Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 680572, 12 pages
doi:10.1155/2010/680572

Research Article

On the Blow-Up Set for Non-Newtonian Equation with a Nonlinear Boundary Condition

Zhilei Liang

School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

Correspondence should be addressed to Zhilei Liang, zhilei0592@gmail.com

Received 9 June 2010; Accepted 15 September 2010

Academic Editor: Nicholas Alikakos

We identify the blow-up set of solutions to the problem $u_t = (|u_x|^{p-2}u_x)_x$, $x > 0$, $t \in (0,T)$, $-|u_x|^{p-2}u_x(0,t) = u^{p-1}(0,t)$, $t \in (0,T)$, and $u(x,0) = u_0(x) \geq 0$, $x > 0$, where $p > 2$. We obtain that the blow up set $B(u)$ satisfies $[0, p(p-1)/(p-2)) \subset B(u) \subset [0, p(p-1)/(p-2)]$. The proof is based on the analysis of the asymptotic behavior of self-similar representation and on the comparison methods.

1. Introduction

Consider a one-dimensional process of diffusion in a medium that occupies the half space $\{x : x \geq 0\}$; that is,

$$\begin{aligned} u_t &= \left(|u_x|^{p-2}u_x\right)_x, \quad x > 0,\ t \in (0,T), \\ -|u_x|^{p-2}u_x(0,t) &= u^q(0,t), \quad t \in (0,T), \\ u(x,0) &= u_0(x) \geq 0, \quad x > 0, \end{aligned} \tag{1.1}$$

where $p > 2$, $q > 0$ and $u_0(x)$ is an appropriately smooth function with some compatibility conditions. Problem (1.1) describes the non-Newtonian fluid with a power dependence of the tangential stress on the velocity of the displacement under nonlinear condition. It has many applications and has been intensively studied; see [1–3] and the references cited therein. For the local in time existence, we refer to [4]. Also it is known that (1.1) has no classical solution in general due to the possible degeneration at $u_x = 0$. So we usually understand the weak solution defined in the following sense.

Definition 1.1. A nonnegative function $u \in C([0,+\infty) \times [0,t))$ with $u_x \in L^\infty([0,+\infty) \times [0,t))$ is said to be a weak solution of (1.1), if the integral identity

$$\int_0^t \int_0^{+\infty} \left(u\phi_t - |u_x|^{p-2} u_x \phi_x \right) dx\, ds + \int_0^{+\infty} u_0(x)\phi(x,0)dx + \int_0^t u^q(0,s)\phi(0,s)ds = 0 \tag{1.2}$$

is fulfilled for all $\phi \in C_0^\infty([0,+\infty) \times [0,t))$.

An interesting phenomenon is that, due to the boundary effect, the solution of (1.1) may exist for $t \in [0,T)$ and becomes unbounded as $t \to T$ for some $T < \infty$. Namely, the solutionoccurs blow-up phenomenon. In this connection, Galaktionov and Levine proved in [5] that the solutions are global in time when $0 < q \leq 2(p-1)/p$ but occur blow-up for the range $2(p-1)/p < q \leq 2(p-1)$, while for $q > 2(p-1)$ blow-up happens or not depending on the size of the initial data. The main concern in this work is on the set of points at which solutions becomes unbounded, that is, the blow-up set, which is defined as

$$B(u) = \{x : \exists\, x_n \longrightarrow x,\ t_n \to T-,\ \text{such that } u(x_n,t_n) \longrightarrow \infty \ \text{ as } \ n \longrightarrow \infty\}. \tag{1.3}$$

A problem which has attracted a lot of attention in the literature is the identification of possible blow-up sets. Numerical analysis hints that the blow-up set should be a single point (single point blow-up) when $q > p-1$, a proper subset of the spatial domain (regional blow-up) when $q = p-1$, and the whole half line (global blow-up) when $0 < q < p-1$. In fact, based on the Gilding and Herrero's work in [6], Quirós and Rossi considered the porous medium type equation

$$\begin{gathered} u_t = (u^m)_{xx}, \quad x > 0,\ t \in (0,T), \\ -(u^m)_x(0,t) = u^q(0,t), \quad t \in (0,T), \\ u(x,0) = u_0(x) \geq 0, \quad x > 0. \end{gathered} \tag{1.4}$$

They proved in [1]: the blow-up set $B(u) = \{0\}$ if $q > m$ but $B(u) = [0,+\infty)$ if $q < m$, however, the blow-up set is regional in case of $q = m$, namely, $0 < B(u) < +\infty$. Afterwards, Cortázar et al. given a detailed description on the regional blow-up set. They proved in [7] that if $q = m$, then the blow-up set satisfies $[0, 2m/m-1) \subset B(u) \subset [0, 2m/m-1]$.

In the light of previous works, we discuss the blow-up set of solutions for the p-Laplacian equation (1.1). In the current paper, we identify the set of blow-up points in case of $q = p-1$. So, in the following we consider

$$\begin{gathered} u_t = \left(|u_x|^{p-2} u_x \right)_x, \quad x > 0,\ t \in (0,T), \\ -|u_x|^{p-2} u_x(0,t) = u^{p-1}(0,t), \quad t \in (0,T), \\ u(x,0) = u_0(x) \geq 0, \quad x > 0, \end{gathered} \tag{1.5}$$

where $p > 2$. We take $u_0(x)$ to be a C^1, nonincreasing and compactly supported function with some compatibility conditions. The below theorem is our main result.

Theorem 1.2. *Assume that* $u_0(x) \in C^1([0,+\infty))$ *is a nonnegative, nonincreasing and compactly supported function, then all the nontrivial solutions* $u(x,t)$ *of problem* (1.5) *occur blow-up; moreover, the blow-up set* $B(u)$ *satisfies*

$$\left[0, \frac{p(p-1)}{p-2}\right) \subset B(u) \subset \left[0, \frac{p(p-1)}{p-2}\right]. \tag{1.6}$$

Remark 1.3. The nonincreasing assumption on u_0 makes the proof much simpler (see also [7]).

Remark 1.4. One could expect that if the solutions of (1.1) occur blow-up, then the blow-up set $B(u) = [0,+\infty)$ if $0 < q < p-1$, but $B(u) = \{0\}$ if $q > p-1$.

2. Proof of Theorem 1.2

In order to study the solution u of (1.5) near the blow-up time T, as in [8] we introduce the rescaled function

$$v(x,\tau) = (T-t)^{1/(p-2)} u(x,t), \tag{2.1}$$

where $\tau = -\ln((T-t)/T) \in [0,+\infty)$. Hence, we obtain the below equations in terms of v by substituting (2.1) into (1.5),

$$\begin{gathered} \frac{\partial v}{\partial \tau} = \left(|v_x|^{p-2} v_x\right)_x - \frac{1}{p-2} v, \quad x > 0,\ \tau > 0, \\ -|v_x|^{p-2} v_x(0,\tau) = v^{p-1}(0,\tau), \quad \tau > 0, \\ v(x,0) = T^{1/(p-2)} u_0(x) \geq 0, \quad x > 0. \end{gathered} \tag{2.2}$$

If v is τ independent, then u is called a self-similar solution. Let $w(x) = (T-t)^{1/(p-2)} u(x,t)$, we have

$$\begin{gathered} \left(|w_x|^{p-2} w_x\right)_x - \frac{1}{p-2} w = 0, \quad x > 0, \\ -|w_x|^{p-2} w_x(0) = w^{p-1}(0). \end{gathered} \tag{2.3}$$

A direct integration shows that the non-negative solution of (2.3) is $w \equiv 0$ or

$$w(x) = \left(\frac{p-2}{p}\right)^{(p-1)/(p-2)} \left(\frac{1}{2(p-1)}\right)^{1/(p-2)} \left(\frac{p(p-1)}{p-2} - x\right)_+^{p/(p-2)}, \tag{2.4}$$

where $a_+ = \max\{0, a\}$.

Theorem 1.2 is a direct consequence of the following two propositions.

Proposition 2.1. *Let $v(x,\tau)$ be a solution of* (2.2), *then there exists a function $w(x)$, such that the limit holds uniformly on* $[0,+\infty)$,

$$v(\cdot,\tau) \longrightarrow w(\cdot), \quad \text{as } \tau \longrightarrow +\infty. \tag{2.5}$$

Moreover, $w(x)$ has the explicit form (2.4) *and solves problem* (2.3).

Proposition 2.1 and the formula (2.1) generate

$$\left[0, \frac{p(p-1)}{p-2}\right) \subset B(u). \tag{2.6}$$

Proposition 2.2. *Suppose that the constants a, b satisfy $p(p-1)/(p-2) < a < b < +\infty$, then there exists a large time point τ_0 such that for all $\tau > \tau_0$*

$$v(x,\tau) \leq Ce^{-\tau/(p-2)}, \quad x \in [a,b], \tag{2.7}$$

where constant C depends on a, b and u_0.

Proposition 2.2, along with (2.1), shows that $u(x,t)$ is uniformly bounded for all $t \in (0,T)$ if $x > p(p-1)/p-2$. This claims

$$B(u) \subset \left[0, \frac{p(p-1)}{p-2}\right]. \tag{2.8}$$

Hence, Theorem 1.2 immediately follows from (2.6) and (2.8). So the remaining task in this paper is to prove the validity of Propositions 2.1 and 2.2. This will be discussed in Section 3.

3. Proof of Propositions 2.1 and 2.2

In this section we prove Propositions 2.1 and 2.2. The argument contains several lemmas.

Lemma 3.1. *All the nontrivial solutions of* (1.5) *occur blow-up.*

The proof is available in [5], (see also [3]).

Lemma 3.2. *If the initial function u_0 is nonincreasing, then so does the solution $u(x,t)$ for all existent times, that is, $u_x \leq 0$, a.e. $(x,t) \in (0,\infty) \times (0,T)$.*

Proof. Consider the regularized problem

$$\frac{\partial u}{\partial t}=\left(\left(u_x^2+\epsilon\right)^{(p-2)/2}u_x\right)_x,\quad x>0,\ t\in(0,T),$$
$$-\left(u_x^2+\epsilon\right)^{(p-2)/2}u_x(0,t)=u^{p-1}(0,t),\quad t\in(0,T),\tag{3.1}$$
$$u(x,0)=u_{0\epsilon}(x)\ge 0,\quad x>0,$$

where $\epsilon>0, u_{0\epsilon}(x)\in C_0^\infty([0,+\infty))$ is nonincreasing and converges to $u_0(x)$ uniformly as $\epsilon\to 0+$. By uniqueness, the solution of (1.5) should be the limit function of that of (3.1). Setting $z(x,t)=(u_x^2+\epsilon)^{(p-2)/2}u_x$ and then differentiating the equation in t to obtain

$$\begin{aligned} z_t&=\left(u_x^2+\epsilon\right)^{(p-2)/2}u_{xt}+(p-2)\left(u_x^2+\epsilon\right)^{(p-2)/2-1}u_x^2u_{xt}\\ &=\left[\left(u_x^2+\epsilon\right)^{(p-2)/2}+(p-2)\left(u_x^2+\epsilon\right)^{(p-2)/2-1}u_x^2\right]z_{xx}.\end{aligned}\tag{3.2}$$

Clearly, (3.2) is a linear parabolic equation with respect to z. On account of the boundary conditions $z(0,t)=-u(0,t)^{p-1}\le 0$ and $z(x,0)=(u_{0\epsilon x}^2+\epsilon)^{(p-2)/2}u_{0\epsilon x}\le 0$, we deduce $z(x,t)\le 0$ via comparison theorem. The proof is completed. □

Remark 3.3. Using the maximum principle for problem (3.1), we obtain

$$\max_{x\ge 0,0\le t\le T}\left\{-\left(u_x^2+\epsilon\right)^{(p-2)/2}u_x\right\}=\max\left\{u^{p-1}(0,t),\ -\left(u_{0\epsilon x}^2+\epsilon\right)^{(p-2)/2}u_{0\epsilon x}\right\}.\tag{3.3}$$

By sending $\epsilon\to 0+$, it follows that

$$\max_{x\ge 0,0\le t\le T}\left\{-|u_x|^{p-2}u_x\right\}=\max\left\{u^{p-1}(0,t),\ -|u_{0x}|^{p-2}u_{0x}\right\}.\tag{3.4}$$

Lemma 3.4. *There exists a constant $C=C(p,u_0)$ such that*

$$\int_0^t u^{p-1}(0,s)ds\le C(T-t)^{-1/(p-2)},\quad \forall t\in[0,T).\tag{3.5}$$

Proof. Integrating $(1.5)_1$, the first equation of (1.5), over $[0,+\infty)\times[0,t)$ yields

$$\int_0^\infty u(x,t)dx-\int_0^\infty u_0dx=\int_0^t u^{p-1}(0,s)ds.\tag{3.6}$$

Now multiplying $(1.5)_1$ by x and integrating the resulting expression once more, we have

$$\begin{aligned}\int_0^\infty xu(x,t)dx - \int_0^\infty xu_0dx &= \int_0^t\int_0^\infty x\left(|u_x|^{p-2}u_x\right)_x dx\,ds\\ &= \int_0^t\int_0^\infty \left(x|u_x|^{p-2}u_x\right)_x dx\,ds - \int_0^t\int_0^\infty |u_x|^{p-2}u_x dx\,ds \\ &= -\int_0^t\int_0^\infty |u_x|^{p-2}u_x dx\,ds.\end{aligned} \tag{3.7}$$

Thanks to Remark 3.3, we estimate

$$\begin{aligned}-\int_0^t\int_0^\infty |u_x|^{p-2}u_x dx\,ds &\le \int_0^t |u_x(0,s)|^{p-2}\int_0^\infty (-u_x)dx\,ds + \int_0^t\int_0^\infty |u_{0x}|^{p-1}\\ &= \int_0^t |u_x(0,s)|^{p-2}u(0,s)ds + C(T,u_{0x})\\ &= \int_0^t u^{p-1}(0,s)ds + C(T,u_{0x}),\end{aligned} \tag{3.8}$$

the last sign comes from the boundary condition. Thus, (3.7) becomes

$$\int_0^\infty xu(x,t)dx - \int_0^\infty xu_0dx \le \int_0^t u^{p-1}(0,s)ds + C(T,u_{0x}). \tag{3.9}$$

Multiplying (3.6) by a constant $K > 1$ and then subtracting (3.9) produces

$$\int_0^\infty (K-x)u(x,t)dx - \int_0^\infty (K-x)u_0dx \ge (K-1)\int_0^t u^{p-1}(0,s)ds - C(T,u_{0x}). \tag{3.10}$$

If we choose $K > 1$ so large such that $\operatorname{supp} u_0 \subset [0,K]$, then

$$\int_0^\infty (K-x)u(x,t)dx - \int_0^\infty (K-x)u_0dx \le \int_0^K (K-x)u(x,t)dx \le K^2u(0,t). \tag{3.11}$$

Inequalities (3.10) and (3.11) lead to

$$K^2u(0,t) \ge (K-1)\int_0^t u^{p-1}(0,s)ds - C(T,u_{0x}). \tag{3.12}$$

Putting $F(t) = \int_0^t u^{p-1}(0,s)ds$ and integrating above inequality over (t,T) brings

$$\begin{aligned}F(t) &\le C(T-t)^{-1/(p-2)} + C(T,u_{0x})\\ &\le C_1(T-t)^{-1/(p-2)}, \quad (t \text{ close to } T).\end{aligned} \tag{3.13}$$

Because u is bounded when time t is away from the blow-up time T, (3.13) and thus (3.5) is valid for all $t \in [0, T)$. □

The next lemma proves that $u(x,t)$ is localized. Namely, the support of $u(\cdot,t)$ is uniformly bounded for all $t \in ([0,T)$.

Lemma 3.5. *There exists a constant M depending on p and u_0 such that if $x > M$, then*

$$u(x,t) \equiv 0, \qquad \forall t \in [0,T). \tag{3.14}$$

Proof. Let us introduce the function

$$q(x,t) = D(T-t)^{-1/(p-2)}(L-x)_+^{p/(p-2)}, \tag{3.15}$$

where D satisfies where $D^{p-2} = (1/2(p-1))(p-2/p)^{p-1}$ and L is an arbitrary positive constant. It is easy to check that $q(x,t)$ satisfies $(1.5)_1$. By differentiating (3.5), we have $u(0,t) \le C(T-t)^{-1/p-2}$. This, together with the monotonicity of u, deduces

$$u(x,t) \le C(T-t)^{-1/(p-2)}, \qquad \forall x \ge 0,\ t \in (0,T). \tag{3.16}$$

Because the initial support is bounded, we choose a suitably large point $x_0 < +\infty$ to guarantee $\operatorname{supp} u_0 \subset [0, x_0]$. Setting $\tilde{u}(x,t) = u(x+x_0,t)$, it has

$$\tilde{u}(x,0) = u(x+x_0,0) \equiv 0 \le q(x,0), \qquad \forall x > 0. \tag{3.17}$$

By virtue of (3.16), there exists some large constant L_0 such that for all $L > L_0$, it holds

$$\begin{aligned} \tilde{u}(0,t) &= u(x_0,t) \\ &\le C(T-t)^{-1/(p-2)} \\ &\le DL^{p/(p-1)}(T-t)^{-1/(p-2)} = q(0,t), \quad \forall t \in (0,T). \end{aligned} \tag{3.18}$$

So comparison theorem concludes

$$\tilde{u}(x,t) \le q(x,t), \qquad \forall x \ge 0,\ t \in (0,T). \tag{3.19}$$

Hence, for every constant L satisfying $L > L_0$, (3.19) ensures $\operatorname{supp} \tilde{u}(\cdot,t) \subset [0,L]$. Thus $\operatorname{supp} u(\cdot,t) \subset [0, x_0 + L]$. Denoted by $M = x_0 + L$, it follows that for all $x > M$,

$$u(x,t) \equiv 0, \quad \forall t \in (0,T). \tag{3.20}$$

This completes the proof. □

Corollary 3.6. *Lemma 3.5 and* (2.1) *claim that $v(x,\tau)$ is localized as well. Moreover,* (2.1) *and* (3.16) *guarantee the boundedness of $v(x,\tau)$ for all $x \geq 0$ and $\tau \geq 0$. By a similar argument as that in Lemma 3.2, one can prove that $|v_x|^{p-2}v_x$ also is bounded.*

In what follows, we analyze the large time behavior of the solution $v(x,\tau)$ of problem (2.2).

Multiplying (2.2) by v and then integrating by parts, we deduce that

$$\int_0^\infty \int_{\tau_1}^{\tau_2} v_\tau^2 dx\, ds + J[v](\tau_2) \leq J[v](\tau_1), \quad \text{a.e.,} \;\; 0 \leq \tau_1 < \tau_2 < \infty, \tag{3.21}$$

where $J[v](\tau)$ is the Liapunov functional and

$$J[v](\tau) = \frac{1}{p}\int_0^\infty |v_x|^p dx - \frac{1}{p}\int_0^\infty v^p(0,\tau) + \frac{1}{2(p-2)}\int_0^\infty v^2(x,\tau)dx. \tag{3.22}$$

Based on the estimates

$$v, v_x \in L^\infty([0,+\infty) \times [0,+\infty)), \quad v_\tau \in L^2([0,+\infty) \times [0,+\infty)), \tag{3.23}$$

the ω-limit set

$$\omega(v_0) = \left\{ w(x) : \exists \tau_n \longrightarrow \infty, \text{ s.t. } v(\cdot,\tau_n) \longrightarrow w(x) \text{ in } L^2 \right\} \tag{3.24}$$

consists of solutions of the problem (2.3). Indeed, using the estimates (3.23) in passing to the limit in (2.2), we have that, given a monotone sequence $\tau_n \rightarrow \infty, v(\cdot,\tau_n+s) \rightarrow w(x;s)$ *in* L^2, where $w(x;s)$ depending on s solves (2.3) in a weak sense, and $w(x;0) \in \omega(v_0)$. It follows from (3.23) that uniformly in $s \in [0,1]$

$$\|v(\cdot,\tau_n+s) - v(\cdot,\tau_n)\| \leq \int_{\tau_n}^{\tau_n+s} \|v_t\|_{L^2}^2 d\tau \longrightarrow 0 \quad (\tau_n \longrightarrow \infty). \tag{3.25}$$

This means that the limit function w does not rely on s and is a weak solution for the (2.3). Finally, the independence of the choice of the sequence $\tau_n \rightarrow \infty$ follows from the nonincreasing of $J[v](\tau)$ in time. Furthermore, the convergence is uniform in x due to the boundness of v_x. In conclusion we arrive to the following lemma.

Lemma 3.7. *The limit holds as $\tau \rightarrow \infty$*

$$v(x,\tau) \longrightarrow w(x), \quad \textit{uniformly for } x \geq 0, \tag{3.26}$$

where $w(x)$ satisfies the problem (2.3).

To finish Proposition 2.1, it remains to confirm that $w(x)$ is not null and thus to be of the form (2.4). The proof will be given at the end of the paper. Now let us turn to Proposition 2.2.

Consider the initial-boundary-value problem

$$\begin{aligned}
&h_\tau = \left(|h_x|^{p-2}h_x\right)_x - \frac{1}{p-2}h, \quad x \in (a-\delta, b+\delta),\ \tau > 0,\\
&h(a-\delta,\tau) = h(b+\delta,\tau) = \epsilon, \quad \tau > 0,\\
&h(x,0) = \epsilon, \quad x \in (a-\delta, b+\delta),
\end{aligned} \tag{3.27}$$

where $a,\ b,\ \epsilon,\ \delta$ are positive constants and $\epsilon,\ \delta$ are suitably small. We have the following.

Lemma 3.8. *Let $D\delta^{p/(p-2)} = \epsilon$ suitably small with D that satisfies $D^{p-2} = (1/2(p-1))((p-1)/p)^{p-1}$. Then there exists a constant $C > 0$ such that for all $\tau \geq 0$,*

$$h(x,\tau) \leq Ce^{-\tau/(p-2)}, \quad \forall x \in [a,b]. \tag{3.28}$$

Proof. It is easy to check that the function of the form

$$j(x) = \begin{cases} D(a-x)^{p/(p-2)}, & x \in \left[a-\delta, a-\dfrac{\delta}{2}\right),\\ 0, & x \in \left[a-\dfrac{\delta}{2}, b+\dfrac{\delta}{2}\right],\\ D(x-b)^{p/(p-2)}, & x \in \left(b+\dfrac{\delta}{2}, b+\delta\right]. \end{cases} \tag{3.29}$$

solves

$$\begin{aligned}
&0 = \left(\left|j_x\right|^{p-2}j_x\right)_x - \frac{1}{p-2}j, \quad x \in (a-\delta, b+\delta),\\
&j(a-\delta) = j(b+\delta) = D\delta^{p/(p-2)}.
\end{aligned} \tag{3.30}$$

Moreover, it has $j(x) \leq D\delta^{p/(p-2)}$ since $D\delta^{p/(p-2)}$ is a super solution to problem (3.30). Choosing $D\delta^{p/(p-2)} = \epsilon$ and subtracting (3.30) from (3.27), we obtain

$$\begin{aligned}
&(h-j)_\tau = \left(|h_x|^{p-2}h_x - \left|j_x\right|^{p-2}j_x\right)_x - \frac{1}{p-2}(h-j), \quad x \in (a-\delta, b+\delta),\ \tau > 0,\\
&h(a-\delta,\tau) - j(a-\delta) = h(b+\delta,\tau) - j(b+\delta) = 0, \quad \tau > 0,\\
&h(x,0) - j(x) \geq 0, \quad x \in (a-\delta, b+\delta).
\end{aligned} \tag{3.31}$$

Integrating (3.31) over $(a-\delta, b+\delta) \times (0,\tau)$ gives rise to

$$\begin{aligned}\int_{a-\delta}^{b+\delta}(h-j)(x,\tau)dx = &\int_{a-\delta}^{b+\delta}(h-j)(x,0)dx \\ &+\int_0^\tau\Big(|h_x|^{p-2}h_x-\left|j_x\right|^{p-2}j_x\Big)(b+\delta,s)ds \\ &-\int_0^\tau\Big(|h_x|^{p-2}h_x-\left|j_x\right|^{p-2}j_x\Big)(a-\delta,s)ds \\ &-\frac{1}{p-2}\int_0^\tau\int_{a-\delta}^{b+\delta}(h-j)(x,s)dx\,ds.\end{aligned} \tag{3.32}$$

Noting that $(h-j)(x,\tau)$ is symmetric with respect to $(a+b)/2$ and nonincreasing on $[a+b/2, b+\delta]$, thus we have for all $\tau > 0$,

$$\operatorname{sign}\Big\{\Big(|h_x|^{p-2}h_x-\left|j_x\right|^{p-2}j_x\Big)(b+\delta,\tau)\Big\} = \operatorname{sign}\{(h_x-j_x)(b+\delta,\tau)\} \le 0, \tag{3.33}$$

where sign is the sign function. Then,

$$\int_0^\tau\Big(|h_x|^{p-2}h_x-\left|j_x\right|^{p-2}j_x\Big)(b+\delta,s)\ ds \le 0. \tag{3.34}$$

Similarly,

$$\int_0^\tau\Big(|h_x|^{p-2}h_x-\left|j_x\right|^{p-2}j_x\Big)(a-\delta,s)ds \ge 0. \tag{3.35}$$

Therefore, (3.32) becomes

$$\int_{a-\delta}^{b+\delta}(h-j)(x,\tau)dx \le \int_{a-\delta}^{b+\delta}(h-j)(x,0)dx - \frac{1}{p-2}\int_0^\tau\int_{a-\delta}^{b+\delta}(h-j)(x,s)dx\,ds. \tag{3.36}$$

Applying the Gronwall inequality yields

$$\int_{a-\delta}^{b+\delta}(h-j)(x,\tau)dx \le Ce^{-(1/(p-2))\tau}. \tag{3.37}$$

On the other hand, since h is symmetric and nondecreasing on $[(a+b)/2, b+\delta]$ with x-variable, for all $x \in [a,b]$ it has

$$\begin{aligned} h(x,\tau) &\leq \frac{1}{\delta}\int_{a-\delta/2}^{b+\delta/2} h(y,\tau)dy = \frac{1}{\delta}\int_{a-\delta/2}^{b+\delta/2}(h-j)(y,\tau)dy \\ &\leq \frac{1}{\delta}\int_{a-\delta}^{b+\delta}(h-j)(x,\tau)dx \leq C_1 e^{-(1/(p-2))\tau}, \end{aligned} \tag{3.38}$$

where we have used the fact supp $j(\cdot) \subset [a-\delta, a-\delta/2] \cup [b+\delta/2, b+\delta]$. That is,

$$h(x,\tau) \leq C_1 e^{-(1/(p-2))\tau}. \tag{3.39}$$

This completes Lemma 3.8. □

We are now in a position to prove Proposition 2.2. From Lemma 3.7 we observe that v tends to the limit function $w(x)$ which is none for all points $x > p(p-1)/(p-2)$. Taking $a > p(p-1)/(p-2)$ and a small constant $\delta > 0$ to satisfy $a - \delta > p(p-1)/(p-2)$. Then, for a given $\epsilon > 0$, the inequality $v(x,\tau) \leq \epsilon$ for all $x \geq a-\delta$ holds provided $\tau > \tau_0$, where τ_0 is a large point which depends on ϵ. Using comparison theorem, one has

$$v(x, \tau+\tau_0) \leq h(x,\tau), \quad \text{for } \tau > \tau_0,\ x > a-\delta. \tag{3.40}$$

The proof ends up with (3.40) and Lemma 3.8.

Finally, We prove that the function $w(x)$ appeared in Lemma 3.7 takes the form (2.4). For this purpose, it is enough to illuminate that $w(x) \equiv 0$ is impossible. We argue by contradiction. Assuming that $w(x) \equiv 0$ for a moment. Then $u(x,t)$ blows up only at the boundary point $x = 0$. By this we can find a sequence $t_n \to \infty$ as $n \to \infty$ such that $u(0,t_n) \to \infty$ as $n \to \infty$. Moreover, By the maximin principle (see Remark 3.3), one has

$$\max_{x\geq 0}\left\{-|u_x|^{p-2}u_x(x,t_n)\right\} = \max\left\{-|u_{0x}|^{p-2}u_{0x},\ u^{p-1}(0,t_n)\right\} = u^{p-1}(0,t_n), \tag{3.41}$$

as long as n is chosen large enough. Recalling that u is nonincreasing with respect to x-variable, from (3.41) we conclude

$$-u_x(x,t_n) \leq u(0,t_n), \quad \forall x \geq 0. \tag{3.42}$$

Integrating (3.42) gives rise to

$$(1-x)u(0,t_n) \leq u(x,t_n). \tag{3.43}$$

This shows that u occurs blow-up at least in the interval $[0,1)$, which contradicts with the assumption that blow-up happens only at $x = 0$.

Now the proof of Theorem 1.2 is completed.

References

[1] F. Quirós and J. D. Rossi, "Blow-up sets and Fujita type curves for a degenerate parabolic system with nonlinear boundary conditions," *Indiana University Mathematics Journal*, vol. 50, no. 1, pp. 629–654, 2001.

[2] A. A. Samarskii, V.A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-Up in Quasilinear Parabolic Equations*, vol. 19 of *de Gruyter Expositions in Mathematics*, Walter de Gruyter, Berlin, Germany, 1995.

[3] Z. Wang, J. Yin, and C. Wang, "Critical exponents of the non-Newtonian polytropic filtration equation with nonlinear boundary condition," *Applied Mathematics Letters*, vol. 20, no. 2, pp. 142–147, 2007.

[4] Z. Wu, J. Zhao, J. Yin, and H. Li, *Nonlinear Diffusion Equations*, World Scientific Publishing, River Edge, NJ, USA, 2001.

[5] V. A. Galaktionov and H. A. Levine, "On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary," *Israel Journal of Mathematics*, vol. 94, pp. 125–146, 1996.

[6] B. H. Gilding and M. A. Herrero, "Localization and blow-up of thermal waves in nonlinear heat conduction with peaking," *Mathematische Annalen*, vol. 282, no. 2, pp. 223–242, 1988.

[7] M. C. Cortázar, M. Elgueta, and O. Venegas, "On the blow-up set for $u_t = (u^m)_{xx}$, $m > 1$, with nonlinear boundary conditions," *Monatshefte für Mathematik*, vol. 142, no. 1-2, pp. 45–56, 2004.

[8] Z. Liang and J. Zhao, "Localization for the evolution p-Laplacian equation with strongly nonlinear source term," *Journal of Differential Equations*, vol. 246, no. 1, pp. 391–407, 2009.