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Research Article

Best Possible Inequalities between Generalized Logarithmic Mean and Classical Means

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We answer the question: for $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma = 1$, what are the greatest value p and the least value q, such that the double inequality $L_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{\gamma}(a,b) < L_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$? Here $L_p(a,b)$, $A(a,b)$, $G(a,b)$, and $H(a,b)$ denote the generalized logarithmic, arithmetic, geometric, and harmonic means of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$ the generalized logarithmic mean $L_p(a,b)$ of two positive numbers a and b with $a \neq b$ is defined by

$$L_p(a,b) = \begin{cases} \left[\dfrac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right]^{1/p}, & p \neq 0,\ p \neq -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0, \\ \dfrac{b-a}{\log b - \log a}, & p = -1. \end{cases} \tag{1.1}$$

It is well known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently, the generalized logarithmic mean has been the subject of intensive research. Many remarkable inequalities and monotonicity results for the generalized logarithmic mean can be found in the literature [1–9]. It might be surprising that the generalized logarithmic mean has applications in physics, economics, and even in meteorology [10–13].

Let $A(a,b) = (a+b)/2$, $I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $L(a,b) = (b-a)/(\log b - \log a)$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers a and b with $a \neq b$, respectively. Then

$$\begin{aligned} \min\{a,b\} < H(a,b) < G(a,b) = L_{-2}(a,b) < L(a,b) = L_{-1}(a,b) \\ < I(a,b) = L_0(a,b) < A(a,b) = L_1(a,b) < \max\{a,b\}. \end{aligned} \tag{1.2}$$

For $p \in \mathbb{R}$, the pth power mean $M_p(a,b)$ of two positive numbers a and b with $a \neq b$ is defined by

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p} (p \neq 0), \qquad M_0(a,b) = (ab)^{1/2}. \tag{1.3}$$

In [14], Alzer and Janous established the following sharp double inequality (see also [15, page 350]):

$$M_{\log 2/\log 3}(a,b) < \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) < M_{2/3}(a,b) \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$.

For $\alpha \in (0,1)$, Janous [16] found the greatest value p and the least value q such that

$$M_p(a,b) < \alpha A(a,b) + (1-\alpha)G(a,b) < M_q(a,b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

In [17–19], the authors presented bounds for $L(a,b)$ and $I(a,b)$ in terms of $G(a,b)$ and $A(a,b)$.

Theorem A. *For all positive real numbers a and b with $a \neq b$, we have*

$$\begin{aligned} L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b), \\ \frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) < I(a,b). \end{aligned} \tag{1.6}$$

The proof of the following Theorem B can be found in [20].

Theorem B. *For all positive real numbers a and b with $a \neq b$, we have*

$$\begin{aligned}\sqrt{G(a,b)A(a,b)} &< \sqrt{L(a,b)I(a,b)} \\ &< \frac{1}{2}(L(a,b)+I(a,b)) \\ &< \frac{1}{2}(G(a,b)+A(a,b)).\end{aligned} \tag{1.7}$$

The following Theorems C–E were established by Alzer and Qiu in [21].

Theorem C. *The inequalities*

$$\alpha A(a,b)+(1-\alpha)G(a,b) < I(a,b) < \beta A(a,b)+(1-\beta)G(a,b) \tag{1.8}$$

hold for all positive real numbers a and b with $a \neq b$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e = 0.73575\cdots$.

Theorem D. *Let a and b be real numbers with $a \neq b$. If $0 < a, b \leq e$, then*

$$[G(a,b)]^{A(a,b)} < [L(a,b)]^{I(a,b)} < [A(a,b)]^{G(a,b)}. \tag{1.9}$$

And if $a, b \geq e$, then

$$[A(a,b)]^{G(a,b)} < [I(a,b)]^{L(a,b)} < [G(a,b)]^{A(a,b)}. \tag{1.10}$$

Theorem E. *For all positive real numbers a and b with $a \neq b$, we have*

$$M_p(a,b) < \frac{1}{2}(L(a,b)+I(a,b)) \tag{1.11}$$

with the best possible parameter $p = \log 2/(1+\log 2) = 0.40938\cdots$.

However, the following problem is still open: for $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma = 1$, what are the greatest value p and the least value q, such that the double inequality

$$L_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{\gamma}(a,b) < L_q(a,b) \tag{1.12}$$

holds for all $a, b > 0$ with $a \neq b$? The purpose of this paper is to give the solution to this open problem.

2. Lemmas

In order to establish our main result, we need four lemmas, which we present in this section.

Lemma 2.1. *If $t > 1$, then*

$$\frac{t-1}{\log t} - \left[\frac{t(t+1)}{2}\right]^{1/3} > 0. \tag{2.1}$$

Proof. If $t > 1$, then it follows from [22, 3.6.16] that

$$\frac{t-1}{\log t} > \frac{t+t^{1/3}}{1+t^{1/3}}. \tag{2.2}$$

It is not difficult to verify that

$$\frac{t+t^{1/3}}{1+t^{1/3}} > \left[\frac{t(t+1)}{2}\right]^{1/3} \tag{2.3}$$

for $t > 1$.

Therefore, Lemma 2.1 follows from inequalities (2.2) and (2.3). □

Lemma 2.2. *If $t > 1$, then*

$$\frac{t}{t-1}\log t - \frac{1}{6}\log t - \frac{2}{3}\log\frac{1+t}{2} - 1 > 0. \tag{2.4}$$

Proof. Let $f(t) = (t/(t-1))\log t - (1/6)\log t - (2/3)\log((1+t)/2) - 1$. Then simple computations lead to

$$\lim_{t\to 1} f(t) = 0, \tag{2.5}$$

$$f'(t) = \frac{g(t)}{6t(t-1)^2(t+1)}, \tag{2.6}$$

where $g(t) = -6t(t+1)\log t + t^3 + 9t^2 - 9t - 1$,

$$g(1) = 0, \tag{2.7}$$

$$g'(t) = -6(2t+1)\log t + 3t^2 + 12t - 15, \tag{2.8}$$

$$g'(1) = 0, \tag{2.9}$$

$$g''(t) = \frac{h(t)}{t}, \tag{2.10}$$

where $h(t) = -12\log t + 6t^2 - 6$,

$$g''(1) = h(1) = 0, \tag{2.11}$$

$$h'(t) = -12\log t + 12t - 12, \tag{2.12}$$

$$h'(1) = 0, \tag{2.13}$$

$$h''(t) = \frac{12}{t}(t-1) > 0 \tag{2.14}$$

for $t > 1$.

Therefore, Lemma 2.2 follows from (2.5)–(2.7), (2.9)–(2.11), (2.13), and (2.14). □

Lemma 2.3. *Let $t > 1$ and $g(t) = (1-(1/2)\lambda)(3\lambda-5)t^{3\lambda-2} + [(\lambda-1)(3\lambda-5)-1]t^{3\lambda-3} - [(1/2)\lambda(3\lambda-5)+1]t^{3\lambda-4} + [(1/2)\lambda(3\lambda-5)+1]t^2 - [(\lambda-1)(3\lambda-5)-1]t - (1-(1/2)\lambda)(3\lambda-5)$. Then*

(1) $g(t) > 0$ *for* $\lambda \in (0,2/3) \cup (1,4/3) \cup (5/3,2)$ *and*

(2) $g(t) < 0$ *for* $\lambda \in (2/3,1) \cup (4/3,5/3)$.

Proof. Simple computations yield

$$\lim_{t\to 1} g(t) = 0, \tag{2.15}$$

$$\begin{aligned} g'(t) = {} & \left(1-\frac{1}{2}\lambda\right)(3\lambda-5)(3\lambda-2)t^{3\lambda-3} + 3(\lambda-1)[(\lambda-1)(3\lambda-5)-1]t^{3\lambda-4} \\ & - (3\lambda-4)\left[\frac{1}{2}\lambda(3\lambda-5)+1\right]t^{3\lambda-5} + [\lambda(3\lambda-5)+2]t \\ & - (\lambda-1)(3\lambda-5)+1, \end{aligned} \tag{2.16}$$

$$g'(1) = 0, \tag{2.17}$$

$$\begin{aligned} g''(t) = {} & 3\left(1-\frac{1}{2}\lambda\right)(\lambda-1)(3\lambda-5)(3\lambda-2)t^{3\lambda-4} \\ & + 3(\lambda-1)(3\lambda-4)[(\lambda-1)(3\lambda-5)-1]t^{3\lambda-5} \\ & - (3\lambda-4)(3\lambda-5)\left[\frac{1}{2}\lambda(3\lambda-5)+1\right]t^{3\lambda-6} + \lambda(3\lambda-5)+2, \end{aligned} \tag{2.18}$$

$$g''(1) = 0, \tag{2.19}$$

$$g'''(t) = t^{3\lambda-7}h(t), \tag{2.20}$$

where $h(t) = 3(1-(1/2)\lambda)(\lambda-1)(3\lambda-5)(3\lambda-4)(3\lambda-2)t^2 + 3(\lambda-1)(3\lambda-4)(3\lambda-5)[(\lambda-1)(3\lambda-5)-1]t - 3(\lambda-2)(3\lambda-4)(3\lambda-5)[(1/2)\lambda(3\lambda-5)+1]$,

$$h(1) = 0, \tag{2.21}$$

$$h'(t) = 81(2-\lambda)\left(\lambda-\frac{2}{3}\right)(\lambda-1)\left(\lambda-\frac{4}{3}\right)\left(\lambda-\frac{5}{3}\right)(t-1). \tag{2.22}$$

(1) If $\lambda \in (0, 2/3) \cup (1, 4/3) \cup (5/3, 2)$, then (2.22) implies

$$h'(t) > 0 \tag{2.23}$$

for $t > 1$.

Therefore, Lemma 2.3(1) follows from (2.15), (2.17), (2.19), (2.20), (2.21), and (2.23).

(2) If $\lambda \in (2/3, 1) \cup (4/3, 5/3)$, then (2.22) leads to

$$h'(t) < 0 \tag{2.24}$$

for $t > 1$.

Therefore, Lemma 2.3(2) follows from (2.15), (2.17), (2.19), (2.20), (2.21), and (2.24). □

Lemma 2.4. *Let $t > 1$ and $G(t) = (1 - 2/\lambda)t^{3-2/\lambda} - (3 - 2/\lambda)t^{2-2/\lambda} + (3 - 2/\lambda)t + (2/\lambda) - 1$. Then*

(1) *$G(t) > 0$ for $\lambda \in (2/3, 1)$ and*

(2) *$G(t) < 0$ for $\lambda \in (0, 2/3) \cup (1, 2)$.*

Proof. From the expression of $G(t)$, we clearly see that

$$\lim_{t \to 1} G(t) = 0, \tag{2.25}$$

$$G'(t) = \left(1 - \frac{2}{\lambda}\right)\left(3 - \frac{2}{\lambda}\right)t^{2-2/\lambda} - 2\left(3 - \frac{2}{\lambda}\right)\left(1 - \frac{1}{\lambda}\right)t^{1-2/\lambda} + 3 - \frac{2}{\lambda}, \tag{2.26}$$

$$G'(1) = 0, \tag{2.27}$$

$$G''(t) = \frac{6}{\lambda^3}\left(\lambda - \frac{2}{3}\right)(\lambda - 1)(\lambda - 2)t^{-2/\lambda}(t - 1). \tag{2.28}$$

From (2.28), we have

$$G''(t) > 0 \tag{2.29}$$

for $\lambda \in (2/3, 1)$ and $t > 1$, and

$$G''(t) < 0 \tag{2.30}$$

for $\lambda \in (0, 2/3) \cup (1, 2)$ and $t > 1$.

Therefore, Lemma 2.4(1) follows from (2.25) and (2.27) together with (2.29), and Lemma 2.4(2) follows from (2.25) and (2.27) together with (2.30). □

3. Main Result

Theorem 3.1. *Let $a, b > 0$ with $a \neq b$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$. Then*

(1) $L_{6\alpha+3\beta-5}(a,b) = A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) = L_{-2/(2\alpha+\beta)}(a,b)$ *for* $2\alpha+\beta \in \{2/3, 1\}$;

(2) $L_{6\alpha+3\beta-5}(a,b) > A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) > L_{-2/(2\alpha+\beta)}(a,b)$ *for* $2\alpha+\beta \in (0,2/3) \cup (1,2)$ *and* $L_{6\alpha+3\beta-5}(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < L_{-2/(2\alpha+\beta)}(a,b)$ *for* $2\alpha+\beta \in (2/3,1)$, *and the parameters* $6\alpha+3\beta-5$ *and* $-2/(2\alpha+\beta)$ *cannot be improved in either case.*

Proof. (1) We divide the proof into two cases.

Case 1. If $2\alpha+\beta = 2/3$, then simple computations lead to

$$\begin{aligned} L_{6\alpha+3\beta-5}(a,b) &= L_{-2/(2\alpha+\beta)}(a,b) = L_{-3}(a,b) \\ &= \left(\frac{2a^2b^2}{a+b}\right)^{1/3} = 2^{1-(2\alpha+\beta)}(ab)^{1-(2\alpha+\beta)/2}(a+b)^{2\alpha+\beta-1} \\ &= A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b). \end{aligned} \tag{3.1}$$

Case 2. If $2\alpha+\beta = 1$, then we clearly see that

$$\begin{aligned} L_{6\alpha+3\beta-5}(a,b) &= L_{-2/(2\alpha+\beta)}(a,b) = L_{-2}(a,b) \\ &= (ab)^{1/2} = A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b). \end{aligned} \tag{3.2}$$

(2) Without loss of generality, we assume that $a > b$ and we put $t = a/b > 1$ and $\lambda = 2\alpha+\beta$.

Firstly, we compare $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ with $L_{3\lambda-5}(a,b)$. We divide the proof into five cases.

Case 1. If $\lambda = 4/3$, then (1.1) leads to

$$\begin{aligned} &L_{3\lambda-5}(a,b) - A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \\ &\quad = \frac{a-b}{\log a - \log b} - \left(\frac{a+b}{2}\right)^{\alpha}(ab)^{\beta/2}\left(\frac{2ab}{a+b}\right)^{1-\alpha-\beta} = b\left[\frac{t-1}{\log t} - \left(\frac{t(t+1)}{2}\right)^{1/3}\right]. \end{aligned} \tag{3.3}$$

Therefore, $L_{3\lambda-5}(a,b) > A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ follows from (3.3) and Lemma 2.1.

Case 2. If $\lambda = 5/3$, then from (1.1) we have

$$\log[L_{3\lambda-5}(a,b)] - \log\left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right] = \frac{t}{t-1}\log t - \frac{1}{6}\log t - \frac{2}{3}\log\frac{1+t}{2} - 1. \tag{3.4}$$

From (3.4) and Lemma 2.2, we clearly see that $L_{3\lambda-5}(a,b) > A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$.

Case 3. If $\lambda \in (0,2/3) \cup (1,4/3) \cup (5/3,2)$, then (1.1) yields

$$\log[L_{3\lambda-5}(a,b)] - \log\left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right] = \frac{1}{3\lambda-5}\log\frac{t^{3\lambda-4}-1}{(3\lambda-4)(t-1)} - (\lambda-1)\log\frac{1+t}{2} - \left(1-\frac{\lambda}{2}\right)\log t. \tag{3.5}$$

Let $f(t) = (1/(3\lambda-5))\log((t^{3\lambda-4}-1)/(3\lambda-4)(t-1)) - (\lambda-1)\log((1+t)/2) - (1-\lambda/2)\log t$. Then simple computations lead to

$$\lim_{t\to 1} f(t) = 0, \tag{3.6}$$

$$f'(t) = \frac{g(t)}{(3\lambda-5)\left(t^{3\lambda-4}-1\right)(t^2-1)t}, \tag{3.7}$$

where $g(t) = (1-(1/2)\lambda)(3\lambda-5)t^{3\lambda-2} + [(\lambda-1)(3\lambda-5)-1]t^{3\lambda-3} - [(1/2)\lambda(3\lambda-5)+1]t^{3\lambda-4} + [(1/2)\lambda(3\lambda-5)+1]t^2 - [(\lambda-1)(3\lambda-5)-1]t - (1-(1/2)\lambda)(3\lambda-5)$.

We clearly see that

$$(3\lambda-5)\left(t^{3\lambda-4}-1\right) > 0. \tag{3.8}$$

Therefore, $L_{3\lambda-5}(a,b) > A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ follows from (3.5)–(3.8) and Lemma 2.3(1).

Case 4. If $\lambda \in (2/3,1)$, then (3.5)–(3.8) again hold, and from (3.5)–(3.8) and Lemma 2.3(2), we know that $L_{3\lambda-5}(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$.

Case 5. If $\lambda \in (4/3,5/3)$, then (3.5)–(3.7) hold and

$$(3\lambda-5)\left(t^{3\lambda-4}-1\right) < 0. \tag{3.9}$$

Therefore, $L_{3\lambda-5}(a,b) > A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ follows from (3.5)–(3.7) and (3.9) together with Lemma 2.3(2).

Secondly, we compare $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)$ with $L_{-2/\lambda}(a,b)$.

From (1.1), we have

$$\log[L_{-2/\lambda}(a,b)] - \log\left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right] = -\frac{\lambda}{2}\log\frac{t^{1-\lambda/2}-1}{(1-\lambda/2)(t-1)} - (\lambda-1)\log\frac{1+t}{2} - \left(1-\frac{\lambda}{2}\right)\log t. \tag{3.10}$$

Let $F(t) = -\lambda/2\log((t^{1-(\lambda/2)}-1)/(1-\lambda/2)(t-1)) - (\lambda-1)\log((1+t)/2) - (1-\lambda/2)\log t$. Then simple computations yield

$$\lim_{t\to 1} F(t) = 0, \tag{3.11}$$

$$F'(t) = \frac{\lambda G(t)}{2t(1-t^{1-2/\lambda})(t^2-1)}, \tag{3.12}$$

where $G(t) = (1-2/\lambda)t^{3-2/\lambda} - (3-2/\lambda)t^{2-2/\lambda} + (3-2/\lambda)t + (2/\lambda) - 1$.

We clearly see that

$$1 - t^{1-2/\lambda} > 0 \tag{3.13}$$

for $\lambda \in (0,2)$.

Therefore, $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < L_{-2/\lambda}(a,b)$ for $\lambda \in (2/3,1)$ follows from (3.10)–(3.13) and Lemma 2.4(1), and $A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) > L_{-2/\lambda}(a,b)$ for $\lambda \in (0,2/3)\cup(1,2)$ follows from (3.10)–(3.13) and Lemma 2.4(2).

At last, we prove that the parameters $-2/\lambda$ and $3\lambda-5$ cannot be improved in either case.

The following two cases will complete the proof for the optimality of parameter $-2/\lambda$.

Case I. If $\lambda \in (0,2/3)\cup(1,2)$, then for any $\epsilon \in (0,2/\lambda-1)$, one has

$$\begin{aligned}
&\lim_{t\to+\infty}\frac{L_{-2/\lambda+\epsilon}(1,t)}{A^{\alpha}(1,t)G^{\beta}(1,t)H^{1-\alpha-\beta}(1,t)}\\
&\quad=\lim_{t\to+\infty}\left\{\left[\frac{2/\lambda-1-\epsilon}{1-t^{1+\epsilon-2/\lambda}}\left(1-\frac{1}{t}\right)\right]^{\lambda/(2-\epsilon\lambda)}\left(\frac{1+t}{2t}\right)^{1-\lambda}t^{\epsilon\lambda^2/2(2-\epsilon\lambda)}\right\}\\
&\quad=+\infty.
\end{aligned}\tag{3.14}$$

Equation (3.14) implies that for any $\epsilon \in (0,2/\lambda-1)$, there exists a sufficiently large $T_1 = T_1(\epsilon,\alpha,\beta) > 1$, such that $L_{-2/\lambda+\epsilon}(1,t) > A^{\alpha}(1,t)G^{\beta}(1,t)H^{1-\alpha-\beta}(1,t)$ for $t \in (T_1,+\infty)$.

Case II. If $\lambda \in (2/3,1)$, then for any $\epsilon > 0$, one has

$$\begin{aligned}
&\lim_{t\to+\infty}\frac{A^{\alpha}(1,t)G^{\beta}(1,t)H^{1-\alpha-\beta}(1,t)}{L_{-2/\lambda-\epsilon}(1,t)}\\
&\quad=\lim_{t\to+\infty}\left\{\left(\frac{2t}{1+t}\right)^{1-\lambda}\left[\frac{1-t^{1-2/\lambda-\epsilon}}{(2/\lambda+\epsilon-1)(1-1/t)}\right]^{\lambda/(2+\epsilon\lambda)}t^{\epsilon\lambda^2/2(2+\epsilon\lambda)}\right\}\\
&\quad=+\infty.
\end{aligned}\tag{3.15}$$

Equation (3.15) implies that for any $\epsilon > 0$, there exists a sufficiently large $T_2 = T_2(\epsilon,\alpha,\beta) > 1$, such that $A^{\alpha}(1,t)G^{\beta}(1,t)H^{1-\alpha-\beta}(1,t) > L_{-2/\lambda-\epsilon}(1,t)$ for $t \in (T_2,+\infty)$.

The following seven cases will complete the proof for the optimality of parameter $3\lambda - 5$.

Case A. If $\lambda = 4/3$, then for any $\epsilon > 0$ and $x > 0$, one has

$$[L_{3\lambda-5-\epsilon}(1,1+x)]^{1+\epsilon} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{1+\epsilon} = \frac{g_1(x)}{(1+x)^{\epsilon}-1}, \tag{3.16}$$

where $g_1(x) = \epsilon x(1+x)^{\epsilon} - (1+x/2)^{(1+\epsilon)/3}(1+x)^{(1+\epsilon)/3}[(1+x)^{\epsilon}-1]$.

Upon letting $x \to 0$, the Taylor expansion leads to

$$g_1(x) = -\frac{1}{24}\epsilon^2(\epsilon+1)x^3 + o\left(x^3\right). \tag{3.17}$$

Equations (3.16) and (3.17) imply that for $\epsilon > 0$, there exists a sufficiently small $\delta_1 = \delta_1(\epsilon) > 0$, such that $L_{3\lambda-5-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x \in (0,\delta_1)$.

Case B. If $\lambda = 5/3$, then for any $\epsilon \in (0,1)$ and $x > 0$ $(x \to 0)$, one has

$$\begin{aligned}
&[L_{3\lambda-5-\epsilon}(1,1+x)]^{\epsilon} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{\epsilon} \\
&\quad = \frac{(1-\epsilon)x}{(1+x)^{1-\epsilon}-1} - (1+x)^{\epsilon/6}\left(1+\frac{x}{2}\right)^{(2/3)\epsilon} \\
&\quad = \frac{1}{(1+x)^{1-\epsilon}-1}\left[-\frac{1}{24}(1-\epsilon)\epsilon^2x^3 + o\left(x^3\right)\right].
\end{aligned} \tag{3.18}$$

Equation (3.18) implies that for any $\epsilon \in (0,1)$, there exists a sufficiently small $\delta_2 = \delta_2(\epsilon) > 0$, such that $L_{3\lambda-5-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x \in (0,\delta_2)$.

Case C. If $\lambda \in (0,2/3)$, then for any $\epsilon > 0$ and $x > 0$ $(x \to 0)$, one has

$$\begin{aligned}
&[L_{3\lambda-5-\epsilon}(1,1+x)]^{5+\epsilon-3\lambda} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{5+\epsilon-3\lambda} \\
&\quad = \frac{(4+\epsilon-3\lambda)x(1+x)^{4+\epsilon-3\lambda}}{(1+x)^{4+\epsilon-3\lambda}-1} - \left[(1+x)^{1-\lambda/2}\left(1+\left(\frac{x}{2}\right)\right)^{\lambda-1}\right]^{5+\epsilon-3\lambda} \\
&\quad = \frac{1}{(1+x)^{4+\epsilon-3\lambda}-1}\left[-\frac{\epsilon}{24}(5+\epsilon-3\lambda)(4+\epsilon-3\lambda)x^3 + o\left(x^3\right)\right].
\end{aligned} \tag{3.19}$$

Equation (3.19) implies that for any $\epsilon > 0$, there exists a sufficiently small $\delta_3 = \delta_3(\epsilon,\alpha,\beta) > 0$, such that $L_{3\lambda-5-\epsilon}(1,1+x) < A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x \in (0,\delta_3)$.

Case D. If $\lambda \in (2/3, 1)$, then for any $\epsilon \in (0, 4-3\lambda)$ and $x > 0$ $(x \to 0)$, one has

$$\begin{aligned}
&[L_{3\lambda-5+\epsilon}(1,1+x)]^{5-3\lambda-\epsilon} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{5-3\lambda-\epsilon} \\
&\quad = \frac{(4-3\lambda-\epsilon)x(1+x)^{4-3\lambda-\epsilon}}{(1+x)^{4-3\lambda-\epsilon}-1} - \left[\frac{(1+x)^{1-\lambda/2}}{(1+x/2)^{1-\lambda}}\right]^{5-3\lambda-\epsilon} \\
&\quad = \frac{(\epsilon/24)(5-3\lambda-\epsilon)(4-3\lambda-\epsilon)x^3 + o\left(x^3\right)}{\left[(1+x)^{4-3\lambda-\epsilon}-1\right](1+x/2)^{(1-\lambda)(5-3\lambda-\epsilon)}}.
\end{aligned} \tag{3.20}$$

Equation (3.20) implies that for any $\epsilon \in (0, 4-3\lambda)$, there exists a sufficiently small $\delta_4 = \delta_4(\epsilon, \alpha, \beta) > 0$, such that $L_{3\lambda-5+\epsilon}(1, 1+x) > A^{\alpha}(1, 1+x)G^{\beta}(1, 1+x)H^{1-\alpha-\beta}(1, 1+x)$ for $x \in (0, \delta_4)$.

Case E. If $\lambda \in (1, 4/3)$, then for any $\epsilon > 0$ and $x > 0$ $(x \to 0)$, one has

$$\begin{aligned}
&[L_{3\lambda-5-\epsilon}(1,1+x)]^{5+\epsilon-3\lambda} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{5+\epsilon-3\lambda} \\
&\quad = \frac{(4+\epsilon-3\lambda)x(1+x)^{4+\epsilon-3\lambda}}{(1+x)^{4+\epsilon-3\lambda}-1} - \left[(1+x)^{1-\lambda/2}\left(1+\left(\frac{x}{2}\right)\right)^{\lambda-1}\right]^{5+\epsilon-3\lambda} \\
&\quad = \frac{1}{(1+x)^{4+\epsilon-3\lambda}-1}\left[-(\epsilon/24)(5+\epsilon-3\lambda)(4+\epsilon-3\lambda)x^3 + o\left(x^3\right)\right].
\end{aligned} \tag{3.21}$$

Equation (3.21) implies that for any $\epsilon > 0$, there exists a sufficiently small $\delta_5 = \delta_5(\epsilon, \alpha, \beta) > 0$, such that $L_{3\lambda-5+\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)G^{\beta}(1, 1+x)H^{1-\alpha-\beta}(1, 1+x)$ for $x \in (0, \delta_5)$.

Case F. If $\lambda \in (4/3, 5/3)$, then for any $\epsilon \in (0, 3\lambda-4)$ and $x > 0$ $(x \to 0)$, one has

$$\begin{aligned}
&[L_{3\lambda-5-\epsilon}(1,1+x)]^{5+\epsilon-3\lambda} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{5+\epsilon-3\lambda} \\
&\quad = \frac{(3\lambda-4-\epsilon)x}{(1+x)^{3\lambda-4-\epsilon}-1} - \left[(1+x)^{1-\lambda/2}\left(1+(\frac{x}{2})\right)^{\lambda-1}\right]^{5+\epsilon-3\lambda} \\
&\quad = \frac{1}{(1+x)^{3\lambda-4-\epsilon}-1}\left[-\frac{\epsilon}{24}(5+\epsilon-3\lambda)(3\lambda-4-\epsilon)x^3 + o\left(x^3\right)\right].
\end{aligned} \tag{3.22}$$

Equation (3.22) implies that for any $\epsilon \in (0, 3\lambda-4)$, there exists a sufficiently small $\delta_6 = \delta_6(\epsilon, \alpha, \beta) > 0$, such that $L_{3\lambda-5+\epsilon}(1, 1+x) < A^{\alpha}(1, 1+x)G^{\beta}(1, 1+x)H^{1-\alpha-\beta}(1, 1+x)$ for $x \in (0, \delta_6)$.

Case G. If $\lambda \in (5/3, 2)$, then for any $\epsilon \in (0, 3\lambda - 5)$ and $x > 0$ $(x \to 0)$, one has

$$\begin{aligned}
&[L_{3\lambda-5-\epsilon}(1,1+x)]^{3\lambda-5-\epsilon} - \left[A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)\right]^{3\lambda-5-\epsilon} \\
&\quad = \frac{(1+x)^{3\lambda-4-\epsilon}-1}{(3\lambda-4-\epsilon)x} - \left[(1+x)^{1-\lambda/2}\left(1+\left(\frac{x}{2}\right)\right)^{\lambda-1}\right]^{3\lambda-5-\epsilon} \\
&\quad = \frac{1}{(3\lambda-4-\epsilon)x}\left[-\frac{\epsilon}{24}(3\lambda-4-\epsilon)(3\lambda-5-\epsilon)x^3 + o\left(x^3\right)\right].
\end{aligned} \tag{3.23}$$

Equation (3.23) implies that for any $\epsilon \in (0, 3\lambda - 5)$, there exists a sufficiently small $\delta_7 = \delta_7(\epsilon, \alpha, \beta) > 0$, such that $L_{3\lambda-5+\epsilon}(1,1+x) < A^{\alpha}(1,1+x)G^{\beta}(1,1+x)H^{1-\alpha-\beta}(1,1+x)$ for $x \in (0, \delta_7)$. □

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