Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 658397, 21 pages
doi:10.1155/2010/658397

Research Article

Multiple Positive Solutions for Semilinear Elliptic Equations in $\mathbb{R}^N$ Involving Concave-Convex Nonlinearities and Sign-Changing Weight Functions

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Received 1 November 2009; Revised 19 March 2010; Accepted 9 June 2010

Academic Editor: Norimichi Hirano

We study the existence and multiplicity of positive solutions for the following semilinear elliptic equation $-\Delta u + u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u$ in $\mathbb{R}^N$, $u \in H^1(\mathbb{R}^N)$, where $\lambda > 0$, $1 < q < 2 < p < 2^*(2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1,2)$, $a(x)$, $b(x)$ satisfy suitable conditions, and $a(x)$ may change sign in $\mathbb{R}^N$.

1. Introduction and Main Results

In this paper, we deal with the existence and multiplicity of positive solutions for the following semilinear elliptic equation:

$$\begin{gathered} -\Delta u + u = \lambda a(x)|u|^{q-2}u + b(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{gathered} \tag{$E_{\lambda a,b}$}$$

where $\lambda > 0$, $1 < q < 2 < p < 2^*$ $(2^* = (2N/(N-2))$ if $N \geq 3$, $2^* = \infty$ if $N = 1,2)$, and a, b are measurable functions and satisfy the following conditions:

(A1) $a \in C(\mathbb{R}^N) \cap L^{q^*}(\mathbb{R}^N)(q^* = p/(p-q))$ with $a^+ = \max\{a, 0\} \not\equiv 0$ in $\mathbb{R}^N$;

(B1) $b \in C(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ and $b^+ = \max\{b, 0\} \not\equiv 0$ in $\mathbb{R}^N$.

Semilinear elliptic equations with concave-convex nonlinearities in bounded domains are widely studied. For example, Ambrosetti et al. [1] considered the following equation:

$$-\Delta u = \lambda u^{q-1} + u^{p-1}, \quad \text{in } \Omega,$$
$$u > 0, \quad \text{in } \Omega, \qquad (E_\lambda)$$
$$u = 0, \quad \text{on } \partial\Omega,$$

where $\lambda > 0$, $1 < q < 2 < p < 2^*$. They proved that there exists $\lambda_0 > 0$ such that (E_λ) admits at least two positive solutions for all $\lambda \in (0, \lambda_0)$ and has one positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$. Actually, Adimurthi et al. [2], Damascelli et al. [3], Ouyang and Shi [4], and Tang [5] proved that there exists $\lambda_0 > 0$ such that (E_λ) in the unit ball $B^N(0;1)$ has exactly two positive solutions for $\lambda \in (0, \lambda_0)$ and has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. For more general results of (E_λ) (involving sign-changing weights) in bounded domains see Ambrosetti et al. [6], García Azorero et al. [7], Brown and Wu [8], Brown and Zhang [9], Cao and Zhong [10], de Figueiredo et al. [11], and their references. However, little has been done for this type of problem in $\mathbb{R}^N$. We are only aware of the works [12–16] which studied the existence of solutions for some related concave-convex elliptic problems (not involving sign-changing weights). Furthermore, we do not know of any results for concave-convex elliptic problems involving sign-changing weight functions except [17]. Wu in [17] has studied the multiplicity of positive solutions for the following equation involving sign-changing weights:

$$-\Delta u + u = a_\lambda(x) u^{q-1} + b_\mu(x) u^{p-1}, \quad \text{in } \mathbb{R}^N,$$
$$u > 0, \quad \text{in } \mathbb{R}^N, \qquad (E_{a_\lambda, b_\mu})$$
$$u \in H^1\left(\mathbb{R}^N\right),$$

where $1 < q < 2 < p < 2^*$, the parameters $\lambda, \mu \geq 0$. He also assumed that $a_\lambda(x) = \lambda a_+(x) + a_-(x)$ is sign-chaning and $b_\mu(x) = c(x) + \mu d(x)$, where c and d satisfy suitable conditions, and proved that (E_{a_λ, b_μ}) has at least four positive solutions.

The main aim of this paper is to study $(E_{\lambda a, b})$ in $\mathbb{R}^N$ involving concave-convex nonlinearities and sign-changing weight functions. We will discuss the Nehari manifold and examine carefully connection between the Nehari manifold and the fibrering maps; then using arguments similar to those used in [18], we will prove the existence of two positive solutions by using Ekeland's variational principle [19].

Set

$$\Lambda_0 = \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}} \right)^{(2-q)/(p-2)} \left(\frac{p-2}{(p-q)\|a^+\|_{L^{q^*}}} \right) S_p^{(p(2-q)/2(p-2))+(q/2)} > 0, \qquad (1.1)$$

where $\|b^+\|_{L^\infty} = \sup_{x \in \mathbb{R}^N} b^+(x)$, $\|a^+\|_{L^{q^*}} = \left(\int_{\mathbb{R}^N} |a^+(x)|^{q^*} dx\right)^{1/q^*}$, and S_p is the best Sobolev constant for the imbedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$. Now, we state the first main result about the existence of positive solution of $(E_{\lambda a, b})$ in $\mathbb{R}^N$.

Theorem 1.1. *Assume that (A1) and (B1) hold. If* $\lambda \in (0, \Lambda_0)$*, then* $(E_{\lambda a,b})$ *admits at least one positive solution in* $H^1(\mathbb{R}^N)$.

Associated with $(E_{\lambda a,b})$, we consider the energy functional $J_{\lambda a,b}$ in $H^1(\mathbb{R}^N)$:

$$J_{\lambda a,b}(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} a(x)|u|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} b(x)|u|^p dx, \tag{1.2}$$

where $\|u\|_{H^1} = (\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)dx)^{1/2}$. By [20, Proposition B.10], $J_{\lambda a,b} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. It is well known that the solutions of $(E_{\lambda a,b})$ are the critical points of the energy functional $J_{\lambda a,b}$ in $H^1(\mathbb{R}^N)$.

Under assumptions (A1), (B1), and $\lambda > 0$, $(E_{\lambda a,b})$ can be regarded as a perturbation problem of the following semilinear elliptic equation:

$$\begin{gathered} -\Delta u + u = b(x)u^{p-1}, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{gathered} \tag{E_b}$$

where $b(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $b(x) > 0$ for all $x \in \mathbb{R}^N$. We denote by S_p^b the best constant which is given by

$$S_p^b = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{H^1}^2}{\left(\int_{\mathbb{R}^N} b(x)|u|^p dx\right)^{2/p}}. \tag{1.3}$$

A typical approach for solving problem of this kind is to use the Minimax method:

$$\alpha_\Gamma^b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_0^b(\gamma(t)), \tag{1.4}$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], H^1\left(\mathbb{R}^N\right)\right) : \gamma(0) = 0, \gamma(1) = e\right\}, \tag{1.5}$$

$J_0^b(e) = 0$, and $e \neq 0$. By the Mountain Pass Lemma due to Ambrosetti and Rabinowitz [21], we called the nonzero critical point $u \in H^1(\mathbb{R}^N)$ of J_0^b a ground state solution of (E_b) in $\mathbb{R}^N$ if $J_0^b(u) = \alpha_\Gamma^b$. We remark that the ground state solutions of (E_b) in $\mathbb{R}^N$ can also be obtained by the Nehari minimization problem

$$\alpha_0^b = \inf_{v \in \mathcal{M}_0^b} J_0^b(v), \tag{1.6}$$

where $\mathcal{M}_0^b = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \|u\|_{H^1}^2 = \int_{\mathbb{R}^N} b(x)|u|^p dx\}$. Note that $\mathcal{M}_0^b$ contains every nonzero solution of (E_b) in $\mathbb{R}^N$ (see Willem [22])

$$\alpha_\Gamma^b = \alpha_0^b = \frac{p-2}{2p}\left(S_p^b\right)^{p/(p-2)} > 0. \tag{1.7}$$

When $b(x) \equiv b^\infty$ is a constant function in $\mathbb{R}^N$, the existence of ground state solutions of (E_b) in $\mathbb{R}^N$ has been established by Berestycki and Lions [23]. Actually, Kwong [24] proved that the positive solution of (E_b) in $\mathbb{R}^N$ is unique.

When $b(x) \not\equiv b^\infty$ and $b(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, it is well known that the existence of ground state solutions of (E_b) has been established by the condition $b(x) \geq b^\infty = \lim_{|x|\to\infty} b(x)$ and the existence of ground state solutions of limit equation

$$\begin{gathered} -\Delta u + u = b^\infty u^{p-1}, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right). \end{gathered} \tag{E_{b^∞}}$$

In order to get the second positive solution of $(E_{\lambda a,b})$ in $\mathbb{R}^N$, we need some additional assumptions for $a(x)$ and $b(x)$. We assume the following conditions on $a(x)$ and $b(x)$:

(B2) $b(x) > 0$ for all $x \in \mathbb{R}^N$, and $b(x)$ satisfies suitable conditions such that (E_b) in $\mathbb{R}^N$ has a positive ground state solution w_0, that is, $J_0^b(w_0) = \alpha_0^b$;

(A2) $\int_{\mathbb{R}^N} a(x)|w_0|^q dx > 0$, where w_0 is a positive ground state solution of (E_b) in $\mathbb{R}^N$.

Theorem 1.2. *Assume that (A1)-(A2) and (B1)-(B2) hold. If $\lambda \in (0, (q/2)\Lambda_0)$, $(E_{\lambda a,b})$ admits at least two positive solutions in $H^1(\mathbb{R}^N)$.*

Remark 1.3. (i) In [17, Theorem 1.1], the author has proved that if

$$\begin{gathered} b_\mu(x) = c(x) + \mu d(x), \qquad a_\lambda(x) = \lambda a_+(x) + a_-(x), \\ \lim_{|x|\to\infty} c(x) = 1, \qquad \lim_{|x|\to\infty} d(x) = 0, \\ 1 \geq c(x) \geq 1 - c_0 \exp(-r_a|x|), \quad \text{for } c_0 < 1,\ \forall x \in \mathbb{R}^N, \\ d(x) \geq d_0 \exp(-r_b|x|), \quad \text{for } d_0 > 0,\ \forall x \in \mathbb{R}^N, \\ a_-(x) \geq -\hat{c}\exp\left(-r_{f_-}|x|\right), \quad \forall x \in \mathbb{R}^N, \end{gathered} \tag{1.8}$$

where $r_b < \min\{r_a, r_{f_-}, q\}$, then for sufficiently small λ and μ,(E_{a_λ,b_μ}),

$$\begin{gathered} -\Delta u + u = a_\lambda(x)u^{q-1} + b_\mu(x)u^{p-1}, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{gathered} \tag{1.9}$$

admits at least two positive solutions in $\mathbb{R}^N$. In particular, b_μ satisfies the following condition:

$$b_\mu(x) = c(x) + \mu d(x) \geq 1 = \lim_{|x| \to \infty} b_\mu(x), \quad \text{for large } |x|. \tag{1.10}$$

(ii) According to Lions' paper, if $b(x) \geq b^\infty = \lim_{|x|\to\infty} b(x)$ for any $x \in \mathbb{R}^N$, then there is a positive ground state solution w_0 of (E_b) in $\mathbb{R}^N$. Supposing $\int_{\mathbb{R}^N} a(x) w_0^p dx = \int_{\mathbb{R}^N} [a_+(x) + a_-(x)] w_0^p dx > 0$, we can prove that for sufficiently small λ,$(E_{\lambda a,b})$,

$$\begin{gathered} -\Delta u + u = \lambda a(x) u^{q-1} + b(x) u^{p-1}, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right) \end{gathered} \tag{1.11}$$

admits at least two positive solutions in $\mathbb{R}^N$. We give an example of $a(x)$ as follows. Let $\eta_1 : \mathbb{R}^N \to [0,1]$ be a C_c^∞-function on $\mathbb{R}^N$ such that $0 \leq \eta_1 \leq 1$ and

$$\eta_1(x) = \begin{cases} 1, & \text{for } |x| \leq 1, \\ 0, & \text{for } |x| \geq 2. \end{cases} \tag{1.12}$$

Since $w_0 \in H^1(\mathbb{R}^N)$, there is a positive number $R > 2$ such that

$$\int_{\{|x| \geq R\}} w_0^p dx < \int_{\{|x| \leq 1\}} w_0^p dx. \tag{1.13}$$

Let $\eta_2 : \mathbb{R}^N \to [0,1]$ be a C^∞-function on $\mathbb{R}^N$ such that $0 \leq \eta_2 \leq 1$ and

$$\eta_2(x) = \begin{cases} 0, & \text{for } |x| \leq R, \\ 1, & \text{for } |x| \geq 2R. \end{cases} \tag{1.14}$$

Define

$$a(x) = \eta_1(x) - \eta_2(x)|x|^{-r}, \quad \text{where } r > 0, \quad N - rq^* < 0, \tag{1.15}$$

then by (1.13), we have that

$$\begin{aligned} \int_{\mathbb{R}^N} a(x) w_0^p dx &\geq \int_{\{|x| \leq 1\}} w_0^p dx - \int_{\{|x| \geq R\}} |x|^{-r} w_0^p dx \\ &\geq \int_{\{|x| \leq 1\}} w_0^p dx - R^{-r} \int_{\{|x| \geq R\}} w_0^p dx > 0. \end{aligned} \tag{1.16}$$

In this case, $a_-(x) = -\eta_2(x)|x|^{-r}$ and $b(x)$ do not satisfy the assumptions of exponential decay in [17].

Throughout this paper, (A1) and (B1) will be assumed. $H^1(\mathbb{R}^N)$ denotes the standard Sobolev space, whose norm $\|\cdot\|_{H^1}$ is induced by the standard inner product. The dual space of $H^1(\mathbb{R}^N)$ will be denoted by $H^{-1}(\mathbb{R}^N)$. $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $H^1(\mathbb{R}^N)$. We denote the norm in $L^s(\mathbb{R}^N)$ by $\|\cdot\|_{L^s}$ for $1 \leq s \leq \infty$. $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$. C, C_i will denote various positive constants, the exact values of which are not important. This paper is organized as follows. In Section 2, we give some properties of Nehari manifold. In Sections 3 and 4, we complete proofs of Theorems 1.1 and 1.2.

2. Nehari Manifold

In this section, we will give some properties of Nehari manifold. As the energy functional $J_{\lambda a,b}$ is not bounded below on $H^1(\mathbb{R}^N)$, it is useful to consider the functional on the Nehari manifold

$$\mathcal{M}_{\lambda a,b} = \left\{u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} : \langle (J_{\lambda a,b})'(u), u\rangle = 0\right\}. \tag{2.1}$$

Thus, $u \in \mathcal{M}_{\lambda a,b}$ if and only if

$$\langle (J_{\lambda a,b})'(u), u\rangle = \|u\|_{H^1}^2 - \lambda \int_{\mathbb{R}^N} a(x)|u|^q dx - \int_{\mathbb{R}^N} b(x)|u|^p dx = 0. \tag{2.2}$$

Note that $\mathcal{M}_{\lambda a,b}$ contains every nonzero solution of $(E_{\lambda a,b})$. Moreover, we have the following results.

Lemma 2.1. *The energy functional $J_{\lambda a,b}$ is coercive and bounded below on $\mathcal{M}_{\lambda a,b}$.*

Proof. If $u \in \mathcal{M}_{\lambda a,b}$, then by (A1), (2.2), and Hölder and Sobolev inequalities

$$J_{\lambda a,b}(u) = \frac{p-2}{2p}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{pq}\right)\int_{\mathbb{R}^N} a(x)|u|^q dx \tag{2.3}$$

$$\geq \frac{p-2}{2p}\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{pq}\right)S_p^{-(q/2)}\|a^+\|_{L^{q^*}}\|u\|_{H^1}^q. \tag{2.4}$$

Thus, $J_{\lambda a,b}$ is coercive and bounded below on $\mathcal{M}_{\lambda a,b}$. □

The Nehari manifold is closely linked to the behavior of the function of the form $\varphi_u : t \to J_{\lambda a,b}(tu)$ for $t > 0$. Such maps are known as fibering maps and were introduced by Drábek and Pohozaev in [25] and are also discussed in [9]. If $u \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned}\varphi_u(t) &= \frac{t^2}{2}\|u\|_{H^1}^2 - \frac{t^q}{q}\lambda\int_{\mathbb{R}^N} a(x)|u|^q dx - \frac{t^p}{p}\int_{\mathbb{R}^N} b(x)|u|^p dx;\\ \varphi_u'(t) &= t\|u\|_{H^1}^2 - t^{q-1}\lambda\int_{\mathbb{R}^N} a(x)|u|^q dx - t^{p-1}\int_{\mathbb{R}^N} b(x)|u|^p dx;\\ \varphi_u''(t) &= \|u\|_{H^1}^2 - (q-1)t^{q-2}\lambda\int_{\mathbb{R}^N} a(x)|u|^q dx - (p-1)t^{p-2}\int_{\mathbb{R}^N} b(x)|u|^p dx.\end{aligned} \tag{2.5}$$

It is easy to see that

$$t\varphi_u'(t) = \|tu\|_{H^1}^2 - \lambda\int_{\mathbb{R}^N} a(x)|tu|^q dx - \int_{\mathbb{R}^N} b(x)|tu|^p dx, \tag{2.6}$$

and so, for $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$, $\varphi_u'(t) = 0$ if and only if $tu \in \mathcal{M}_{\lambda a,b}$, that is, the critical points of φ_u correspond to the points on the Nehari manifold. In particular, $\varphi_u'(1) = 0$ if and only if $u \in \mathcal{M}_{\lambda a,b}$. Thus, it is natural to split $\mathcal{M}_{\lambda a,b}$ into three parts corresponding to local minima, local maxima and points of inflection. Accordingly, we define

$$\begin{aligned}\mathcal{M}_{\lambda a,b}^{+} &= \{u \in \mathcal{M}_{\lambda a,b} : \varphi_u''(1) > 0\},\\ \mathcal{M}_{\lambda a,b}^{0} &= \{u \in \mathcal{M}_{\lambda a,b} : \varphi_u''(1) = 0\},\\ \mathcal{M}_{\lambda a,b}^{-} &= \{u \in \mathcal{M}_{\lambda a,b} : \varphi_u''(1) < 0\}\end{aligned} \tag{2.7}$$

and note that if $u \in \mathcal{M}_{\lambda a,b}$, that is, $\varphi_u'(1) = 0$, then

$$\varphi_u''(1) = (2-p)\|u\|_{H^1}^2 - (q-p)\lambda\int_{\mathbb{R}^N} a(x)|u|^q dx \tag{2.8}$$

$$= (2-q)\|u\|_{H^1}^2 - (p-q)\int_{\mathbb{R}^N} b(x)|u|^p dx. \tag{2.9}$$

We now derive some basic properties of $\mathcal{M}_{\lambda a,b}^{+}$, $\mathcal{M}_{\lambda a,b}^{0}$, and $\mathcal{M}_{\lambda a,b}^{-}$.

Lemma 2.2. *Assume that u_λ is a local minimizer for $J_{\lambda a,b}$ on $\mathcal{M}_{\lambda a,b}$ and $u_\lambda \notin \mathcal{M}_{\lambda a,b}^{0}$. Then $(J_{\lambda a,b})'(u_\lambda) = 0$ in $H^{-1}(\mathbb{R}^N)$.*

Proof. Our proof is almost the same as that in Brown and Zhang [9, Theorem 2.3] (or see Binding et al. [26]). □

Lemma 2.3. *One has the following.*

(i) *If* $u \in \mathcal{M}^+_{\lambda a,b} \cup \mathcal{M}^0_{\lambda a,b}$, *then* $\int_{\mathbb{R}^N} a(x)|u|^q dx > 0$;

(ii) *If* $u \in \mathcal{M}^-_{\lambda a,b}$, *then* $\int_{\mathbb{R}^N} b(x)|u|^p dx > 0$.

Proof. The proof is immediate from (2.8) and (2.9). □

Moreover, we have the following result.

Lemma 2.4. *If* $\lambda \in (0, \Lambda_0)$, *then* $\mathcal{M}^0_{\lambda a,b} = \emptyset$, *where* Λ_0 *is the same as in* (1.1).

Proof. Suppose the contrary. Then there exists $\lambda \in (0, \Lambda_0)$ such that $\mathcal{M}^0_{\lambda a,b} \neq \emptyset$. Then for $u \in \mathcal{M}^0_{\lambda a,b}$ by (2.8) and Sobolev inequality, we have

$$\frac{2-q}{p-q}\|u\|^2_{H^1} = \int_{\mathbb{R}^N} b(x)|u|^p dx \leq \|b^+\|_{L^\infty} S_p^{-(p/2)} \|u\|^p_{H^1} \tag{2.10}$$

and so

$$\|u\|_{H^1} \geq \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{1/(p-2)} S_p^{p/2(p-2)}. \tag{2.11}$$

Similarly, using (2.9) and Hölder and Sobolev inequalities, we have

$$\|u\|^2_{H^1} = \lambda \frac{p-q}{p-2} \int_{\mathbb{R}^N} a(x)|u|^q dx \leq \lambda \frac{p-q}{p-2} \|a^+\|_{L^{q^*}} S_p^{-(q/2)} \|u\|^q_{H^1} \tag{2.12}$$

which implies that

$$\|u\|_{H^1} \leq \left(\lambda \frac{p-q}{p-2} \|a^+\|_{L^{q^*}}\right)^{1/(2-q)} S_p{}^{-(q/2(2-q))}. \tag{2.13}$$

Hence, we must have

$$\lambda \geq \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{(2-q)/(p-2)} \left(\frac{p-2}{(p-q)\|a^+\|_{L^{q^*}}}\right) S_p^{(p(2-q)/2(p-2))+(q/2)} = \Lambda_0, \tag{2.14}$$

which is a contradiction. This completes the proof. □

In order to get a better understanding of the Nehari manifold and fibering maps, we consider the function $\psi_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi_u(t) = t^{2-q}\|u\|^2_{H^1} - t^{p-q} \int_{\mathbb{R}^N} b(x)|u|^p dx, \quad \text{for } t > 0. \tag{2.15}$$

Clearly $tu \in \mathcal{M}_{\lambda a,b}$ if and only if $\psi_u(t) = \lambda \int_{\mathbb{R}^N} a(x)|u|^q dx$. Moreover,

$$\psi_u'(t) = (2-q)t^{1-q}\|u\|_{H^1}^2 - (p-q)t^{p-q-1}\int_{\mathbb{R}^N} b(x)|u|^p dx, \quad \text{for } t > 0, \tag{2.16}$$

and so it is easy to see that, if $tu \in \mathcal{M}_{\lambda a,b}$, then $t^{q-1}\psi_u'(t) = \varphi_u''(t)$. Hence, $tu \in \mathcal{M}_{\lambda a,b}^+$ (or $tu \in \mathcal{M}_{\lambda a,b}^-$) if and only if $\psi_u'(t) > 0$ (or $\psi_u'(t) < 0$).

Let $\mathcal{Z} = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} b(x)|u|^p dx = 0\}$. Suppose that $u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z}$. Then, by (2.16), ψ_u has a unique critical point at $t = t_{\max}(u)$, where

$$t_{\max}(u) = \left(\frac{(2-q)\|u\|_{H^1}^2}{(p-q)\int_{\mathbb{R}^N} b(x)|u|^p dx}\right)^{1/(p-2)} > 0, \tag{2.17}$$

and clearly ψ_u is strictly increasing on $(0, t_{\max}(u))$ and strictly decreasing on $(t_{\max}(u), \infty)$ with $\lim_{t\to\infty} \psi_u(t) = -\infty$. Moreover, if $\lambda \in (0, \Lambda_0)$, then

$$\begin{aligned}
\psi_u(t_{\max}(u)) &= \left[\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} - \left(\frac{2-q}{p-q}\right)^{(p-q)/(p-2)}\right] \frac{\|u\|_{H^1}^{2(p-q)/(p-2)}}{\left(\int_{\mathbb{R}^N} b(x)|u|^p dx\right)^{(2-q)/(p-2)}} \\
&= \|u\|_{H^1}^q \left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} \left(\frac{\|u\|_{H^1}^p}{\int_{\mathbb{R}^N} b(x)|u|^p dx}\right)^{(2-q)/(p-2)} \\
&\geq \|u\|_{H^1}^q \left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} S_p^{p(2-q)/2(p-2)} \\
&> \lambda \|a^+\|_{L^{q^*}} S_p^{-(q/2)} \|u\|_{H^1}^q \|b^+\|_{L^\infty} \\
&\geq \lambda \int_{\mathbb{R}^N} a^+(x)|u|^q dx \\
&\geq \lambda \int_{\mathbb{R}^N} a(x)|u|^q dx.
\end{aligned} \tag{2.18}$$

Therefore, we have the following lemma.

Lemma 2.5. *Let $\lambda \in (0, \Lambda_0)$. For each $u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z}$, one has the following.*

(i) *If $\lambda \int_{\mathbb{R}^N} a(x)|u|^q dx \leq 0$, then there exists a unique $t^- = t^-(u) > t_{\max}(u)$ such that $t^- u \in \mathcal{M}_{\lambda a,b}^-$, ψ_u is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover,*

$$J_{\lambda a,b}(t^- u) = \sup_{t \geq 0} J_{\lambda a,b}(tu). \tag{2.19}$$

(ii) *If* $\lambda\int_{\mathbb{R}^N} a(x)|u|^q dx > 0$, *then there exists unique* $0 < t^+ = t^+(u) < t_{\max}(u) < t^- = t^-(u)$ *such that* $t^+u \in \mathcal{M}^+_{\lambda a,b}$, $t^-u \in \mathcal{M}^-_{\lambda a,b}$, φ_u *is decreasing on* $(0, t^+)$, *increasing on* (t^+, t^-), *and decreasing on* (t^-, ∞)

$$J_{\lambda a,b}(t^+u) = \inf_{0\le t\le t_{\max}(u)} J_{\lambda a,b}(tu); \qquad J_{\lambda a,b}(t^-u) = \sup_{t\ge t^+} J_{\lambda a,b}(tu). \tag{2.20}$$

(iii) $\mathcal{M}^-_{\lambda a,b} = \{u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z} : (1/\|u\|_{H^1})t^-(u/\|u\|_{H^1}) = 1\}$.

(iv) *There exists a continuous bijection between* $U = \{u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z} : \|u\|_{H^1} = 1\}$ *and* $\mathcal{M}^-_{\lambda a,b}$. *In particular,* t^- *is a continuous function for* $u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z}$.

Proof. Fix $u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z}$.

(i) Suppose $\lambda\int_{\mathbb{R}^N} a(x)|u|^q dx \le 0$. Then $\psi_u(t) = \lambda\int_{\mathbb{R}^N} a(x)|u|^q dx$ has a unique solution $t^- > t_{\max}(u)$ such that $\psi'_u(t^-) < 0$ and $\varphi'_u(t^-) = 0$. Thus, by $t^{q-1}\psi'_u(t) = \varphi''_u(t)$, φ_u has a unique critical point at $t = t^-$ and $\varphi''_u(t^-) < 0$. Therefore, $t^-u \in \mathcal{M}^-_{\lambda a,b}$ and (2.19) holds.

(ii) Suppose $\lambda\int_{\mathbb{R}^N} a(x)|u|^q dx > 0$. Since $\psi_u(t_{\max}(u)) > \lambda\int_{\mathbb{R}^N} a(x)|u|^q dx$, the equation $\psi_u(t) = \lambda\int_{\mathbb{R}^N} a(x)|u|^q dx$ has exactly two solutions $t^+ < t_{\max}(u) < t^-$ such that $\psi'_u(t^+) > 0$ and $\psi'_u(t^-) < 0$. Thus, there exist exactly two multiples of u lying in $H^1(\mathbb{R}^N)$, that is, $t^+u \in \mathcal{M}^+_{\lambda a,b}$ and $t^-u \in \mathcal{M}^-_{\lambda a,b}$. Therefore, by $t^{q-1}\psi'_u(t) = \varphi''_u(t)$, φ_u has critical points at $t = t^+$ and $t = t^-$ with $\varphi''_u(t^+) > 0$ and $\varphi''_u(t^-) < 0$. Therefore, φ_u is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). This implies that (2.20) holds.

(iii) For $u \in \mathcal{M}^-_{\lambda a,b}$. By Lemma 2.3(ii) and, considering $w = u/\|u\|_{H^1}$, we have $u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z}$. By (i) and (ii), there exists a unique $t^-(w) > 0$ such that $t^-(w)w \in \mathcal{M}^-_{\lambda a,b}$, that is, $t^-(u/\|u\|_{H^1})(1/\|u\|_{H^1})u \in \mathcal{M}^-_{\lambda a,b}$. Since $u \in \mathcal{M}^-_{\lambda a,b}$, we have $(1/\|u\|_{H^1})t^-(u/\|u\|_{H^1}) = t^-(u) = 1$. Therefore

$$\mathcal{M}^-_{\lambda a,b} \subset \left\{u \in H^1\left(\mathbb{R}^N\right) \setminus \mathcal{Z} : \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) = 1\right\}. \tag{2.21}$$

Conversely, if $u \in H^1(\mathbb{R}^N) \setminus \mathcal{Z}$ is such that $(1/\|u\|_{H^1})t^-(u/\|u\|_{H^1}) = t^-(u) = 1$, then by the uniqueness of $t^-(u)$, we get that $u \in \mathcal{M}^-_{\lambda a,b}$. Thus, we have

$$\mathcal{M}^-_{\lambda a,b} = \left\{u \in H^1\left(\mathbb{R}^N\right) : \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) = 1\right\}. \tag{2.22}$$

(iv) Fix $u \in U$ arbitrary. Define $G_u : (0, \infty) \times U \to \mathbb{R}$ by

$$G_u(t, w) = \left\langle (J_{\lambda a,b})'(tw), tw \right\rangle = \phi_{\lambda a,b}(tw), \tag{2.23}$$

where $\phi_{\lambda a,b} : H^1(\mathbb{R}^N) \to \mathbb{R}$ is defined by $\phi_{\lambda a,b}(u) = \langle (J_{\lambda a,b})'(u), u\rangle$. Since $G_u(t^-(u), u) = \langle (J_{\lambda a,b})'(t^-(u)u), t^-(u)u\rangle = 0$, and

$$\frac{\partial G_u}{\partial t}(t^-(u), u) = \left[t^-(u)\right]^{-1}\langle (\phi_{\lambda a,b})'(t^-(u)u), t^-(u)u\rangle < 0, \tag{2.24}$$

then by the implicit function theorem, there is a neighborhood W_u of u in U and a unique continuous function $H_u : W_u \to (0,\infty)$ such that $G_u(H_u(w), w) = 0$ for all $w \in W_u$, in particular $H_u(u) = t^-(u)$. Since $u \in U$ is arbitrary, we obtain that the function $H : U \to (0,\infty)$, given by $H(u) = t^-(u)$, is continuous and one to one. By $H^- : U \to \mathcal{M}^-_{\lambda a,b}$, where $H^-(u) = t^-(u)u$, we have that H^- is continuous and one to one. Now if $u \in \mathcal{M}^-_{\lambda a,b}$, then by (iii) we have that $H^-(w) = u$, where $w = u/\|u\|_{H^1}$. Since t^- is continuous on U, it follows that t^- is continuous on $H^1(\mathbb{R}^N) \setminus \mathfrak{Z}$. This completes the proof. □

3. Proof of Theorem 1.1

First, we remark that it follows from Lemma 2.4 that

$$\mathcal{M}_{\lambda a,b} = \mathcal{M}^+_{\lambda a,b} \cup \mathcal{M}^-_{\lambda a,b} \tag{3.1}$$

for all $\lambda \in (0, \Lambda_0)$. Furthermore, by Lemma 2.5 it follows that $\mathcal{M}^+_{\lambda a,b}$ and $\mathcal{M}^-_{\lambda a,b}$ are nonempty, and by Lemma 2.1 we may define

$$\alpha_{\lambda a,b} = \inf_{u\in\mathcal{M}_{\lambda a,b}} J_{\lambda a,b}(u);\ \alpha^+_{\lambda a,b} = \inf_{u\in\mathcal{M}^+_{\lambda a,b}} J_{\lambda a,b}(u);\ \alpha^-_{\lambda a,b} = \inf_{u\in\mathcal{M}^-_{\lambda a,b}} J_{\lambda a,b}(u). \tag{3.2}$$

Then we get the following result.

Theorem 3.1. *One has the following.*

(i) *If $\lambda \in (0, \Lambda_0)$, then one has $\alpha^+_{\lambda a,b} < 0$.*

(ii) *If $\lambda \in (0, (q/2)\Lambda_0)$, then $\alpha^-_{\lambda a,b} > d_0$ for some $d_0 > 0$.*

In particular, for each $\lambda \in (0, (q/2)\Lambda_0)$, one has $\alpha^+_{\lambda a,b} = \alpha_{\lambda a,b}$.

Proof. (i) Let $u \in \mathcal{M}^+_{\lambda a,b}$. By (2.8)

$$\frac{2-q}{p-q}\|u\|^2_{H^1} > \int_{\mathbb{R}^N} b(x)|u|^p dx \tag{3.3}$$

and so

$$\begin{aligned}
J_{\lambda a,b}(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|_{H^1}^2 + \left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbb{R}^N} b(x)|u|^p dx \\
&< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{p}\right)\left(\frac{2-q}{p-q}\right)\right]\|u\|_{H^1}^2 \\
&= -\frac{(p-2)(2-q)}{2pq}\|u\|_{H^1}^2 < 0.
\end{aligned} \tag{3.4}$$

Therefore, $\alpha_{\lambda a,b}^{+} < 0$.

(ii) Let $u \in \mathcal{M}_{\lambda a,b}^{-}$. By (2.8)

$$\frac{2-q}{p-q}\|u\|_{H^1}^2 < \int_{\mathbb{R}^N} b(x)|u|^p dx. \tag{3.5}$$

Moreover, by (B1) and Sobolev inequality

$$\int_{\mathbb{R}^N} b(x)|u|^p dx \leq \|b^+\|_{L^\infty} S_p^{-(p/2)}\|u\|_{H^1}^p. \tag{3.6}$$

This implies that

$$\|u\|_{H^1} > \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{1/(p-2)} S_p^{p/2(p-2)}, \quad \forall u \in \mathcal{M}_{\lambda a,b}^{-}. \tag{3.7}$$

By (2.4) and (3.7), we have

$$\begin{aligned}
J_{\lambda a,b}(u) &\geq \|u\|_{H^1}^q \left[\frac{p-2}{2p}\|u\|_{H^1}^{2-q} - \lambda\left(\frac{p-q}{pq}\right)S_p^{-(q/2)}\|a^+\|_{L^{q^*}}\right] \\
&> \left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{q/(p-2)} S_p^{pq/2(p-2)} \\
&\quad \times \left[\frac{p-2}{2p}S_p^{p(2-q)/2(p-2)}\left(\frac{2-q}{(p-q)\|b^+\|_{L^\infty}}\right)^{(2-q)/(p-2)} - \lambda\left(\frac{p-q}{pq}\right)S_p^{-(q/2)}\|a^+\|_{L^{q^*}}\right].
\end{aligned} \tag{3.8}$$

Thus, if $\lambda \in (0, (q/2)\Lambda_0)$, then

$$J_{\lambda a,b}(u) > d_0, \quad \forall u \in \mathcal{M}_{\lambda a,b}^{-}, \tag{3.9}$$

for some positive constant d_0. This completes the proof. □

Remark 3.2. (i) If $\lambda \in (0, \Lambda_0)$, then by (2.8) and Hölder and Sobolev inequalities, for each $u \in \mathcal{M}^+_{\lambda a,b}$ we have

$$\begin{aligned} \|u\|^2_{H^1} &< \frac{p-q}{p-2}\lambda \int_{\mathbb{R}^N} a(x)|u|^q dx \\ &\le \frac{p-q}{p-2}\lambda \|a\|_{L^{q^*}} S_p^{-(q/2)} \|u\|^q_{H^1} \\ &< \frac{p-q}{p-2}\Lambda_0 \|a\|_{L^{q^*}} S_p^{-(q/2)} \|u\|^q_{H^1}, \end{aligned} \tag{3.10}$$

and so

$$\|u\|_{H^1} \le \left(\frac{p-q}{p-2}\Lambda_0 \|a\|_{L^{q^*}} S_p^{-(q/2)}\right)^{1/(2-q)}, \quad \forall u \in \mathcal{M}^+_{\lambda a,b}. \tag{3.11}$$

(ii) If $\lambda \in (0, (q/2)\Lambda_0)$, then by Lemma 2.5(i), (ii) and Theorem 3.1(ii), for each $u \in \mathcal{M}^-_{\lambda a,b}$ we have

$$J_{\lambda a,b}(u) = \sup_{t\ge 0} J_{\lambda a,b}(tu). \tag{3.12}$$

We define the Palais-Smale (simply by (PS)) sequences, (PS) values, and (PS) conditions in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$ as follows.

Definition 3.3. (i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$ if $J_{\lambda a,b}(u_n) = c + o_n(1)$ and $(J_{\lambda a,b})'(u_n) = o_n(1)$ strongly in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$.

(ii) $c \in \mathbb{R}$ is a (PS) value in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$ if there exists a $(\mathrm{PS})_c$ sequence in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$.

(iii) $J_{\lambda a,b}$ satisfies the $(\mathrm{PS})_c$-condition in $H^1(\mathbb{R}^N)$ if any $(\mathrm{PS})_c$ sequence $\{u_n\}$ in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$ contains a convergent subsequence.

Now, we use the Ekeland variational principle [19] to get the following results.

Proposition 3.4. (i) *If $\lambda \in (0, \Lambda_0)$, then there exists a $(PS)_{\alpha_{\lambda a,b}}$ sequence $\{u_n\} \subset \mathcal{M}_{\lambda a,b}$ in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$.*

(ii) *If $\lambda \in (0, (q/2)\Lambda_0)$, then there exists a $(PS)_{\alpha^-_{\lambda a,b}}$ sequence $\{u_n\} \subset \mathcal{M}^-_{\lambda a,b}$ in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$.*

Proof. The proof is almost the same as that in [27, Proposition 9]. □

Now, we establish the existence of a local minimum for $J_{\lambda a,b}$ on $\mathcal{M}^+_{\lambda a,b}$.

Theorem 3.5. *Assume that (A1) and (B1) hold. If $\lambda \in (0, \Lambda_0)$, then $J_{\lambda a,b}$ has a minimizer u_λ in $\mathcal{M}^+_{\lambda a,b}$, and it satisfies the following:*

(i) $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b} = \alpha^+_{\lambda a,b}$;

(ii) *u_λ is a positive solution of $(E_{\lambda a,b})$ in $\mathbb{R}^N$;*

(iii) *$\|u_\lambda\|_{H^1} \to 0$ as $\lambda \to 0^+$.*

Proof. By Proposition 3.4(i), there is a minimizing sequence $\{u_n\}$ for $J_{\lambda a,b}$ on $\mathcal{M}_{\lambda a,b}$ such that

$$J_{\lambda a,b}(u_n) = \alpha_{\lambda a,b} + o_n(1), \quad (J_{\lambda a,b})'(u_n) = o_n(1), \quad \text{in } H^{-1}\left(\mathbb{R}^N\right). \tag{3.13}$$

Since $J_{\lambda a,b}$ is coercive on $\mathcal{M}_{\lambda a,b}$ (see Lemma 2.1), we get that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Going if necessary to a subsequence, we can assume that there exists $u_\lambda \in H^1(\mathbb{R}^N)$ such that

$$\begin{gathered} u_n \rightharpoonup u_\lambda, \quad \text{weakly in } H^1\left(\mathbb{R}^N\right), \\ u_n \longrightarrow u_\lambda, \quad \text{almost everywhere in } \mathbb{R}^N, \\ u_n \longrightarrow u_\lambda, \quad \text{strongly in } L^s_{\text{loc}}\left(\mathbb{R}^N\right),\ \forall 1 \le s < 2^*. \end{gathered} \tag{3.14}$$

By (A1), Egorov theorem, and Hölder inequality, we have

$$\lambda \int_{\mathbb{R}^N} a(x)|u_n|^q dx = \lambda \int_{\mathbb{R}^N} a(x)|u_\lambda|^q dx + o_n(1), \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

First, we claim that u_λ is a nonzero solution of $(E_{\lambda a,b})$. By (3.13) and (3.14), it is easy to see that u_λ is a solution of $(E_{\lambda a,b})$. From $u_n \in \mathcal{M}_{\lambda a,b}$ and (2.3), we deduce that

$$\lambda \int_{\mathbb{R}^N} a(x)|u_n|^q dx = \frac{q(p-2)}{2(p-q)}\|u_n\|^2_{H^1} - \frac{pq}{p-q}J_{\lambda a,b}(u_n). \tag{3.16}$$

Let $n \to \infty$ in (3.16); by (3.13), (3.15), and $\alpha_{\lambda a,b} < 0$, we get

$$\lambda \int_{\mathbb{R}^N} a(x)|u_\lambda|^q dx \ge -\frac{pq}{p-q}\alpha_{\lambda a,b} > 0. \tag{3.17}$$

Thus, $u_\lambda \in \mathcal{M}_{\lambda a,b}$ is a nonzero solution of $(E_{\lambda a,b})$. Now we prove that $u_n \to u_\lambda$ strongly in $H^1(\mathbb{R}^{\mathrm{N}})$ and $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b}$. By (3.16), if $u \in \mathcal{M}_{\lambda a,b}$, then

$$J_{\lambda a,b}(u) = \frac{p-2}{2p}\|u\|^2_{H^1} - \frac{p-q}{pq}\lambda \int_{\mathbb{R}^N} a(x)|u|^q dx. \tag{3.18}$$

In order to prove that $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b}$, it suffices to recall that $u_n, u_\lambda \in \mathcal{M}_{\lambda a,b}$, by (3.18) and applying Fatou's lemma to get

$$\begin{aligned} \alpha_{\lambda a,b} \le J_{\lambda a,b}(u_\lambda) &= \frac{p-2}{2p}\|u_\lambda\|^2_{H^1} - \frac{p-q}{pq}\lambda \int_{\mathbb{R}^N} a(x)|u_\lambda|^q dx \\ &\le \liminf_{n\to\infty}\left(\frac{p-2}{2p}\|u_n\|^2_{H^1} - \frac{p-q}{pq}\lambda \int_{\mathbb{R}^N} a(x)|u_n|^q dx\right) \\ &\le \liminf_{n\to\infty} J_{\lambda a,b}(u_n) = \alpha_{\lambda a,b}. \end{aligned} \tag{3.19}$$

This implies that $J_{\lambda a,b}(u_\lambda) = \alpha_{\lambda a,b}$ and $\lim_{n\to\infty}\|u_n\|_{H^1}^2 = \|u_\lambda\|_{H^1}^2$. Let $v_n = u_n - u_\lambda$, then Brezis-Lieb lemma [28] implies that

$$\|v_n\|_{H^1}^2 = \|u_n\|_{H^1}^2 - \|u_\lambda\|_{H^1}^2 + o_n(1). \tag{3.20}$$

Therefore, $u_n \to u_\lambda$ strongly in $H^1(\mathbb{R}^N)$. Moreover, we have $u_\lambda \in \mathcal{M}^+_{\lambda a,b}$. On the contrary, if $u_\lambda \in \mathcal{M}^-_{\lambda a,b}$, then by Lemma 2.5, there are unique t_0^+ and t_0^- such that $t_0^+ u_\lambda \in \mathcal{M}^+_{\lambda a,b}$ and $t_0^- u_\lambda \in \mathcal{M}^-_{\lambda a,b}$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_{\lambda a,b}(t_0^+ u_\lambda) = 0, \qquad \frac{d^2}{dt^2}J_{\lambda a,b}(t_0^+ u_\lambda) > 0, \tag{3.21}$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_{\lambda a,b}(t_0^+ u_\lambda) < J_{\lambda a,b}(\bar{t}u_\lambda)$. By Lemma 2.5,

$$J_{\lambda a,b}(t_0^+ u_\lambda) < J_{\lambda a,b}\left(\bar{t}u_\lambda\right) \le J_{\lambda a,b}(t_0^- u_\lambda) = J_{\lambda a,b}(u_\lambda), \tag{3.22}$$

which is a contradiction. Since $J_{\lambda a,b}(u_\lambda) = J_{\lambda a,b}(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{M}^+_{\lambda a,b}$, by Lemma 2.2 we may assume that u_λ is a nonzero nonnegative solution of $(E_{\lambda a,b})$. By Harnack inequality [29] we deduce that $u_\lambda > 0$ in $\mathbb{R}^N$. Finally, by (2.3) and Hölder and Sobolev inequalities,

$$\|u_\lambda\|_{H^1}^{2-q} < \lambda\frac{p-q}{p-2}\|a^+\|_{L^{q^*}} S_p^{-(q/2)} \tag{3.23}$$

and so $\|u_\lambda\|_{H^1} \to 0$ as $\lambda \to 0^+$. □

Now, we begin the proof of Theorem 1.1: By Theorem 3.5, we obtain that $(E_{\lambda a,b})$ has a positive solution u_λ in $H^1(\mathbb{R}^N)$.

4. Proof of Theorem 1.2

In this section, we will establish the existence of the second positive solution of $(E_{\lambda a,b})$ by proving that $J_{\lambda a,b}$ satisfies the $(\mathrm{PS})_{\alpha^-_{\lambda a,b}}$-condition.

Lemma 4.1. *Assume that (A1) and (B1) hold. If $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a $(PS)_c$ sequence for $J_{\lambda a,b}$, then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. We argue by contradiction. Assume that $\|u_n\|_{H^1} \to \infty$. Let $\widehat{u}_n = u_n/\|u_n\|_{H^1}$. We may assume that $\widehat{u}_n \rightharpoonup \widehat{u}$ weakly in $H^1(\mathbb{R}^N)$. This implies that $\widehat{u}_n \to \widehat{u}$ strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 \le s < 2^*$. By (A1), Egorov theorem, and Hölder inequality, we have

$$\frac{\lambda}{q}\int_{\mathbb{R}^N} a(x)|\widehat{u}_n|^q dx = \frac{\lambda}{q}\int_{\mathbb{R}^N} a(x)|\widehat{u}|^q dx + o_n(1). \tag{4.1}$$

Since $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence for $J_{\lambda a,b}$ and $\|u_n\|_{H^1} \to \infty$, there hold

$$\begin{gathered}
\frac{1}{2}\|\widehat{u}_n\|_{H^1}^2 - \frac{\lambda\|u_n\|_{H^1}^{q-2}}{q}\int_{\mathbb{R}^N} a(x)|\widehat{u}_n|^q dx - \frac{\|u_n\|_{H^1}^{p-2}}{p}\int_{\mathbb{R}^N} b(x)|\widehat{u}_n|^p dx = c + o_n(1), \\
\|\widehat{u}_n\|_{H^1}^2 - \lambda\|u_n\|_{H^1}^{q-2}\int_{\mathbb{R}^N} a(x)|\widehat{u}_n|^q dx - \|u_n\|_{H^1}^{p-2}\int_{\mathbb{R}^N} b(x)|\widehat{u}_n|^p dx = o_n(1).
\end{gathered} \tag{4.2}$$

From (4.1)-(4.2), we can deduce that

$$\|\widehat{u}_n\|_{H^1}^2 = \frac{2(p-q)}{q(p-2)}\|u_n\|^{q-2}\lambda\int_{\mathbb{R}^N} a(x)|\widehat{u}|^q dx + o_n(1). \tag{4.3}$$

Since $1 < q < 2$ and $\|u_n\|_{H^1} \to \infty$, (4.3) implies that

$$\|\widehat{u}_n\|_{H^1}^2 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{4.4}$$

which contradicts with the fact $\|\widehat{u}_n\|_{H^1} = 1$ for all n. □

We assume that condition (B2) holds and recall

$$S_p^b = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{H^1}^2}{\left(\int_{\mathbb{R}^N} b(x)|u|^p dx\right)^{2/p}}. \tag{4.5}$$

Lemma 4.2. *Assume that (A1) and (B1)-(B2) hold. If $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a $(PS)_c$ sequence for $J_{\lambda a,b}$ with $c \in (0, \alpha_0^b)$, then there exists a subsequence of $\{u_n\}$ converging weakly to a nonzero solution of $(E_{\lambda a,b})$ in $\mathbb{R}^N$.*

Proof. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a $(\mathrm{PS})_c$ sequence for $J_{\lambda a,b}$ with $c \in (0, \alpha_0^b)$. We know from Lemma 4.1 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and then there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_0 \in H^1(\mathbb{R}^N)$ such that

$$\begin{gathered}
u_n \rightharpoonup u_0, \quad \text{weakly in } H^1\left(\mathbb{R}^N\right), \\
u_n \longrightarrow u_0, \quad \text{almost everywhere in } \mathbb{R}^N, \\
u_n \longrightarrow u_0, \quad \text{strongly in } L^s_{\text{loc}}\left(\mathbb{R}^N\right),\ \forall 1 \le s < 2^*.
\end{gathered} \tag{4.6}$$

It is easy to see that $(J_{\lambda a,b})'(u_0) = 0$, and by (A1), Egorov theorem, and Hölder inequality, we have

$$\lambda\int_{\mathbb{R}^N} a(x)|u_n|^q dx = \lambda\int_{\mathbb{R}^N} a(x)|u_0|^q dx + o_n(1). \tag{4.7}$$

Next we verify that $u_0 \not\equiv 0$. Arguing by contradiction, we assume that $u_0 \equiv 0$. We set

$$l = \lim_{n\to\infty} \int_{\mathbb{R}^N} b(x)|u_n|^p dx. \tag{4.8}$$

Since $(J_{\lambda a,b})'(u_n) = o_n(1)$ and $\{u_n\}$ is bounded, then by (4.7), we can deduce that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle (J_{\lambda a,b})'(u_n), u_n \rangle \\ &= \lim_{n\to\infty} \left(\|u_n\|_{H^1}^2 - \int_{\mathbb{R}^N} a(x)|u_n|^q dx - \int_{\mathbb{R}^N} b(x)|u_n|^p dx \right) \\ &= \lim_{n\to\infty} \|u_n\|_{H^1}^2 - l, \end{aligned} \tag{4.9}$$

that is,

$$\lim_{n\to\infty} \|u_n\|_{H^1}^2 = l. \tag{4.10}$$

If $l = 0$, then we get $c = \lim_{n\to\infty} J_{\lambda a,b}(u_n) = 0$, which contradicts with $c > 0$. Thus we conclude that $l > 0$. Furthermore, by the definition of S_p^b we obtain

$$\|u_n\|_{H^1}^2 \geq S_p^b \left(\int_{\mathbb{R}^N} b(x)|u_n|^p dx \right)^{2/p}. \tag{4.11}$$

Then as $n \to \infty$ we have

$$l = \lim_{n\to\infty} \|u_n\|_{H^1}^2 \geq S_p^b l^{2/p}, \tag{4.12}$$

which implies that

$$l \geq \left(S_p^b \right)^{p/(p-2)}. \tag{4.13}$$

Hence, from (1.7) and (4.7)–(4.13) we get

$$\begin{aligned} c &= \lim_{n\to\infty} J_{\lambda a,b}(u_n) \\ &= \frac{1}{2} \lim_{n\to\infty} \|u_n\|_{H^1}^2 - \frac{\lambda}{q} \lim_{n\to\infty} \int_{\mathbb{R}^N} a(x)|u_n|^q dx - \frac{1}{p} \lim_{n\to\infty} \int_{\mathbb{R}^N} b(x)|u_n|^p dx \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) l \geq \frac{p-2}{2p} \left(S_p^b \right)^{p/(p-2)} = \alpha_0^b. \end{aligned} \tag{4.14}$$

This is a contradiction to $c < \alpha_0^b$. Therefore u_0 is a nonzero solution of $(E_{\lambda a,b})$. □

Lemma 4.3. *Assume that (A1)-(A2) and (B1)-(B2) hold. Let w_0 be a positive ground state solution of (E_b), then*

(i) $\sup_{t\geq 0} J_{\lambda a,b}(tw_0) < \alpha_0^b$ *for all* $\lambda > 0$;

(ii) $\alpha_{\lambda a,b}^- < \alpha_0^b$ *for all* $\lambda \in (0, \Lambda_0)$.

Proof. (i) First, we consider the functional $Q : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$Q(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} b(x)|u|^p dx, \quad \forall u \in H^1\left(\mathbb{R}^N\right). \tag{4.15}$$

Then, from (1.3) and (1.7), we conclude that

$$\sup_{t\geq 0} Q(tw_0) = \frac{p-2}{2p}\left(\frac{\|w_0\|_{H^1}^2}{\left(\int_{\mathbb{R}^N} b(x)|w_0|^p dx\right)^{2/p}}\right)^{p/(p-2)} = \frac{p-2}{2p}\left(S_p^b\right)^{p/(p-2)} = \alpha_0^b, \tag{4.16}$$

where the following fact has been used:

$$\sup_{t\geq 0}\left(\frac{t^2}{2}A - \frac{t^p}{p}B\right) = \frac{p-2}{2p}\left(\frac{A}{B^{2/p}}\right)^{p/(p-2)}, \quad \text{where } A, B > 0. \tag{4.17}$$

Using the definitions of $J_{\lambda a,b}$, w_0, and $b(x) > 0$ for all $x \in \mathbb{R}^N$, for any $\lambda > 0$ we have

$$J_{\lambda a,b}(tw_0) \longrightarrow -\infty, \quad \text{as } t \longrightarrow \infty. \tag{4.18}$$

From this we know that there exists $t_0 > 0$ such that

$$\sup_{t\geq 0} J_{\lambda a,b}(tw_0) = \sup_{0\leq t\leq t_0} J_{\lambda a,b}(tw_0). \tag{4.19}$$

By the continuity of $J_{\lambda a,b}(tw_0)$ as a function of $t \geq 0$ and $J_{\lambda a,b}(0) = 0$, we can find some $t_1 \in (0, t_0)$ such that

$$\sup_{0\leq t\leq t_1} J_{\lambda a,b}(tw_0) < \alpha_0^b. \tag{4.20}$$

Thus, we only need to show that

$$\sup_{t_1\leq t\leq t_0} J_{\lambda a,b}(tw_0) < \alpha_0^b. \tag{4.21}$$

To this end, by (A2) and (4.16), for all $\lambda > 0$ we have

$$\sup_{t_1\leq t\leq t_0} J_{\lambda a,b}(tw_0) \leq \sup_{t\geq 0} Q(tw_0) - \frac{t_1^q}{q}\lambda\int_{\mathbb{R}^N} a(x)|w_0|^q dx < \alpha_0^b. \tag{4.22}$$

Hence (i) holds.

(ii) By (A1), (A2), and the definition of w_0, we have

$$\int_{\mathbb{R}^N} b(x)|w_0|^p dx > 0, \qquad \int_{\mathbb{R}^N} a(x)|w_0|^q dx > 0. \tag{4.23}$$

Combining this with Lemma 2.5(ii), from the definition of $\alpha^-_{\lambda a,b}$ and part (i), for all $\lambda \in (0, \Lambda_0)$, we obtain that there exists $t_0 > 0$ such that $t_0 w_0 \in \mathcal{M}^-_{\lambda a,b}$ and

$$\alpha^-_{\lambda a,b} \le J_{\lambda a,b}(t_0 w_0) \le \sup_{t\ge 0} J_{\lambda a,b}(t w_0) < \alpha^b_0. \tag{4.24}$$

Therefore (ii) holds. □

Now, we establish the existence of a local minimum of $J_{\lambda a,b}$ on $\mathcal{M}^-_{\lambda a,b}$.

Theorem 4.4. *Assume that (A1)-(A2), (B1), and* $(\mathbb{R}^N_b)$ *hold. If* $\lambda \in (0, (q/2)\Lambda_0)$*, then* $J_{\lambda a,b}$ *has a minimizer* U_λ *in* $\mathcal{M}^-_{\lambda a,b}$*, and it satisfies the following:*

(i) $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$;

(ii) U_λ *is a positive solution of* $(E_{\lambda a,b})$ *in* $\mathbb{R}^N$.

Proof. If $\lambda \in (0, (q/2)\Lambda_0)$, then by Theorem 3.1(ii), Proposition 3.4(ii), and Lemma 4.3(ii), there exists a $(\mathrm{PS})_{\alpha^-_{\lambda a,b}}$ sequence $\{u_n\} \subset \mathcal{M}^-_{\lambda a,b}$ in $H^1(\mathbb{R}^N)$ for $J_{\lambda a,b}$ with $\alpha^-_{\lambda a,b} \in (0, \alpha^b_0)$. From Lemma 4.2, there exist a subsequence still denoted by $\{u_n\}$ and a nonzero solution $U_\lambda \in H^1(\mathbb{R}^N)$ of $(E_{\lambda a,b})$ such that $u_n \rightharpoonup U_\lambda$ weakly in $H^1(\mathbb{R}^N)$. Now we prove that $u_n \to U_\lambda$ strongly in $H^1(\mathbb{R}^N)$ and $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$. By (3.18), if $u \in \mathcal{M}_{\lambda a,b}$, then

$$J_{\lambda a,b}(u) = \frac{p-2}{2p}\|u\|^2_{H^1} - \frac{p-q}{pq}\lambda \int_{\mathbb{R}^N} a(x)|u|^q dx. \tag{4.25}$$

First, we prove that $U_\lambda \in \mathcal{M}^-_{\lambda a,b}$. On the contrary, if $U_\lambda \in \mathcal{M}^+_{\lambda a,b}$, then by the definition of

$$\mathcal{M}^-_{\lambda a,b} = \left\{u \in \mathcal{M}_{\lambda a,b} : \varphi''_u(1) < 0\right\} \tag{4.26}$$

and Lemma 2.4, we have $\|U_\lambda\|^2_{H^1} < \liminf_{n\to\infty}\|u_n\|^2_{H^1}$. From Lemma 2.3(i) and $b(x) > 0$ for all $x \in \mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N} a(x)|U_\lambda|^q dx > 0, \qquad \int_{\mathbb{R}^N} b(x)|U_\lambda|^p dx > 0. \tag{4.27}$$

By Lemma 2.5(ii), there exists a unique t^-_λ such that $t^-_\lambda U_\lambda \in \mathcal{M}^-_{\lambda a,b}$. Since $u_n \in \mathcal{M}^-_{\lambda a,b}$, $J_{\lambda a,b}(u_n) \ge J_{\lambda a,b}(t u_n)$ for all $t \ge 0$, and by (4.25), we have

$$\alpha^-_{\lambda a,b} \le J_{\lambda a,b}\left(t^-_\lambda U_\lambda\right) < \lim_{n\to\infty} J_{\lambda a,b}\left(t^-_\lambda u_n\right) \le \lim_{n\to\infty} J_{\lambda a,b}(u_n) = \alpha^-_{\lambda a,b}, \tag{4.28}$$

and this is a contradiction.

In order to prove that $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$, it suffices to recall that $u_n, U_\lambda \in \mathcal{M}^-_{\lambda a,b}$ for all n, by (4.25) and applying Fatou's lemma to get

$$\begin{aligned}
\alpha^-_{\lambda a,b} \le J_{\lambda a,b}(U_\lambda) &= \frac{p-2}{2p}\|U_\lambda\|^2_{H^1} - \frac{p-q}{pq}\lambda\int_{\mathbb{R}^N} a(x)|U_\lambda|^q dx \\
&\le \liminf_{n\to\infty}\left(\frac{p-2}{2p}\|u_n\|^2_{H^1} - \frac{p-q}{pq}\lambda\int_{\mathbb{R}^N} a(x)|u_n|^q dx\right) \\
&\le \liminf_{n\to\infty} J_{\lambda a,b}(u_n) = \alpha^-_{\lambda a,b}.
\end{aligned} \tag{4.29}$$

This implies that $J_{\lambda a,b}(U_\lambda) = \alpha^-_{\lambda a,b}$ and $\lim_{n\to\infty}\|u_n\|^2_{H^1} = \|U_\lambda\|^2_{H^1}$. Let $v_n = u_n - U_\lambda$, then Brézis and Lieb lemma [28] implies that

$$\|v_n\|^2_{H^1} = \|u_n\|^2_{H^1} - \|U_\lambda\|^2_{H^1} + o_n(1). \tag{4.30}$$

Therefore, $u_n \to U_\lambda$ strongly in $H^1(\mathbb{R}^N)$.

Since $J_{\lambda a,b}(U_\lambda) = J_{\lambda a,b}(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{M}^-_{\lambda a,b}$, by Lemma 2.2 we may assume that U_λ is a nonzero nonnegative solution of $(E_{\lambda a,b})$. Finally, By the Harnack inequality [29] we deduce that $U_\lambda > 0$ in $\mathbb{R}^N$.

Now, we complete the proof of Theorem 1.2. By Theorems 3.5 and 4.4, we obtain that $(E_{\lambda a,b})$ has two positive solutions u_λ and U_λ such that $u_\lambda \in \mathcal{M}^+_{\lambda a,b}$, $U_\lambda \in \mathcal{M}^-_{\lambda a,b}$. Since $\mathcal{M}^+_{\lambda a,b} \cap \mathcal{M}^-_{\lambda a,b} = \emptyset$, this implies that u_λ and U_λ are distinct. It completes the proof of Theorem 1.2. □

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