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Research Article

On an Integral-Type Operator Acting between Bloch-Type Spaces on the Unit Ball

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Let $\mathbb{B}$ denote the open unit ball of $\mathbb{C}^n$. For a holomorphic self-map φ of $\mathbb{B}$ and a holomorphic function g in $\mathbb{B}$ with $g(0) = 0$, we define the following integral-type operator: $I_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)(dt/t)$, $z \in \mathbb{B}$. Here $\Re f$ denotes the radial derivative of a holomorphic function f in $\mathbb{B}$. We study the boundedness and compactness of the operator between Bloch-type spaces $\mathcal{B}_\omega$ and $\mathcal{B}_\mu$, where ω is a normal weight function and μ is a weight function. Also we consider the operator between the little Bloch-type spaces $\mathcal{B}_{\omega,0}$ and $\mathcal{B}_{\mu,0}$.

1. Introduction

Let $\mathbb{B}$ denote the open unit ball of the n-dimensional complex vector space $\mathbb{C}^n$ and $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$. For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\gamma|\ge 0} a_\gamma z^\gamma$, let

$$\Re f(z) = \sum_{|\gamma|\ge 0} |\gamma| a_\gamma z^\gamma \tag{1.1}$$

be the radial derivative of f, where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, and $z^\gamma = z_1{}^{\gamma_1} \ldots z_n{}^{\gamma_n}$. It is well known that

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \langle \nabla f(z), \overline{z} \rangle, \tag{1.2}$$

where ∇ is the usual gradient on $\mathbb{C}^n$.

Let φ be a holomorphic self-map of $\mathbb{B}$ and $g \in H(\mathbb{B})$ with $g(0) = 0$. Then φ and g define an operator I_φ^g on $H(\mathbb{B})$ as follows:

$$I_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), z \in \mathbb{B}. \tag{1.3}$$

The following important formula involving $\Re$ and $I_\varphi^g f$ was proved, for example, in [1]

$$\Re\left[I_\varphi^g f\right](z) = \Re f(\varphi(z)) g(z), \quad z \in \mathbb{B}. \tag{1.4}$$

Motivated by papers [2, 3], operators I_φ^g were introduced by the first author of the present paper and Zhu in [1, 4–6], where its boundedness and compactness from the α-Bloch space, the Zygmund space, the mixed-norm space, and the generalized weighted Bergman space into the Bloch-type space on the unit ball are studied. In our previous work [7], we studied the boundedness and compactness of I_φ^g acting between weighted-type spaces. For related operators on $\mathbb{C}^n$ see, for example, [8–21] and the references therein.

Let ω be a strictly positive continuous function on $\mathbb{B}$ (*weight*). If $\omega(z) = \omega(|z|)$ for every $z \in \mathbb{B}$, we call it *radial weight*. A weight ω is called *normal* ([9, 22]) if it is radial and there are a and b, $0 < a < b < \infty$ such that $\omega(r)/(1-r)^a$ is decreasing on $[0,1)$, $\omega(r)/(1-r)^b$ is increasing on $[0,1)$,

$$\lim_{r\to 1} \frac{\omega(r)}{(1-r)^a} = 0, \qquad \lim_{r\to 1} \frac{\omega(r)}{(1-r)^b} = \infty. \tag{1.5}$$

A radial weight ω is called *typical* if it is nonincreasing with respect to $|z|$ and $\omega(z) \to 0$ as $|z| \to 1^-$. If ω is normal, then by the monotonicity of $\omega(r)/(1-r)^a$, for $0 \le r_1 < r < 1$, we have that

$$\omega(r) = (1-r)^a \frac{\omega(r)}{(1-r)^a} < (1-r)^a \frac{\omega(r_1)}{(1-r_1)^a} < \omega(r_1), \tag{1.6}$$

that is, ω is decreasing on $[0,1)$. On the other hand, from the first equality in (1.5), we have that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < \omega(r) < \varepsilon(1-r)^a, \quad (\delta < r < 1), \tag{1.7}$$

which implies $\lim_{r\to 1^-} \omega(r) = 0$. Hence every normal weight ω is also typical.

For a weight ω, the *associated weight* $\tilde{\omega}$ ([23]) is defined by

$$\tilde{\omega}(z) := \frac{1}{\sup\left\{ |f(z)| : f \in H_\omega^\infty, \|f\|_{H_\omega^\infty} \le 1 \right\}}, \quad z \in \mathbb{B}. \tag{1.8}$$

Here H_ω^∞ denotes the *weighted-type space* consisting of all $f \in H(\mathbb{B})$ with

$$\|f\|_{H_\omega^\infty} = \sup_{z\in\mathbb{B}} \omega(z)|f(z)| < \infty \tag{1.9}$$

(see, e.g., [23, 24]). Associated weights assist us in studying of weighted-type spaces of holomorphic functions. It is known that associated weights are also continuous, $0 < \omega \le \widetilde{\omega}$, and for each $z \in \mathbb{B}$, we can find an $f_z \in H_\omega^\infty, \|f_z\|_{H_\omega^\infty} \le 1$ such that $f_z(z) = 1/\widetilde{\omega}(z)$. Let $H_{\omega,0}^\infty$ be the *little weighted-type space*, that is, the space of all $f \in H(\mathbb{B})$ such that $\omega(z)|f(z)| \to 0$ as $|z| \to 1^-$. If ω is typical, then the unit ball $B_{H_\omega^\infty}$ is the closure of $B_{H_{\omega,0}^\infty}$ for the compact open topology. Hence we have

$$\widetilde{\omega}(z) = \frac{1}{\sup\left\{|f(z)| : f \in H_{\omega,0}^\infty, \|f\|_{H_\omega^\infty} \le 1\right\}} \tag{1.10}$$

and so for each $z \in \mathbb{B}$, we can choose an $f_z \in B_{H_{\omega,0}^\infty}$ such that $f_z(z) = 1/\widetilde{\omega}(z)$. A weight ω is called *essential* if it satisfies that $\widetilde{\omega} \le C\omega$ for some positive constant C. By the arguments in [25], we see that a normal weight function is also essential. For some examples of essential weights, see, for example, [25]. Related results can also be found in [22, 26].

The *Bloch-type space* $\mathcal{B}_\omega$ is the space of all holomorphic functions f on $\mathbb{B}$ such that

$$b_\omega(f) = \sup_{z\in\mathbb{B}} \omega(z)|\Re f(z)| < \infty, \tag{1.11}$$

where ω is a weight (see, e.g., [20]). The *little Bloch-type space* $\mathcal{B}_{\omega,0}$ consists of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1^-} \omega(z)|\Re f(z)| = 0. \tag{1.12}$$

Both spaces $\mathcal{B}_\omega$ and $\mathcal{B}_{\omega,0}$ are Banach spaces with the norm

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + b_\omega(f), \tag{1.13}$$

and $\mathcal{B}_{\omega,0}$ is a closed subspace of $\mathcal{B}_\omega$. When $\omega(r) = 1 - r^2$, the space $\mathcal{B}_\omega$ is a classical Bloch space.

The purpose of this paper is to characterize the boundedness and compactness of the operators $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ and $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$.

Throughout this paper, we assume that φ is a holomorphic self-map of $\mathbb{B}$ and $g \in H(\mathbb{B})$ with $g(0) = 0$. Furthermore, some constants are denoted by C; they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there exists a positive constant C such that $a \le Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary Results

Here we formulate and prove some auxiliary results which are used in the proofs of the main ones.

The following lemma was proved in [20, Theorem 2.1].

Lemma 2.1. *Let ω be a normal weight function and $f \in H(\mathbb{B})$. Then $f \in \mathcal{B}_\omega$ if and only if $\sup_{z\in\mathbb{B}} \omega(z)|\nabla f(z)| < \infty$ and it holds that*

$$\|f\|_{\mathcal{B}_\omega} \asymp |f(0)| + \sup_{z\in\mathbb{B}} \omega(z)|\nabla f(z)|. \tag{2.1}$$

Moreover, $f \in \mathcal{B}_{\omega,0}$ if and only if $\lim_{|z|\to 1^-} \omega(z)|\nabla f(z)| = 0$.

As an application of Lemma 2.1, we have the following result.

Lemma 2.2. *Let ω be a normal weight function and $f \in \mathcal{B}_\omega$. Then $f \in \mathcal{B}_{\omega,0}$ if and only if it holds that $\lim_{r\to 1^-} \|f_r - f\|_{\mathcal{B}_\omega} = 0$, where $f_r(z) = f(rz)$.*

Proof. Take an $f \in \mathcal{B}_{\omega,0}$. For a fixed $\varepsilon > 0$, by Lemma 2.1, we can choose a $\delta_0 \in (0,1)$ such that

$$\omega(z)|\nabla f(z)| < \frac{\varepsilon}{2} \tag{2.2}$$

for any $z \in \mathbb{B} \setminus {\delta_0}^2\overline{\mathbb{B}}$. Since $(\partial f_r/\partial z_j)(z) = r(\partial f/\partial z_j)\,(rz)$ for $j \in \{1,\ldots,n\}$, $r \in (0,1)$, and $z \in \mathbb{B}$, we have

$$\begin{aligned}\|f_r - f\|_{\mathcal{B}_\omega} &\asymp \sup_{z\in\mathbb{B}} \omega(z)|r\nabla f(rz) - \nabla f(z)| \\ &\le \sup_{z\in\mathbb{B}\setminus\delta_0\overline{\mathbb{B}}} \omega(z)|r\nabla f(rz) - \nabla f(z)| \\ &\quad + \sup_{z\in\delta_0\overline{\mathbb{B}}} \omega(z)|r\nabla f(rz) - \nabla f(z)|.\end{aligned} \tag{2.3}$$

Since

$$\max_{|z|\le\delta_0} |r\nabla f(rz) - \nabla f(z)| \longrightarrow 0, \quad \text{as } r \longrightarrow 1^-, \tag{2.4}$$

we see that the second term in (2.3) converges to 0 as $r \to 1^-$.

If $r \in (\delta_0, 1)$ and $z \in \mathbb{B} \setminus \delta_0\overline{\mathbb{B}}$, then by (2.2) we have

$$\omega(rz)|\nabla f(rz)| < \frac{\varepsilon}{2}. \tag{2.5}$$

By (1.6) we have that $\omega(z) \le \omega(rz)$ for $r, |z| \in [0,1)$.

Hence we have

$$\sup_{z\in\mathbb{B}\setminus\delta_0\overline{\mathbb{B}}} \omega(z)|r\nabla f(rz) - \nabla f(z)| \le \sup_{z\in\mathbb{B}\setminus\delta_0\overline{\mathbb{B}}} \omega(rz)|\nabla f(rz)| + \sup_{z\in\mathbb{B}\setminus\delta_0\overline{\mathbb{B}}} \omega(z)|\nabla f(z)| < \varepsilon, \tag{2.6}$$

for all $r \in (\delta_0, 1)$. This proves that $\lim_{r\to 1^-} \|f_r - f\|_{\mathcal{B}_\omega} = 0$ whenever $f \in \mathcal{B}_{\omega,0}$.

Conversely, the normality of ω implies that for any $\varepsilon > 0$ we have

$$\omega(z)\left|\nabla f(rz)\right| \leq \varepsilon(1-|z|)^{a} \sup_{|w|\leq r}\left|\nabla f(w)\right| \longrightarrow 0, \quad \text{as } |z| \longrightarrow 1^{-}, \tag{2.7}$$

so that $f_r \in \mathcal{B}_{\omega,0}$ for any $r \in (0,1)$. On the other hand, by the assumption $\lim_{r\to 1^-}\|f_r - f\|_{\mathcal{B}_\omega} = 0$, we have that for every $\varepsilon > 0$ there is an $r_1 \in (0,1)$ such that

$$\left\|f_r - f\right\|_{\mathcal{B}_\omega} < \varepsilon, \tag{2.8}$$

for $r \in (r_1, 1)$.

By letting $|z| \to 1^-$ in the following inequality, which easily follows from Lemma 2.1:

$$\omega(z)\left|\nabla f(z)\right| \preccurlyeq \omega(z)\left|\nabla f(rz)\right| + \left\|f_r - f\right\|_{\mathcal{B}_\omega}, \tag{2.9}$$

then using (2.7) and (2.8), we get $f \in \mathcal{B}_{\omega,0}$, as claimed. □

Corollary 2.3. *Let ω be a normal weight function. Then the set of all holomorphic polynomials is dense in $\mathcal{B}_{\omega,0}$.*

Proof. For the homogeneous expansion $f = \sum_{k=0}^{\infty} F_k$ of an $f \in \mathcal{B}_{\omega,0}$, we set $P^j = \sum_{k=0}^{j} F_k$ for each $j \in \mathbb{N}$. Since $P^j \to f$ uniformly on compact subsets of $\mathbb{B}$ as $j \to \infty$, we see that $\mathfrak{R}[P_r^j] \to \mathfrak{R}[f_r]$ uniformly on $\mathbb{B}$ for any $r \in (0,1)$. Moreover, we have

$$\begin{aligned}\left\|P_r^j - f\right\|_{\mathcal{B}_\omega} &\leq \left\|P_r^j - f_r\right\|_{\mathcal{B}_\omega} + \left\|f_r - f\right\|_{\mathcal{B}_\omega} \\ &\leq \sup_{z\in r\overline{\mathbb{B}}}\omega(z) \sup_{z\in\mathbb{B}}\left|\mathfrak{R}\left[P_r^j\right](z) - \mathfrak{R}\left[f_r\right](z)\right| + \left\|f_r - f\right\|_{\mathcal{B}_\omega}.\end{aligned} \tag{2.10}$$

Combining this with Lemma 2.2, we get the desired result. □

The following lemma can be found in [1, Lemma 3]. Its proof is similar to the proof of the corresponding one-dimensional result in [27], for the case of the little Bloch space $\mathcal{B}_{(1-r),0}$. Hence we omit the proof.

Lemma 2.4. *A closed subset K in $\mathcal{B}_{\omega,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1^-} \sup_{f\in K} \omega(z)\left|\mathfrak{R}f(z)\right| = 0. \tag{2.11}$$

The following lemma is very useful for estimating the norm of the Bloch-type space.

Lemma 2.5. *Assume that m is a positive integer and ω is normal. Then for every $f \in H(\mathbb{B})$,*

$$\sup_{z\in\mathbb{B}} \omega(z)\left|f(z)\right| \asymp \left|f(0)\right| + \sup_{z\in\mathbb{B}}(1-|z|)^m \omega(z)\left|\mathfrak{R}^m f(z)\right|. \tag{2.12}$$

Proof. For the details of the proof, we can refer [9] or [28]. □

3. The Boundedness of Operator I_φ^g

In this section we consider the boundedness of the operator $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ or $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$.

Theorem 3.1. *Let ω be a normal weight function and μ a weight function. Then the following conditions are equivalent:*

(a) $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ *is bounded;*

(b) $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ *is bounded;*

(c) φ *and* g *satisfy*

$$\sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} < \infty. \tag{3.1}$$

Moreover, if $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is bounded, then

$$\left\|I_\varphi^g\right\|_{\mathcal{B}_\omega \to \mathcal{B}_\mu} \asymp \sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))}. \tag{3.2}$$

Proof. The implication (a) $\Rightarrow$ (b) is clear, so we only prove (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c): assume that $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ is bounded and fix $z \in \mathbb{B}$. We may assume that $\varphi(z) \neq 0$. For $w := \varphi(z)$, there exists $h_w \in H_{\omega,0}^\infty$ such that $\|h_w\|_{H_\omega^\infty} \le 1$ and $h_w(w) = 1/\widetilde{\omega}(w)$. We define the function f_w as follows:

$$f_w(v) = \int_0^1 h_w(tv) \frac{\langle tv, w\rangle}{|w|} \frac{dt}{t}, \quad v \in \mathbb{B}. \tag{3.3}$$

Since $\Re f_w(v) = h_w(v)(\langle v, w\rangle/|w|)$, we see that $f_w \in \mathcal{B}_{\omega,0}$ and $\|f_w\|_{\mathcal{B}_\omega} \le 1$. Hence, by (1.4), we have

$$\left\|I_\varphi^g\right\|_{\mathcal{B}_\omega \to \mathcal{B}_\mu} \ge \left\|I_\varphi^g f_w\right\|_{\mathcal{B}_\mu} \ge \mu(z)|g(z)||\Re f_w(\varphi(z))| = \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))}, \tag{3.4}$$

and so condition (3.1) is true.

(c) $\Rightarrow$ (a): we assume (3.1) and take an $f \in \mathcal{B}_\omega$. Since ω is an essential weight (due to its normality), (1.4) gives

$$\begin{aligned} \mu(z)\left|\Re\left[I_\varphi^g f\right](z)\right| &= \mu(z)|g(z)||\Re f(\varphi(z))| \\ &\le \mu(z)|g(z)||\varphi(z)| \frac{\widetilde{\omega}(\varphi(z))|\nabla f(\varphi(z))|}{\widetilde{\omega}(\varphi(z))} \\ &\preceq \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} \sup_{w\in\mathbb{B}} \omega(w)|\nabla f(w)|, \end{aligned} \tag{3.5}$$

for any $z \in \mathbb{B}$. By Lemma 2.1, we have $\sup_{w\in\mathbb{B}} \omega(w)|\nabla f(w)| \le \|f\|_{\mathcal{B}_\omega}$, and so we obtain

$$\left\| I_\varphi^g f \right\|_{\mathcal{B}_\mu} \le \sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} \|f\|_{\mathcal{B}_\omega}. \tag{3.6}$$

This implies that $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is bounded. The relation (3.2) follows from (3.4) and (3.6). This completes the proof. □

Theorem 3.2. *Let ω be a normal weight function and μ a weight function. Then the following conditions are equivalent:*

(a) $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$ *is bounded;*

(b) φ *and* g *satisfy*

$$\lim_{|z|\to 1^-} \mu(z)|g(z)||\varphi(z)| = 0, \qquad \sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} < \infty. \tag{3.7}$$

Proof. (a) $\Rightarrow$ (b): as in the proof of Theorem 3.1, for fixed $z \in \mathbb{B}$ and $w = \varphi(z)$, we see that φ and g satisfy the condition

$$\sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} < \infty. \tag{3.8}$$

On the other hand, since the normality of ω implies that the function $\pi_j(z) := z_j$ $(1 \le j \le n)$ belongs to $\mathcal{B}_{\omega,0}$, we obtain that $\mu(z)|g(z)|\,|\varphi_j(z)| \to 0$ for each j, and so $\mu(z)|g(z)||\varphi(z)| \to 0$ as $|z| \to 1^-$.

(b) $\Rightarrow$ (a): the assumption $\lim_{|z|\to 1^-} \mu(z)|g(z)||\varphi(z)| = 0$ shows that $I_\varphi^g p \in \mathcal{B}_{\mu,0}$ for any polynomial p. For each $f \in \mathcal{B}_{\omega,0}$, by Corollary 2.3, we can choose a sequence of polynomials $\{p_j\}_{j\in\mathbb{N}}$ such that $\|f - p_j\|_{\mathcal{B}_\omega} \to 0$ as $j \to \infty$. Furthermore, the assumption

$$\sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} < \infty \tag{3.9}$$

shows that $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ is bounded by Theorem 3.1. Thus we obtain

$$0 \le \left\| I_\varphi^g f - I_\varphi^g p_j \right\|_{\mathcal{B}_\mu} \le \left\| I_\varphi^g \right\|_{\mathcal{B}_\omega \to \mathcal{B}_\mu} \|f - p_j\|_{\mathcal{B}_\omega} \longrightarrow 0 \quad (\text{as } j \longrightarrow \infty). \tag{3.10}$$

Since $I_\varphi^g f \in \mathcal{B}_\mu$, $\{I_\varphi^g p_j\}_{j\in\mathbb{N}} \subset \mathcal{B}_{\mu,0}$, and $\mathcal{B}_{\mu,0}$ is closed in $\mathcal{B}_\mu$, we have $I_\varphi^g f \in \mathcal{B}_{\mu,0}$ for any $f \in \mathcal{B}_{\omega,0}$. Hence $I_\varphi^g(\mathcal{B}_{\omega,0}) \subseteq \mathcal{B}_{\mu,0}$ which means that $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$ is bounded. The proof is accomplished. □

The following corollary is an immediate consequence of Theorems 3.1 and 3.2.

Corollary 3.3. *Let ω be a normal weight function and μ a weight function. Then $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$ is bounded if and only if $\lim_{|z|\to 1^-} \mu(z)|g(z)||\varphi(z)| = 0$ and $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is bounded.*

4. The Compactness of Operator I_φ^g

In this section we characterize the compactness of $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ or $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$. To do this, we need the following standard lemma (see, e.g., [13, Lemma 3]).

Lemma 4.1. *Let ω and μ be weight functions. Suppose that the operator $I_\varphi^g : \mathcal{B}_\omega$ (or $\mathcal{B}_{\omega,0}$) $\to \mathcal{B}_\mu$ is bounded. Then $I_\varphi^g : \mathcal{B}_\omega$ (or $\mathcal{B}_{\omega,0}$) $\to \mathcal{B}_\mu$ is compact if and only if for every bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{B}_\omega$ (or $\mathcal{B}_{\omega,0}$) which converges to 0 uniformly on compact subsets of $\mathbb{B}$, $\|I_\varphi^g f_j\|_{\mathcal{B}_\mu} \to 0$ as $j \to \infty$.*

Theorem 4.2. *Let ω and μ be weight functions. Suppose that φ is a holomorphic self-map of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$ and the operator $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is bounded. Then $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is compact. Here $\|\varphi\|_\infty$ denotes the supremum $\sup_{z\in\mathbb{B}} |\varphi(z)|$.*

Proof. Since $\|\varphi\|_\infty < 1$, we see that $|\varphi(z)| \le r$ for some $r \in (0,1)$ and any $z \in \mathbb{B}$. From the proof of Theorem 3.1, we see that the boundedness of $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ implies

$$M := \sup_{z\in\mathbb{B}} \frac{\mu(z)\left|g(z)\right|\left|\varphi(z)\right|}{\widetilde{\omega}(\varphi(z))} < \infty. \tag{4.1}$$

Thus we obtain that

$$\sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right|\left|\varphi(z)\right| \le M \sup_{w\in r\overline{\mathbb{B}}} \widetilde{\omega}(w) < \infty. \tag{4.2}$$

Take a bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{B}_\omega$ such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $j \to \infty$. By (1.4), we have

$$\begin{aligned} \left\|I_\varphi^g f_j\right\|_{B_\mu} &= \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right|\left|\Re f_j(\varphi(z))\right| \\ &\le \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right|\left|\varphi(z)\right|\left|\nabla f_j(\varphi(z))\right| \\ &\le M \sup_{w\in r\overline{\mathbb{B}}} \widetilde{\omega}(w) \sup_{w\in r\overline{\mathbb{B}}} \left|\nabla f_j(w)\right|. \end{aligned} \tag{4.3}$$

Since $\partial f_j/\partial z_k$ $(1 \le k \le n)$ also converges to 0 uniformly on $r\overline{\mathbb{B}}$ as $j \to \infty$, (4.2) and (4.3) show that $\|I_\varphi^g f_j\|_{\mathcal{B}_\mu} \to 0$ as $j \to \infty$. From Lemma 4.1, it follows that $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is compact, and so we get the assertions. □

Lemma 4.3. *Suppose that ω is a weight function. Then there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ in the closed unit ball of $\mathcal{B}_\omega$ such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$.*

Proof. Let $\{w_k\}_{k\in\mathbb{N}} \subset \mathbb{B}$ with $|w_k| \to 1^-$ as $k \to \infty$. For each w_k, there exists $h_k := h_{w_k} \in H_\omega^\infty$ such that $\|h_k\|_{H_\omega^\infty} \le 1$ and $h_k(w_k) = 1/\widetilde{\omega}(w_k)$. We define f_k as follows:

$$f_k(z) = \int_0^1 h_k(tz)\left\{\frac{\langle tz, w_k\rangle}{|w_k|}\right\}^{1/(1-|w_k|)} \frac{dt}{t}, \quad z \in \mathbb{B}. \tag{4.4}$$

Since $f_k(0) = 0$ and $|\Re f_k(z)| \le |h_k(z)|$, we have $\{f_k\}_{k\in\mathbb{N}} \subset \mathcal{B}_\omega$ and $\|f_k\|_{\mathcal{B}_\omega} \le 1$ for each $k \in \mathbb{N}$. For any compact subset $\mathcal{K}$ of $\mathbb{B}$, we can choose an $r \in (0,1)$ such that $\mathcal{K} \subset r\overline{\mathbb{B}}$. Hence we obtain that for any $z \in \mathcal{K}$

$$\left|f_k(z)\right| \le \int_0^1 |h_k(tz)| t^{|w_k|/(1-|w_k|)} dt \le \|h_k\|_{H^\infty_{\tilde{\omega}}} \int_0^1 \frac{1}{\omega(tz)} t^{|w_k|/(1-|w_k|)} dt \le \max_{w\in r\overline{\mathbb{B}}} \frac{1}{\omega(w)} (1 - |w_k|). \tag{4.5}$$

From the above inequality, it follows that f_k converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. This completes the proof. □

Remark 4.4. If we assume that ω is typical in Lemma 4.3, then we can choose $h_k \in H^\infty_{\omega,0}$. In this case, hence, we see that f_k belongs to $\mathcal{B}_{\omega,0}$ for each $k \in \mathbb{N}$.

Theorem 4.5. *Let ω be a normal weight function and μ a weight function. Suppose that the operator $I^g_\varphi : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is bounded and $\|\varphi\|_\infty = 1$. Then the following conditions are equivalent:*

(a) *$I^g_\varphi : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is compact;*

(b) *$I^g_\varphi : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ is compact;*

(c) *φ and g satisfy*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(z)\left|g(z)\right|\left|\varphi(z)\right|}{\widetilde{\omega}\left(\varphi(z)\right)} = 0. \tag{4.6}$$

Proof. (a) ⇒ (b): this implication is obvious.

(b) ⇒ (c): take a sequence $\{z_k\}_{k\in\mathbb{N}}$ in $\mathbb{B}$ with $|\varphi(z_k)| \to 1^-$ as $k \to \infty$ and put $w_k = \varphi(z_k)$ for each k. Then, by Remark 4.4 after Lemma 4.3, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{B}_{\omega,0}$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}_\omega} \le 1$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. By Lemma 4.1, the compactness of $I^g_\varphi : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ implies that $\|I^g_\varphi f_k\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$.

On the other hand, (1.4) gives $\Re[I^g_\varphi f_k](z) = \Re f_k(\varphi(z))g(z)$, and so we have

$$\left\|I^g_\varphi f_k\right\|_{\mathcal{B}_\mu} \ge \mu(z_k)\left|\Re f_k\left(\varphi(z_k)\right)\right|\left|g(z_k)\right| \ge \mu(z_k)\left|\Re f_k\left(\varphi(z_k)\right)\right|\left|g(z_k)\right|\left|\varphi(z_k)\right|. \tag{4.7}$$

From the construction (4.4) of f_k, we obtain

$$\Re f_k\left(\varphi(z_k)\right) = \frac{\left|\varphi(z_k)\right|^{1/(1-|\varphi(z_k)|)}}{\widetilde{\omega}\left(\varphi(z_k)\right)}, \tag{4.8}$$

for each $k \in \mathbb{N}$. Combining this with (4.7), we have

$$\left\|I^g_\varphi f_k\right\|_{\mathcal{B}_\mu} \ge \frac{\mu(z_k)\left|g(z_k)\right|\left|\varphi(z_k)\right|}{\widetilde{\omega}\left(\varphi(z_k)\right)} \left|\varphi(z_k)\right|^{1/(1-|\varphi(z_k)|)}. \tag{4.9}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \frac{\mu(z_k)\left|g(z_k)\right|\left|\varphi(z_k)\right|}{\widetilde{\omega}(\varphi(z_k))} = 0, \tag{4.10}$$

for any sequence $\{z_k\}_{k\in\mathbb{N}}$ with $|\varphi(z_k)| \to 1^-$. This proves that (4.6) is true.

(c) $\Rightarrow$ (a): we will prove the following estimate:

$$\left\| I_\varphi^g \right\|_e \preceq \limsup_{|\varphi(z)|\to 1^-} \frac{\mu(z)\left|g(z)\right|\left|\varphi(z)\right|}{\widetilde{\omega}(\varphi(z))}. \tag{4.11}$$

Here $\|I_\varphi^g\|_e$ denotes the essential norm of $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$, namely,

$$\left\| I_\varphi^g \right\|_e = \inf\left\{ \left\| I_\varphi^g + \mathcal{K} \right\|_{\mathcal{B}_\omega \to \mathcal{B}_\mu} \mid \mathcal{K} : \mathcal{B}_\omega \longrightarrow \mathcal{B}_\mu \text{ is compact} \right\}. \tag{4.12}$$

Now we take a sequence $\{r_l\}_{l\in\mathbb{N}} \subset (0,1)$ which increasingly converges to 1 and put

$$I_{r_l\varphi}^g f(z) = \int_0^1 \Re f(r_l\varphi(tz)) g(tz) \frac{dt}{t}. \tag{4.13}$$

Since $\|r_l\varphi\|_\infty \le r_l < 1$, Theorem 4.2 implies that $I_{r_l\varphi}^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is compact for each $l \in \mathbb{N}$. For any $f \in \mathcal{B}_\omega$ with $\|f\|_{\mathcal{B}_\omega} \le 1$, from (1.4) it follows that

$$\begin{aligned} \left\| I_\varphi^g f - I_{r_l\varphi}^g f \right\|_{\mathcal{B}_\omega} &= \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right|\left|\Re f(\varphi(z)) - \Re f(r_l\varphi(z))\right| \\ &\le \sup_{R<|\varphi(z)|<1} \mu(z)\left|g(z)\right|\left|\Re f(\varphi(z)) - \Re f(r_l\varphi(z))\right| \\ &\quad + \sup_{|\varphi(z)|\le R} \mu(z)\left|g(z)\right|\left|\Re f(\varphi(z)) - \Re f(r_l\varphi(z))\right|, \end{aligned} \tag{4.14}$$

for some fixed $R \in (0,1)$. The essentiality of ω and Lemma 2.1 give

$$\begin{aligned} \mu(z)\left|g(z)\right|\left|\Re f(\varphi(z))\right| &\le \frac{\mu(z)\left|g(z)\right|\left|\varphi(z)\right|}{\widetilde{\omega}(\varphi(z))} \widetilde{\omega}(\varphi(z))\left|\nabla f(\varphi(z))\right| \\ &\preceq \frac{\mu(z)\left|g(z)\right|\left|\varphi(z)\right|}{\widetilde{\omega}(\varphi(z))} \sup_{w\in\mathbb{B}} \omega(w)\left|\nabla f(w)\right| \\ &\preceq \frac{\mu(z)\left|g(z)\right|\left|\varphi(z)\right|}{\widetilde{\omega}(\varphi(z))}. \end{aligned} \tag{4.15}$$

Similarly, we also have

$$\mu(z)|g(z)||\Re f(r_l\varphi(z))| \preceq \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(r_l\varphi(z))}, \tag{4.16}$$

for each $l \in \mathbb{N}$. The normality of ω implies that

$$\frac{\omega(r_l\varphi(z))}{(1-|r_l\varphi(z)|)^a} \geq \frac{\omega(\varphi(z))}{(1-|\varphi(z)|)^a}, \tag{4.17}$$

for each $l \in \mathbb{N}$ and some $a > 0$, and so by the essentiality,

$$\frac{\widetilde{\omega}(r_l\varphi(z))}{(1-|r_l\varphi(z)|)^a} \succeq \frac{\widetilde{\omega}(\varphi(z))}{(1-|\varphi(z)|)^a}. \tag{4.18}$$

Thus (4.16) and (4.18) give

$$\mu(z)|g(z)||\Re f(r_l\varphi(z))| \preceq \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))}, \tag{4.19}$$

for each $l \in \mathbb{N}$. By (4.15) and (4.19), we obtain

$$\sup_{R<|\varphi(z)|<1} \mu(z)|g(z)||\Re f(\varphi(z)) - \Re f(r_l\varphi(z))| \preceq \sup_{R<|\varphi(z)|<1} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))}. \tag{4.20}$$

When $|\varphi(z)| \leq R$, by using the mean value theorem, we have

$$\begin{aligned} \mu(z)|g(z)||\Re f(\varphi(z)) - \Re f(r_l\varphi(z))| &\leq (1-r_l)\mu(z)|g(z)||\varphi(z)|\sup_{|w|\leq R}|\nabla[\Re f](w)| \\ &\leq \frac{1-r_l}{1-R}\max_{w\in R\overline{\mathbb{B}}}\frac{1}{\omega(w)}\sup_{z\in\mathbb{B}}\mu(z)|g(z)||\varphi(z)| \\ &\quad\times \sup_{w\in\mathbb{B}}\omega(w)(1-|w|)|\nabla[\Re f](w)|. \end{aligned} \tag{4.21}$$

Since $\omega(w)(1-|w|)$ is also normal, by Lemmas 2.1 and 2.5, we have

$$\begin{aligned} \sup_{w\in\mathbb{B}}\omega(w)(1-|w|)|\nabla[\Re f](w)| &\asymp \sup_{w\in\mathbb{B}}\omega(w)(1-|w|)|\Re^2 f(w)| \\ &\asymp \sup_{w\in\mathbb{B}}\omega(w)|\Re f(w)|. \end{aligned} \tag{4.22}$$

Hence we obtain

$$\sup_{|\varphi(z)|\le R} \mu(z)|g(z)||\Re f(\varphi(z)) - \Re f(r_l\varphi(z))| \le \frac{1-r_l}{1-R} \max_{w\in r\overline{\mathbb{B}}} \frac{1}{\omega(w)} \sup_{z\in\mathbb{B}} \mu(z)|g(z)||\varphi(z)|. \tag{4.23}$$

Since the boundedness of $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ implies $\sup_{z\in\mathbb{B}} \mu(z)|g(z)||\varphi(z)| < \infty$, letting $l \to \infty$ in the above inequality, we have

$$\sup_{\|f\|_{\mathcal{B}_\omega}\le 1} \sup_{|\varphi(z)|\le R} \mu(z)|g(z)||\Re f(\varphi(z)) - \Re f(r_l\varphi(z))| \longrightarrow 0. \tag{4.24}$$

By using (4.14), (4.20), and (4.24) and letting $R \to 1^-$, we obtain the desired estimate

$$\left\|I_\varphi^g\right\|_e \le \limsup_{|\varphi(z)|\to 1^-} \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))}. \tag{4.25}$$

So if condition (4.6) is true, then $\|I_\varphi^g\|_e = 0$, which means that $I_\varphi^g : \mathcal{B}_\omega \to \mathcal{B}_\mu$ is compact. Our proof is accomplished. □

Theorem 4.6. *Let ω be a normal weight function and μ a weight function. Suppose that the operator $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$ is bounded. Then $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1^-} \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))} = 0. \tag{4.26}$$

Proof. Suppose that (4.26) holds. For any $f \in \mathcal{B}_{\omega,0}$, by Lemma 2.1 and (1.4), we have

$$\begin{aligned} \mu(z)\left|\Re\left[I_\varphi^g f\right](z)\right| &= \mu(z)|\Re f(\varphi(z))||g(z)| \\ &\le \mu(z)|g(z)||\varphi(z)||\nabla f(\varphi(z))| \\ &\le \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))}\omega(\varphi(z))|\nabla f(\varphi(z))| \\ &\le \frac{\mu(z)|g(z)||\varphi(z)|}{\tilde{\omega}(\varphi(z))}\|f\|_{\mathcal{B}_\omega}. \end{aligned} \tag{4.27}$$

Combining this with (4.26), we obtain

$$\lim_{|z|\to 1^-} \sup_{\|f\|_{\mathcal{B}_\omega}\le 1} \mu(z)\left|\Re\left[I_\varphi^g f\right](z)\right| = 0. \tag{4.28}$$

Hence it follows from Lemma 2.4 that $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$ is compact.

Conversely, we assume that $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_{\mu,0}$ is compact. By Theorem 3.2, we see that

$$\lim_{|z|\to 1^-} \mu(z)|g(z)||\varphi(z)| = 0. \tag{4.29}$$

Thus this implies (4.26) if $\|\varphi\|_\infty < 1$.

Now assume $\|\varphi\|_\infty = 1$. We claim that

$$\limsup_{|z|\to 1^-} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} = \limsup_{|\varphi(z)|\to 1^-} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))}. \tag{4.30}$$

Further assume that $\{z_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that

$$\limsup_{|z|\to 1^-} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} = \lim_{k\to\infty} \frac{\mu(z_k)|g(z_k)||\varphi(z_k)|}{\widetilde{\omega}(\varphi(z_k))}. \tag{4.31}$$

If $\sup_{k\in\mathbb{N}} |\varphi(z_k)| < 1$, then from this and (4.29) we have that both limits in (4.30) are equal to zero. If $\sup_{k\in\mathbb{N}} |\varphi(z_k)| = 1$, then there is a subsequence $\{\varphi(z_{k_l})\}_{l\in\mathbb{N}}$ such that $|\varphi(z_{k_l})| \to 1^-$ as $l \to \infty$. Hence we have

$$\limsup_{|z|\to 1^-} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))} = \lim_{l\to\infty} \frac{\mu(z_{k_l})|g(z_{k_l})||\varphi(z_{k_l})|}{\widetilde{\omega}(\varphi(z_{k_l}))} \le \limsup_{|\varphi(z)|\to 1^-} \frac{\mu(z)|g(z)||\varphi(z)|}{\widetilde{\omega}(\varphi(z))}, \tag{4.32}$$

and so (4.30) holds.

Since $I_\varphi^g : \mathcal{B}_{\omega,0} \to \mathcal{B}_\mu$ is also compact, by Theorem 4.5, we see that the second limit in (4.30) is equal to zero, so that (4.26) holds. This completes the proof. □

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