.4), we obtain the similarity solution

$$\Phi(\eta) = \frac{q_0^*\sqrt{\pi}}{2G(\mu_0,\Lambda)}\left[\operatorname{erf}(\mu_0)G(\eta,\Lambda) - \operatorname{erf}(\eta)G(\mu_0,\Lambda)\right], \quad 0 < \eta < \mu_0, \tag{3.8}$$

where

$$G(x,\Lambda) = 1 + \Lambda \int_0^x f_1(r)dr = 1 + \Lambda\varphi_1(x), \tag{3.9}$$

and f_1 is the Dawson's integral and φ_1 is given by (2.23).

By condition (3.5), the unknown parameter $\mu_0 = \mu_0(\Lambda, l, c, q_0^*)$ must be solution of the following equation:

$$\Lambda \operatorname{erf}(x) f_1(x) = \frac{2}{\sqrt{\pi}} G(x,\Lambda) \left[\exp\left(-x^2\right) - \frac{2l}{cq_0^*}x\right], \quad x > 0, \tag{3.10}$$

which is equivalent to the following one:

$$H_2(x) = H_3(x), \quad x > 0, \tag{3.11}$$

where the real functions H_2 and H_3 are defined by

$$H_2(x) = \Lambda \operatorname{erf}(x) f_1(x), \tag{3.12}$$

$$H_3(x) = \frac{2}{\sqrt{\pi}} G(x,\Lambda) H_1(x), \tag{3.13}$$

$$H_1(x) = \left[\exp\left(-x^2\right) - \frac{2l}{cq_0^*}x\right]. \tag{3.14}$$

Remark 3.1. If $\Lambda = 0$ (i.e., $\lambda_0 = 0$), we have the solution

$$\Phi(\eta) = \frac{q_0^*\sqrt{\pi}}{2}\left[\operatorname{erf}(\mu_{00}) - \operatorname{erf}(\eta)\right], \quad 0 < \eta < \mu_{00}, \tag{3.15}$$

where μ_{00} is the unique solution of the following equation:

$$\exp\left(-x^2\right) = \frac{2l}{cq_0^*}x. \tag{3.16}$$

In order to solve (3.11), we consider properties of Dawson's integral, error function, and some auxiliary functions, and then we obtain the following result.

Theorem 3.2. *For each $\lambda_0 < \rho c/2\gamma$, the free boundary problem* (3.1)*, where F is defined by* (1.9)*, has a unique similarity solution of the type*

$$u(x,t,\lambda_0) = \frac{q_0 a\sqrt{\pi}}{kG(\mu_0(\lambda_0),4\gamma\lambda_0/\rho c)}\left[\operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)G\left(\mu_0(\lambda_0),\frac{4\gamma\lambda_0}{\rho c}\right) - \operatorname{erf}(\mu_0(\lambda_0))G\left(\frac{x}{2a\sqrt{t}},\frac{4\gamma\lambda_0}{\rho c}\right)\right],$$
$$0 < \frac{x}{2a\sqrt{t}} < \mu_0(\lambda_0), \quad t > 0, \tag{3.17}$$
$$s(t,\lambda_0) = 2a\mu_0(\lambda_0)\sqrt{t}, \quad t > 0,$$

where $\mu_0 = \mu_0(\lambda_0)$ is the unique solution of (3.11), $0 < \mu_0(\lambda_0) < \mu_{00}$.

Proof. We follow a similar method developed in Theorem 2.7. □

4. Explicit Solution to a One-Phase Stefan Problem for a Nonclassical Heat Equation with Control Function of the Type $F(u(0,t),t) = (\lambda_0/t)u(0,t)$ and a Convective Condition at the Fixed Face

In this section, we consider a similar problem to the one given in Section 3 for a convective boundary condition [22, 26] on the fixed face given by

$$\begin{aligned}
&\rho c u_t - k u_{xx} = -\gamma F(u(0,t),t), \quad 0 < x < s(t), \ t > 0,\\
&k u_x(0,t) = \frac{h_0}{\sqrt{t}}(u(0,t) - f) > 0, \quad t > 0,\\
&u(s(t),t) = 0, \quad t > 0,\\
&k u_x(s(t),t) = -\rho l \dot{s}(t), \quad t > 0,\\
&s(0) = 0,
\end{aligned} \tag{4.1}$$

where F is defined by (1.9) and h_0 characterizes the heat transfer coefficients [22, 26]. To solve this problem, we consider again a similarity type solution given by (2.2). Then, the problem (4.1) and (1.9) are equivalent to the following one:

$$\Phi''(\eta) + 2\eta\Phi'(\eta) = \Lambda\Phi(0), \quad 0 < \eta < \mu_0, \tag{4.2}$$

$$\Phi'(0) = h_0^*(\Phi(0) - f), \quad h_0^* = \frac{2ah_0}{k}, \tag{4.3}$$

$$\Phi(\mu_0) = 0, \tag{4.4}$$

$$\Phi'(\mu_0) = -\frac{2l}{c}\mu_0, \tag{4.5}$$

where the dimensionless parameter Λ is defined by (3.6) and

$$s(t) = 2a\mu_0\sqrt{t} \tag{4.6}$$

is the free boundary, where μ_0 is an unknown parameter to be determined. We obtain the solution

$$\Phi(\eta) = \frac{h_0^* f\sqrt{\pi}}{2}\frac{[\operatorname{erf}(\mu_0)G(\eta,\Lambda) - \operatorname{erf}(\eta)G(\mu_0,\Lambda)]}{G(\mu_0,\Lambda) + (h_0^*\sqrt{\pi}/2)\operatorname{erf}(\mu_0)}, \quad 0 < \eta < \mu_0, \tag{4.7}$$

where $G(x,\Lambda)$ is given by (3.9). Taking into account the condition (4.5), the unknown parameter $\mu_0 = \mu_0(\Lambda, l, c, h_0^*)$ must be the solution of the following equation:

$$\Lambda \operatorname{erf}(x) f_1(x) + \frac{2}{\mathrm{Ste}}\operatorname{erf}(x)x = \frac{2}{\sqrt{\pi}}G(x,\Lambda)\left[\exp\left(-x^2\right) - \frac{2}{h_0^*\mathrm{Ste}}x\right], \quad x > 0, \tag{4.8}$$

which is equivalent to

$$H_2^*(x) = H_3^*(x), \quad x > 0, \tag{4.9}$$

where

$$\begin{aligned} H_2^*(x) &= H_2(x) + \frac{2}{\mathrm{Ste}}\operatorname{erf}(x)x, \quad x > 0, \\ H_3^*(x) &= \frac{2}{\sqrt{\pi}}G(x,\Lambda)\left[\exp\left(-x^2\right) - \frac{2}{h_0^*\mathrm{Ste}}x\right], \quad x > 0, \end{aligned} \tag{4.10}$$

and the function H_2 is defined by (3.12).

Similarly to the previous cases, we can enunciate the following result.

Theorem 4.1. (a) *For each* $\Lambda < 2$ ($\lambda_0 < \rho c/2\gamma$), *the free boundary problem* (4.1), *where* F *is defined by* (1.9), *has a unique similarity solution given by*

$$\begin{aligned} u(x,t,\lambda_0) = \frac{-h_0 a f\sqrt{\pi}}{k}\Bigg[&\frac{\operatorname{erf}\left(x/2a\sqrt{t}\right)G(\mu_0(\lambda_0), 4\gamma\lambda_0/\rho c)}{(h_0 a f\sqrt{\pi}/k)\operatorname{erf}(\mu_0(\lambda_0)) + G(\mu_0(\lambda_0), 4\gamma\lambda_0/\rho c)} \\ &- \frac{\operatorname{erf}(\mu_0(\lambda_0))G\left(x/2a\sqrt{t}, 4\gamma\lambda_0/\rho c\right)}{(h_0 a f\sqrt{\pi}/k)\operatorname{erf}(\mu_0(\lambda_0)) + G(\mu_0(\lambda_0), 4\gamma\lambda_0/\rho c)}\Bigg], \\ &0 < \frac{x}{2a\sqrt{t}} < \mu_0(\lambda_0), \quad t > 0, \end{aligned} \tag{4.11}$$

$$s(t,\lambda_0) = 2a\mu_0(\lambda_0)\sqrt{t}, \quad t > 0,$$

where $\mu_0 = \mu_0(\lambda_0)$ *is the unique solution of* (4.9).

(b) *Let* $M(x) = \Lambda f_1(x)$ *and* $N(x) = 2xG(x, \Lambda)$ *be, there exists a unique solution* $x^* > 0$ *of the equation* $M(x) = N(x)$.

For each $\Lambda > 2(\lambda_0 > \rho c/2\gamma)$ *such that* $M(\alpha(\Lambda)) - N(\alpha(\Lambda)) < 2/h_0^* Ste$*, where* $0 < \alpha(\Lambda) < x^*$ *satisfies* $M'(\alpha(\Lambda)) - N'(\alpha(\Lambda)) = 0$*,there exists a unique similarity solution to the free boundary problem* (3.1)*, where* F *is defined by* (1.9)*. The solution is given by* (4.11)*.*

Acknowledgments

This paper has been partially sponsored by Projects PIP no. 0460 from CONICET-UA (Rosario, Argentina) and Fondo de Ayuda a la Investigacion de la Universidad Austral (Argentina). The authors would like to thank the three referees for their constructive comments which improved the readability of the paper.

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