Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 156725, 18 pages
doi:10.1155/2010/156725

Research Article

Global Stability and Oscillation of a Discrete Annual Plants Model

S. H. Saker[1, 2]

[1] *Department of Mathematics, College of Science, King Saud University, Riyadh 11451, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

Correspondence should be addressed to S. H. Saker, shsaker@mans.edu.eg

Received 26 September 2010; Accepted 2 November 2010

Academic Editor: Nicholas Alikakos

The objective of this paper is to systematically study the stability and oscillation of the discrete delay annual plants model. In particular, we establish some sufficient conditions for global stability of the unique positive fixed point and establish an explicit sufficient condition for oscillation of the positive solutions about the fixed point. Some illustrative examples and numerical simulations are included to demonstrate the validity and applicability of the results.

1. Introduction

Most populations live in seasonal environments and, because of this, have annual rhythms of reproduction and death. In addition, measurements are often made annually because interest is centered on population changes from year to year rather than on the obvious and predictable changes that occur seasonally. Continuous differential equations are not well suited to these kinds of processes and data. Thus, practical ecologists have long employed discrete-time difference equations for studying the dynamics of resource and pest populations. In particular one can consider the difference equation

$$N(n+1) = f(N(n)), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

as a measure of the population growth, where $N(n+1)$ is the size of the population at time $n+1$, $N(n)$ is the size of the population at time n, and the function $f(N)$ is the density-dependent growth rate from generation to generation and in general it is a nonlinear function of N. The skills in modelling a specific population's growth lie in determining the appropriate form of $f(N)$ to reflect the known observations or the facts of the species under consideration.

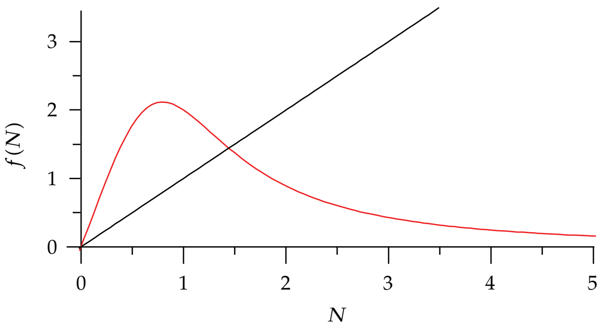


Figure 1: Population model shape.

Density dependent is a dependence of per capita population growth rate on present and/or past population densities. Hassell [1] proposed that the models of the population dynamics in a limited environment are based on the following two fundamentals:

(1) population have the potential to increase exponentially;

(2) there is a density-dependent feedback that progressively reduces the actual rate of increase.

In fact, in population dynamics, there is a tendency for that variable $N(n)$ to increase from one generation to the next when it is small, and decrease when it is large. In [2] Cull showed that for population dynamics models the nonlinear function $f(N)$ often has the following properties: $f(0) = 0$ and there is a unique positive fixed-point $\overline{N}$ such that $f(\overline{N}) = \overline{N}$, $f(N) > N$ for $0 < N < \overline{N}$, and $f(N) < \overline{N}$ for $N < \overline{N}$, and such that if $f(N)$ has a maximum N_M in $(0, \overline{N})$ then $f(N)$ decreases monotonically as N increases beyond $N = N_M$ such that $f(N) > 0$, (see Figure 1).

In recent decades the dynamics of discrete models in different areas have been extensively investigated by many authors. For contributions, we refer the reader to [2–17] and the references cited therein.

For population models of plants, Watkinson [18] assumed that the function $f(N)$ represents the number of seeds produced per parent plant which survived to flowering in the next generation and reproduce seasonality and have effectively nonoverlapping generations, even if a seed bank is present [18]. Using these assumptions Watkinson derived some different forms of the function $f(N)$ for seven different cases. For the annual plants Watkinson [18] assumed that the density-dependent function is given by

$$f(N) := \frac{\lambda N}{(1 + aN)^{\gamma} + \lambda m N}, \tag{1.2}$$

where λ is the growth rate and m represents the reciprocal of the asymptotic value of N when the initial plant density tends to infinity and it is called the degree of self-thinning. The parameter a has the dimension of the area and $1/a$ can be considered as the density of plants at which mutual interference between individuals becomes appreciable and γ is the density-dependent parameter where the biological significance is rather unclear. In [18] the author proposed that $\gamma > 1$, which reflects the fact that an increasing density leads to a less-efficient use of the resources with a given area in terms of total dry matter population.

Combining (1.1) and (1.2), we see that the Watkinson model of the annual plants is given by the difference equation

$$N(n+1) = \frac{\lambda N(n)}{(1+aN(n))^{\gamma} + \lambda m N(n)}, \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

In this model the density-independent mortality is not included and the growth of the population occurs only during the vegetative phase of the life cycle.

Watkinson in [18] assumed that density-independent mortality during the seed phase of the life cycle can easily be incorporated by multiplying λ by the probability that a seed will survive from the time of its formation to germination and establishment. Also in this model it is clear that the past history of the population is ignored, that is, the growth of the population is governed by a principle of causality, that is, the future state of population is independent of the past and is determined solely by the present. In fact in a single species population there is a time delay because of the time it takes a female animal or a plant to mature before it can begin to reproduce. A more realistic model must include some of the past history of population. Accordingly Kocić and Ladas [19] considered the model

$$N(n+1) = \frac{\lambda N(n)}{(1+aN(n-1))^{\gamma} + m N(n-1)}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

and proved that if $N(-1) \geq 0, N(0) > 0$ and

$$\lambda \in (1, \infty), \qquad a \in [0, \infty), \qquad \gamma \in [0, 1], \qquad m \in (0, \infty), \tag{1.5}$$

then $\lim_{n \to \infty} N(n) = \overline{N}$, where $\overline{N}$ is the unique fixed point of (1.4). Note that the assumption $\gamma \leq 1$ is different from the assumption $\gamma > 1$ that has been proposed by Watkinson [18], which reflects the fact that an increasing density leads to a less efficient use of the resources with a given area in terms of total dry matter population.

In [11] the authors considered the general equation with two delays of the form

$$N(n+1) = \frac{\lambda N(n)}{(1+aN(n-k))^{\gamma} + \lambda m N(n-l)}, \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

where

$$\lambda \in (1, \infty), \quad a, \gamma, m \in (0, \infty), \quad l, k \in \{0, 1, 2, 3, \ldots\}. \tag{1.7}$$

The authors in [11, Theorem 6.3.1] proved that if

$$\frac{\overline{N}\gamma a \left(1 + a\overline{N}\right)^{\gamma - 1}}{\left(1 + a\overline{N}\right)^{\gamma} + \lambda m \overline{N}} + m\overline{N} + l + k \neq 1, \tag{1.8}$$

then every solution of (1.6) oscillates about $\overline{N}$ if and only if every solution of the linearized delay difference equation

$$y(n+1)-y(n)+\frac{\overline{N}\gamma a\left(1+a\overline{N}\right)^{\gamma-1}}{\lambda}y(n-k)+m\overline{N}y(n-l)=0, \tag{1.9}$$

oscillates. In [11, Open Problem 6.3.1] the authors mentioned that the global asymptotic stability of the fixed-point $\overline{N}$ of (1.6) has not been investigated yet. Our aims in this paper is to consider this open problem, when $k = l$, and establish some sufficient conditions for the global stability of the positive fixed point of the delay difference equation

$$N(n+1)=\frac{\lambda N(n)}{(1+aN(n-k))^{\gamma}+\lambda mN(n-k)}, \quad n=0,1,2,\ldots, \tag{1.10}$$

where $N(n)$ in (1.10) represents the number of mature population in the nth and the function

$$F(N(n-k)):=\frac{\lambda}{(1+aN(n-k))^{\gamma}+\lambda mN(n-k)}, \tag{1.11}$$

represents the number of mature population that were produced in the $(n-k)$th cycle and survived to maturity in the nth cycle. We also establish an explicit sufficient condition for oscillation of all solutions of (1.10) about the fixed point. We note that when $m = 0$, and $k = 0$ (1.10) reduces to

$$N(n+1)=\frac{\lambda N(n)}{(1+aN(n))^{\gamma}}, \quad n=0,1,2,\ldots. \tag{1.12}$$

This equation has been proposed by Hassell [1] to describe the growth of the population of insects. On the other hand, when $\gamma = 1$ and $m = 0$, (1.10) becomes the Pielou equation [20]

$$N(n+1)=\frac{\lambda N(n)}{1+aN(n-k)}, \quad n=0,1,2,\ldots. \tag{1.13}$$

By the biological interpretation, we assume that the initial condition of (1.10) is given by

$$N(-k),N(-k+1),N(-k+2),\ldots,N(1)\in[0,\infty), \quad N(0)>0. \tag{1.14}$$

By a solution of (1.10), we mean a sequence $N(n)$ which is defined for $n \geq -k$ and satisfies (1.10) for $n \geq 0$ and by a positive solution, we mean that the terms of the sequence $\{N(n)\}_{n=1}^{\infty}$ are all positive. Then, it is easy to see that the initial value problem (1.10) and (1.14) has a unique positive solution $N(n)$. In the sequel, we will only consider positive solutions of (1.10). We say that $\overline{N}$ is a fixed of (1.10) if

$$\overline{N}\left(\left(1+a\overline{N}\right)^{\gamma}+\lambda m\overline{N}\right)=\lambda\overline{N}, \tag{1.15}$$

that is, the constant sequence $\{N(n)\}_{n=-k}^{\infty}$ with $N(n) = \overline{N}$ for all $n \geq -k$ is a solution of (1.10). In the following, we prove that (1.10) has a unique positive fixed point. Let

$$g(N) := (1 + aN)^{\gamma} + \lambda mN - \lambda. \tag{1.16}$$

Then $g(0) = -\lambda < 0$ and $g(\infty) = \infty$, so that there exists $\overline{N} > 0$ such that $g(\overline{N}) = 0$. Also

$$g'(N) = \gamma a(1 + aN)^{\gamma-1} + \lambda m > 0, \quad \forall N > 0. \tag{1.17}$$

It follows that $g(N) = 0$ has exactly one solution and so (1.10) has a unique positive fixed point which is denoted by $\overline{N}$ and obtained from the solution of

$$\left(1 + a\overline{N}\right)^{\gamma} + \lambda m\overline{N} - \lambda = 0. \tag{1.18}$$

The stability of equilibria is one of the most important issues in the studies of population dynamics. The fixed-point $\overline{N}$ of (1.10) is locally stable if the solution of the population model $N(n)$ approaches $\overline{N}$ as time increases for all $N(0)$ in some neighborhood of $\overline{N}$. The fixed point $\overline{N}$ of (1.10) is globally stable if for all positive initial values the solution of the model approaches $\overline{N}$ as time increases. A model is locally or globally stable if its positive fixed point is locally or globally stable. The fixed-point $\overline{N}$ is globally asymptotically stable if its locally and globally stable. A solution $\{N(n)\}$ of (1.10) is said to be oscillatory about $\overline{N}$ if $N(n)-\overline{N}$ is oscillatory, where the sequence $N(n)-\overline{N}$ is said to be oscillatory if $N(n)-\overline{N}$ is not eventually positive or eventually negative.

For the delay equations, for completeness, we present some global stability conditions of the zero solution of the delay difference equation

$$x(n+1) - x(n) + A(n)x(n-k) = 0, \tag{1.19}$$

that we will use in the proof of the main global stability results. Györi and Pituk [21], proved that if

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} A(i) < 1, \tag{1.20}$$

then the zero solution is globally stable. Erbe et al. [22] improved (1.20) and proved that if

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} A(i) < \frac{3}{2} + \frac{1}{2(k+1)}, \tag{1.21}$$

then the zero solution is globally stable. Also Kovácsvölgy [23] proved that if

$$\limsup_{n\to\infty} \sum_{i=n-2k}^{n-1} A(i) < \frac{7}{4}, \tag{1.22}$$

then every zero solution of (1.19) is globally stable, whereas the result by Yu and Cheng [24] gives an improvement over (1.22) to

$$\lim_{n\to\infty} \sup \sum_{i=n-2k}^{n} A(i) < 2. \tag{1.23}$$

The paper is organized as follows: in Section 2, we establish some sufficient conditions for global stability of $\overline{N}$. The results give a partial answer to the open problem posed by Kocić and Ladas [11, Open problem 6.3.1] and improve the results that has been established by Kocić and Ladas [19] in the sense that the restrictive condition $\gamma \in [0,1]$ is not required. In Section 3, we will use an approach different from the method used in [11] and establish an explicit sufficient condition for oscillation of the positive solutions of the delay equation (1.10) about $\overline{N}$. Some illustrative examples and simulations are presented throughout the paper to demonstrate the validity and applicability of the results.

2. Global Stability Results

In this section, we establish some sufficient conditions for local and global stability of the positive fixed-point $\overline{N}$. First, we establish a sufficient condition for local stability of (1.10). The linearized equation associated with (1.10) at $\overline{N}$ is given by

$$y(n+1) - y(n) + \frac{\overline{N}\gamma a\left(1+a\overline{N}\right)^{\gamma-1} + \lambda m\overline{N}}{\left(1+a\overline{N}\right)^{\gamma} + \lambda m\overline{N}} y(n-k) = 0. \tag{2.1}$$

Applying the local stability result of Levin and May [25] on (2.1), we have the following result.

Theorem 2.1. *Assume that* (1.14) *holds and* $\gamma, \lambda > 1$. *If*

$$L\left(a,\gamma,\lambda,m,k,\overline{N}\right) = \frac{\overline{N}\gamma a\left(1+a\overline{N}\right)^{\gamma-1} + \lambda m\overline{N}}{\left(1+a\overline{N}\right)^{\gamma} + \lambda m\overline{N}} < 2\cos\frac{\pi k}{2k+1}, \tag{2.2}$$

then the fixed-point $\overline{N}$ *of* (1.10) *is locally asymptotically stable.*

To prove the main global stability results for (1.10), we need to find some upper and lower bounds for positive solutions of (1.10) which oscillate about $\overline{N}$.

Theorem 2.2. *Assume that* (1.14) *holds and* $\gamma, \lambda > 1$. *Let* $N(n)$ *be a positive solution of* (1.10) *which oscillates about* $\overline{N}$. *Then there exists* $n_1 > 0$ *such that for all* $n \geq n_1$, *one has*

$$Y_1 := \frac{\lambda^k \overline{N}}{\left(\left(1+a\overline{N}\lambda^k\right)^{\gamma} + m\overline{N}\lambda^{k+1}\right)^k} \leq N(n) \leq \overline{N}\lambda^k := Y_2, \tag{2.3}$$

Proof. First we will show the upper bound in (2.3). The sequence $N(n)$ is oscillatory about the positive periodic solution $\overline{N}$ in the sense that there exists a sequence of positive integers $\{n_l\}$ for $l = 1, 2, \ldots$ such that $k \le n_1 < n_2 < \cdots < n_l < \cdots$ with $\lim_{l \to \infty} n_l = \infty$, $N(n_l) < \overline{N}$ and $N(n_l + 1) \ge \overline{N}$. We assume that some of the terms $N(j)$ with $n_l < j \le n_{l+1}$ are greater than $\overline{N}$ and some are less than $\overline{N}$. Our strategy is to show that the upper bound holds in each interval $[n_l, n_{l+1}]$. For each $l = 1, 2, \ldots$, let ζ_l be the integer in the interval $[n_l, n_{l+1}]$ such that

$$N(\zeta_l + 1) = \max\{N(j) : n_l < j \le n_{l+1}\}. \tag{2.4}$$

Then $N(\zeta_l + 1) \ge N(\zeta_l)$ which implies that $\Delta N(\zeta_l) \ge 0$. To show the upper bound on (2.3), it suffices to show that

$$N(\zeta_l) \le \overline{N}\lambda^k = \Upsilon_2. \tag{2.5}$$

We assume that $N(\zeta_l) > \overline{N}$, otherwise there is nothing to prove. Now, since $\Delta N(\zeta_l) \ge 0$, it follows from (1.10) that

$$1 \le \frac{N(\zeta_l + 1)}{N(\zeta_l)} = \frac{\lambda}{(1 + aN(\zeta_l - k))^\gamma + \lambda m N(\zeta_l - k)}, \tag{2.6}$$

and hence

$$\frac{\lambda}{(1 + aN(n - k))^\gamma + \lambda m N(n - k)} > 1 = \frac{\lambda}{\left(1 + a\overline{N}\right)^\gamma + \lambda m \overline{N}}. \tag{2.7}$$

This implies that $N(\zeta_l - k) < \overline{N}$. Now, since $N(\zeta_l) > \overline{N}$ and $N(\zeta_l - k) < \overline{N}$, there exists an integer $\overline{\zeta}_l$ in the interval $[\zeta_l - k, \zeta_l]$, such that $N(\overline{\zeta}_l) \le \overline{N}$ and $N(j) > \overline{N}$ for $j = \overline{\zeta}_l + 1, \ldots, \zeta_l$. From (1.10), we see that

$$N(n + 1) = \frac{\lambda N(n)}{(1 + aN(n - k))^\gamma + \lambda m N(n - k)} < \lambda N(n), \tag{2.8}$$

so that

$$N(n + 1) < \lambda N(n). \tag{2.9}$$

Multiplying this inequality from $\overline{\zeta}_l$ to $\zeta_l - 1$, we have

$$N(\zeta_l) < N\left(\overline{\zeta}_l\right)(\lambda)^{\zeta_l - \overline{\zeta}_l}, \tag{2.10}$$

and so

$$N(\zeta_l) < \overline{N}\lambda^k, \tag{2.11}$$

which immediately gives (2.5). Hence, there exists an $n_1 > 0$ such that $N(n) \leq Y_2$ for all $n \geq n_1$. Now, we show the lower bound in (2.3) for $n \geq n_1 + k$. For this, let μ_l be the integer in the interval $[n_l, n_{l+1}]$ such that

$$N(\mu_l + 1) = \min\{N(j) : n_l < j \leq n_{l+1}\}. \tag{2.12}$$

Then $N(\mu_l+1) \leq N(\mu_l)$ which implies that $\Delta N(\mu_l) \leq 0$. We assume that $N(\mu_l) < \overline{N}$, otherwise there is nothing to prove. Then, it suffices to show that

$$N(\mu_l) \geq \frac{\lambda^k \overline{N}}{\left(\left(1 + a\overline{N}\lambda^k\right)^\gamma + m\overline{N}\lambda^{k+1}\right)^k}. \tag{2.13}$$

Since $\Delta N(\mu_l) \leq 0$, we have from (1.10) that

$$1 \geq \frac{N(\mu_l + 1)}{N(\mu_l)} = \frac{\lambda}{(1 + aN(\mu_l - k))^\gamma + \lambda m N(\mu_l - k)}, \tag{2.14}$$

which implies that

$$\frac{\lambda}{(1 + aN(\mu_l - k))^\gamma + \lambda m N(\mu_l - k)} < 1 = \frac{\lambda}{\left(1 + a\overline{N}\right)^\gamma + \lambda m \overline{N}}. \tag{2.15}$$

This leads to $N(\mu_l - k) > \overline{N}$. Now, since $N(\mu_l) < \overline{N}$ and $N(\mu_l - k) > \overline{N}$, then there exists a $\overline{\mu}_l \in [\mu_l - k, \mu_l]$ such that $N(\overline{\mu}_l) \geq \overline{N}$ and $N(j) < \overline{N}$ for $j = \overline{\mu}_l + 1, \ldots, \mu_l$. From (1.10) and (2.3), we have

$$\begin{aligned} N(n+1) &= \frac{\lambda N(n)}{(1 + aN(n-k))^\gamma + \lambda m N(n-k)} \\ &\geq \frac{\lambda N(n)}{\left(1 + a\overline{N}\lambda^k\right)\big)^\gamma + \lambda m \overline{N}\lambda^k}. \end{aligned} \tag{2.16}$$

Multiplying the last inequality from $\overline{\mu}_l$ to $\mu_l - 1$, we have

$$N(\mu_l) > N(\overline{\mu}_l)\left(\frac{\lambda}{\left(1 + a\overline{N}\lambda^k\right)\big)^\gamma + \lambda m \overline{N}\lambda^k}\right)^{(\mu_l - \overline{\mu}_l)}, \tag{2.17}$$

and this implies that

$$N(\mu_l) > \frac{\lambda^k \overline{N}}{\left(\left(1 + a\overline{N}\lambda^k\right)^\gamma + m\overline{N}\lambda^{k+1}\right)^k}, \tag{2.18}$$

which immediately leads to (2.13). The proof is complete. □

One of the techniques used in the proof of the global stability of the zero solution of the nonlinear equation

$$\Delta z(n) + h(z(n-k)) = 0, \tag{2.19}$$

is the application of what is called a linear method (see [15, 16]). To apply this method, we have to prove that the solution is bounded and the solution of (1.10), say $z(n)$, is a solution of the corresponding linear equation. This can be done by using the main value theorem, which implies

$$h(z(n-k)) = h(0) + z(n-k)h'(\zeta_n), \tag{2.20}$$

where ζ_n lies between zero and $z(n-k)$. Therefore, we obtain

$$\Delta z(n) + h'(\zeta_n)z(n-k) = 0. \tag{2.21}$$

Applying the global stability results presented in Section 1, we can obtain some sufficient conditions for global stability provided that the solution is bounded. With this idea and using the fact that the solutions are bounded, we are now ready to state and prove the main global stability results for (1.10).

Theorem 2.3. *Assume that* (1.14) *holds,* $\lambda, \gamma > 1$ *and* $N(n)$ *is a positive solution of* (1.10). *If*

$$G\left(a, \gamma, \lambda, m, k, \overline{N}\right) = \frac{\gamma a\overline{N}\lambda^k\left(1 + a\overline{N}\lambda^k\right)^{\gamma-1} + m\overline{N}\lambda^{k+1}}{\left(1 + a\overline{N}\lambda^k\right)^{\gamma} + m\overline{N}\lambda^{k+1}} < \frac{1}{k}, \tag{2.22}$$

then

$$\lim_{n\to\infty} N(n) = \overline{N}. \tag{2.23}$$

Proof. First, we prove that every positive solution $N(n)$ which does not oscillate about $\overline{N}$ satisfies (2.23). Assume that $N(n) > \overline{N}$ for n sufficiently large (the proof when $N(n) < \overline{N}$ is similar and will be omitted since $uh(u) >$ for $u \neq 0$ see below). Let

$$N(n) = \overline{N}e^{z(n)}. \tag{2.24}$$

To prove that (2.23) holds it suffices to prove that $\lim_{n\to\infty} z(n) = 0$. From (1.10) and (2.24), we see that $z(n) > 0$ and satisfies

$$z(n+1) - z(n) + h(z(n-k)) = 0, \tag{2.25}$$

where

$$h(u) := \ln\left(\frac{\left(1 + a\overline{N}e^{u}\right)^{\gamma} + \lambda m\overline{N}e^{u}}{\left(1 + a\overline{N}\right)^{\gamma} + \lambda m\overline{N}}\right). \tag{2.26}$$

Note that $h(0) = 0$ and $h(u) > 0$ for $u > 0$. It follows from (2.25) that

$$z(n+1) - z(n) = -h(z(n-k)) < 0. \tag{2.27}$$

Hence, $z(n)$ is decreasing and there exists a nonnegative real number $\alpha \geq 0$ such that $\lim_{n\to\infty} z(n) = \alpha$. If $\alpha > 0$, then there exists a positive integer $n_2 > n_1$ such that $(\alpha/2) \leq z(n-k) \leq (3\alpha/2)$ for $n > n_2$. This implies from (2.25) that

$$z(n+1) - z(n) \leq -\eta, \quad \text{for } n > n_2, \tag{2.28}$$

where $\eta = \min_{\alpha/2 \leq u \leq 3\alpha/2} h(u) > 0$. Thus summing up the last inequality from n_2 to $n-1$, we obtain

$$z(n) \leq z(n_2) - \eta(n - n_2) \longrightarrow -\infty, \quad \text{as } n \longrightarrow \infty. \tag{2.29}$$

This contradicts the fact that $z(n)$ is positive. Then $\alpha = 0$, that is, $\lim_{n\to\infty} z(n) = 0$. Thus (2.23) holds, and any positive solution of (1.10) which does not oscillate about $\overline{N}$ satisfies (2.23). To complete the global attractivity results we prove that every oscillatory about $\overline{N}$ satisfies (2.23). From the transformation (2.24) it is clear that $N(n)$ oscillates about $\overline{N}$ if and only if $z(n)$ oscillates about zero. So to complete the proof, we have to demonstrate that $\lim_{n\to\infty} z(n) = 0$, where $z(n)$ is a solution of (2.25). Equation (2.25) is the same as

$$\Delta z(n) + h(z(n-k)) - h(0) = 0. \tag{2.30}$$

Clearly, by the mean value theorem (2.30) can be written as

$$\Delta z(n) + Az(n-k) = 0, \tag{2.31}$$

where

$$\begin{aligned} A = \left.\frac{dh(u)}{du}\right|_{u=\zeta_n} &= \frac{\gamma a\overline{N}e^{\zeta_n}\left(1 + a\overline{N}e^{\zeta_n}\right)^{\gamma-1} + \lambda m\overline{N}e^{\zeta_n}}{\left(1 + a\overline{N}e^{\zeta_n}\right)^{\gamma} + \lambda m\overline{N}e^{\zeta_n}} \\ &= \frac{\gamma a\eta_n\left((1 + a\eta_n)^{\gamma-1} + \lambda m\eta_n\right.}{(1 + a\eta_n)^{\gamma} + \lambda m\eta_n}, \end{aligned} \tag{2.32}$$

and η_n lies between $\overline{N}$ and $N(n-k)$. In Theorem 2.2, we proved that the oscillatory solutions of (1.10) are bounded and the upper bound of solution is given by Υ_2. Using this, since the values of A are increasing in η, we have

$$\max_{\eta_n\in[\Upsilon_1,\Upsilon_2]} \frac{\gamma a\eta_n\left((1+a\eta_n)^{\gamma-1}+\lambda m\eta_n\right.}{(1+a\eta_n)^{\gamma}+\lambda m\eta_n} = \frac{\gamma a\overline{N}\lambda^k\left(1+a\overline{N}\lambda^k\right)^{\gamma-1}+\lambda m\overline{N}\lambda^k}{\left(1+a\overline{N}\lambda^k\right)^{\gamma}+\lambda m\overline{N}\lambda^k}. \tag{2.33}$$

From the last inequality and the assumption (2.22), we see that

$$Ak < \frac{\gamma a\overline{N}\lambda^k\left(1+a\overline{N}\lambda^k\right)^{\gamma-1}+\lambda m\overline{N}\lambda^k}{\left(1+a\overline{N}\lambda^k\right)^{\gamma}+\lambda m\overline{N}\lambda^k}k < 1. \tag{2.34}$$

Then by the results of Györi and Pituk [21], we deduce that the zero solution of (2.31) is globally stable that is, $\lim_{n\to\infty} z(n) = 0$, and hence $\lim_{n\to\infty} N(n) = \overline{N}$. The proof is complete. □

Remark 2.4. From Theorem 2.3, it is clear that the global attractivity of $\overline{N}$ is equivalent to the global attractivity of zero solution of the linear difference equation (2.31). By employing the results by Györi and Pituk [21], we showed that if (2.22) holds then (2.23) is satisfied. Now, we apply different result which improves the condition (2.22) based on the improvement of the global attractivity condition that has been given Györi and Pituk [21] for the difference equation (2.31). Applying the result by Erbe et al. [22] we have the following result.

Corollary 2.5. *Assume that* (1.14) *holds,* $\lambda, \gamma > 1$. *If*

$$G_1\left(a,\gamma,\lambda,m,k,\overline{N}\right) = \frac{\gamma a\overline{N}\lambda^k\left(1+a\overline{N}\lambda^k\right)^{\gamma-1}+m\overline{N}\lambda^{k+1}}{\left(1+a\overline{N}\lambda^k\right)^{\gamma}+m\overline{N}\lambda^{k+1}} < \frac{3k+4}{2(k+1)^2}, \tag{2.35}$$

then every positive solution of $N(n)$ *of* (1.10) *satisfies* (2.23).

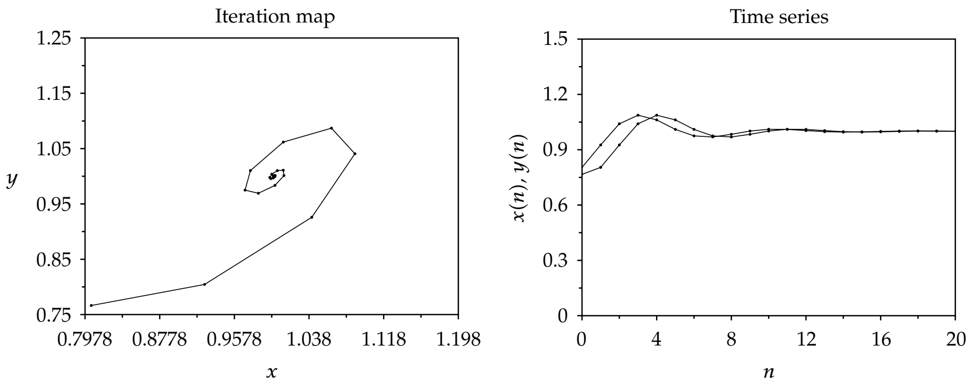
Also by employing the result due to Kovácsvölgy [23], we have the following result.

Corollary 2.6. *Assume that* (1.14) *holds,* $\lambda, \gamma > 1$. *If*

$$G_1\left(a,\gamma,\lambda,m,k,\overline{N}\right) = \frac{\gamma a\overline{N}\lambda^k\left(1+a\overline{N}\lambda^k\right)^{\gamma-1}+m\overline{N}\lambda^{k+1}}{\left(1+a\overline{N}\lambda^k\right)^{\gamma}+m\overline{N}\lambda^{k+1}} < \frac{7}{8k}, \tag{2.36}$$

then every positive solution of $N(n)$ *of* (1.10) *satisfies* (2.23).

The result by Yu and Cheng [24] when applied gives us the following result.



(a) The iterations in the phase plane

(b) Time series

Figure 2

Corollary 2.7. *Assume that* (1.14) *holds,* $\lambda, \gamma > 1$. *If*

$$G_1\left(a, \gamma, \lambda, m, k, \overline{N}\right) = \frac{\gamma a \overline{N} \lambda^k \left(1 + a\overline{N}\lambda^k\right)^{\gamma-1} + m\overline{N}\lambda^{k+1}}{\left(1 + a\overline{N}\lambda^k\right)^{\gamma} + m\overline{N}\lambda^{k+1}} < \frac{2}{2k+1}, \tag{2.37}$$

then every positive solution of $N(n)$ *of* (1.10) *satisfies* (2.23).

We illustrate the main results with the following examples.

Example 2.8. Consider the model

$$N(n+1) = \frac{2N(n)}{(1 + (0.4)N(n-1))^2 + 0.04N(n-1)}. \tag{2.38}$$

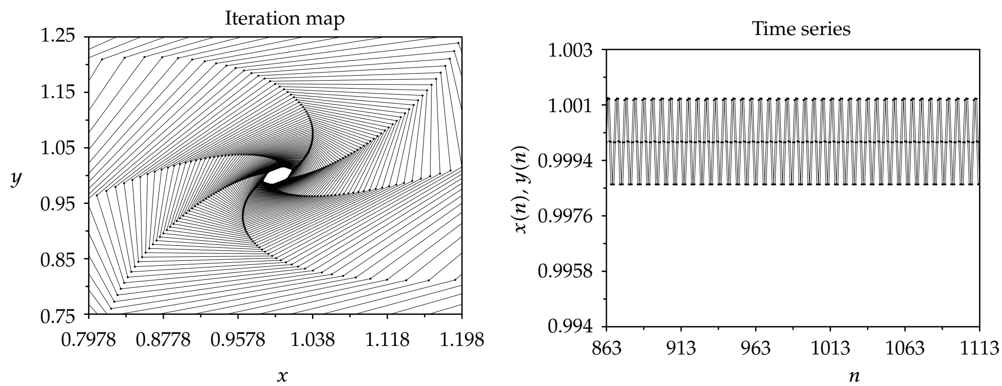
In this model $a = 0.4$, $m = 0.02$, $k = 1$, $\gamma = 2$, $\lambda = 2$, and $\overline{N} = 1$. Now, we apply Theorem 2.3. For (2.38) the condition (2.22) of Theorem 2.3 reads

$$G(0.4, 2, 2, 0.02, 1, 1) = 0.89157 < \frac{1}{k} = 1. \tag{2.39}$$

Then the condition (2.22) of Theorem 2.3 is satisfied and then the fixed-point $\overline{N} = 1$ is globally stable. For illustration, we plotted the iterations in the phase plane and the time series ($x = N(n)$, $y = N(n-1)$, $(\overline{x}, \overline{y}) = (\overline{N}, \overline{N})$), where we found that the solutions oscillate and converge to the fixed-point $(\overline{x}, \overline{y}) = (\overline{N}, \overline{N}) = (1, 1)$, see Figure 2.

Example 2.9. Consider the model

$$N(n+1) = \frac{(3.98)N(n)}{(1 + (0.97494)N(n-1))^2 + 0.0796N(n-1)}, \tag{2.40}$$



(a) The iterations in the phase plane of (2.40)

(b) The time series of (2.40)

Figure 3

where in this case $a = 0.97494$, $m = 0.02$, $k = 1$, $\gamma = 2$, and $\lambda = (1+a)^2/(1-m) = 3.98$ then $\overline{N} = 1$, and the condition (2.22) in Theorem 2.3 reads

$$G(0.97494, 2, 3.98, 0.02, 1, 1) = 1.5824 > \frac{1}{k} = 1. \tag{2.41}$$

Then the condition (2.22) of Theorem 2.3 is not satisfied which means that the fixed-point $\overline{N} = 1$ is not globally stable. Also the condition (2.35) of Corollary 2.5 which reads in this case

$$G_1\left(a, \gamma, \lambda, m, k, \overline{N}\right) = 1.5824 > \frac{7}{8}, \tag{2.42}$$

is not satisfied and then the fixed-point $\overline{N} = 1$ is not globally stable, see Figure 3.

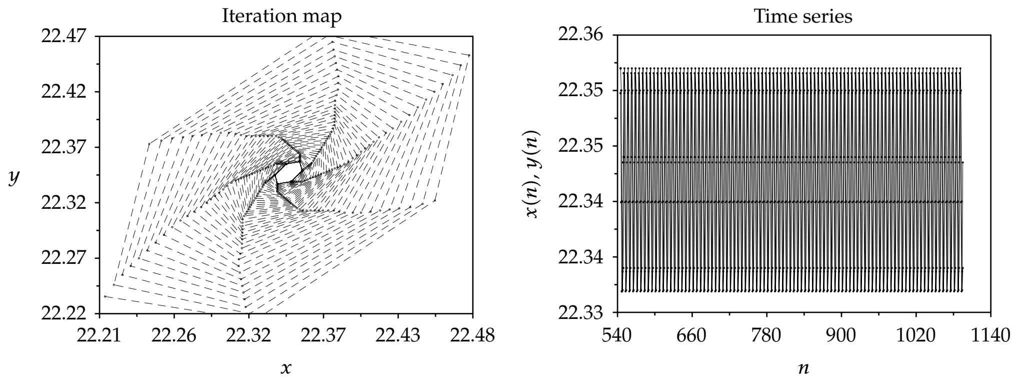
Remark 2.10. Note that the condition of local stability of (2.40) is

$$L(0.97494, 2, 3.98, 0.02, 1, 1) = 0.98756 < 2\cos\frac{\pi}{3} = 1. \tag{2.43}$$

Then the fixed point is a locally stable, but this condition is not a sufficient condition for global stability, since the model is not globally stable.

Example 2.11. Consider the model

$$N(n+1) = \frac{30N(n)}{(1+(0.0521)N(n-1))^{1.5} + 1.2N(n-1)}. \tag{2.44}$$



(a) The iterations in the phase plane of (2.44)

(b) The time series of (2.44)

Figure 4

Here, we take $a = 0.0521$, $m = 0.04$, $\gamma = 1.5$, $\lambda = 30$, and $\overline{N} = 22.347$. One can easily check that the condition (2.22) of Theorem 2.3 is not satisfied and then the fixed-point $\overline{N} = 22.347$ is not globally stable. For illustration, we plotted the iterations and the time series where we found that the solutions oscillate and does not converge to the fixed-point $\overline{N} = 22.347$, see Figure 4.

Remark 2.12. We note that the results in [19] cannot be applied on (2.44), since $\gamma = 1.5 > 1$. Note also that the condition of local stability for (2.44) which reads

$$L(0.0521, 1.5, 30, 0.04, 1, 22.347) = 0.97951 < 2\cos\frac{\pi}{3} = 1, \tag{2.45}$$

is already satisfied but this condition is not a sufficient condition for global stability, since the model is not globally stable.

3. Oscillation Results

In this section, we establish an explicit sufficient condition for oscillation of (1.10) about the positive fixed-point $\overline{N}$. In (1.13) Pielou assumed that there is a delay k in the response of the growth rate per individual to density changes. Pielou showed that the tendency to oscillate is a property of the populations themselves and is independent of any extrinsic factors. That is, population size oscillates even though the environment remains constant according to Pielou; oscillation can be set up in a population if its growth rate is governed by a density-dependent mechanism and if there is a delay in the response of the growth rate to density changes. In this section, we consider (1.10) and also prove that under some conditions on the parameters the solutions oscillate about the fixed-point $\overline{N}$ even though the environment remains constant. Oscillatory behavior of the solutions is very significant which implies the prevalence of the mature plants around the positive fixed point.

We, first prove the following theorem which proves that oscillation of (1.10) about the positive fixed point is equivalent to oscillation of a linear difference equation about zero. Note that when $k \geq 1$ then the condition (1.8) is already satisfied.

Theorem 3.1. *Assume that* (1.14) *holds. Furthermore, assume that there exists* $\varepsilon > 0$*, such that every solution of*

$$z(n+1) - z(n) + \frac{\left(\gamma a\overline{N}\left(1 + a\overline{N}\right)^{\gamma-1} + \lambda m\overline{N}\right)}{\lambda}(1-\varepsilon)z(n-k) = 0, \tag{3.1}$$

oscillates. Then every positive solution of (1.10) *oscillates about* $\overline{N}$.

Proof. Without loss of generality we assume that (1.10) has a solution $N(n)$ exceeding $\overline{N}$ and define $z(n)$ as in (2.24). From the transformation (2.24) it is clear that $N(n)$ oscillates about $\overline{N}$ if and only if $z(n)$ oscillates about zero and transforms (1.10) to

$$z(n+1) - z(n) + \frac{\left(\gamma a\overline{N}\left(1 + a\overline{N}\right)^{\gamma-1} + \lambda m\overline{N}\right)}{\lambda}f(z(n-k)) = 0, \tag{3.2}$$

where

$$f(u) = \frac{\lambda}{\gamma a\overline{N}\left(1 + a\overline{N}\right)^{\gamma-1} + \lambda m\overline{N}} \ln\left(\frac{\left(1 + a\overline{N}e^{u}\right)^{\gamma} + \lambda m\overline{N}e^{u}}{\left(1 + a\overline{N}\right)^{\gamma} + \lambda m\overline{N}}\right). \tag{3.3}$$

Note that $f(0) = 0$, and

$$uf(u) > 0, \quad \text{for } u \neq 0, \qquad \lim_{u \to 0}\frac{f(u)}{u} = 1. \tag{3.4}$$

From (3.4) it follows that for any given arbitrarily small $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < u < \delta$, we have $f(u) \geq (1-\varepsilon)u$ (similarly, for all $-\delta < u < 0$, we have $f(u) \leq (1-\varepsilon)u$). In view of Theorem 2.2, since $z(n) \to 0$, thus for sufficiently large n we can use this estimate in (3.2), to conclude that $z(n)$ is a positive solution of the differential inequality

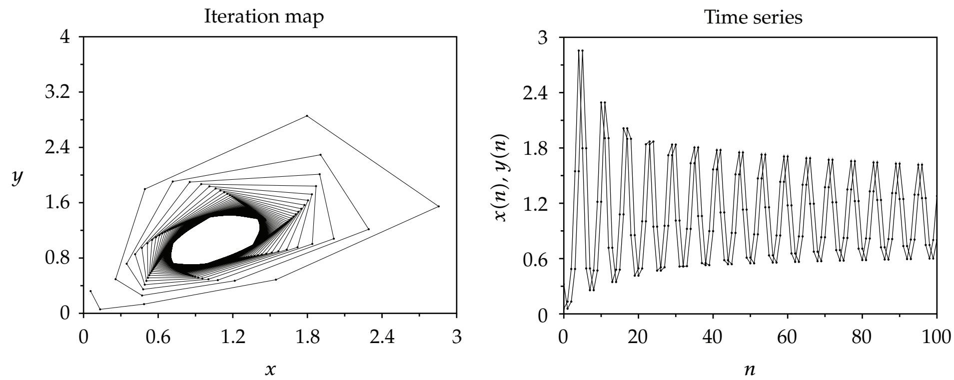
$$z(n+1) - z(n) + \frac{(1-\varepsilon)}{\lambda}\left(\gamma a\overline{N}\left(1 + a\overline{N}\right)^{\gamma-1} + \lambda m\overline{N}\right)z(n-k) \leq 0. \tag{3.5}$$

Then, by Lemma 1 in [26] the delay difference equation (3.1) also has an eventually positive solution, which contradicts the assumption that every solution of (3.1) oscillates. Thus every positive solution of (1.10) oscillates about $\overline{N}$, which completes the proof. □

To establish the condition for oscillation of all positive solution of (1.10) about the positive fixed point, we need the following result which is extracted from [27].

Lemma 3.2. *If* $k \geq 1$ *and*

$$p > \frac{k^{k}}{(k+1)^{k+1}}, \tag{3.6}$$



(a) Iterations of (3.9)

(b) Time series of (3.9)

Figure 5

then every solution of

$$z(n+1) - z(n) + pz(n-k) = 0, \tag{3.7}$$

oscillates.

Theorem 3.1 and Lemma 3.2 immediately imply the following oscillation result for (1.10).

Theorem 3.3. *Assume that* (1.14) *holds. If* $k \geq 1$ *and*

$$\frac{\gamma a\overline{N}\left(1 + a\overline{N}\right)^{\gamma-1} + \lambda m\overline{N}}{\lambda} > \frac{k^k}{(k+1)^{k+1}}, \tag{3.8}$$

then every solution of (1.10) *oscillates about the positive fixed-point* $\overline{N}$.

To illustrate the main result of Theorem 3.3, we consider the following example.

Example 3.4. Consider the model

$$N(n+1) = \frac{(4.0816)N(n)}{(1 + N(n-1))^2 + 8.1633 \times 10^{-2} N(n-1)}. \tag{3.9}$$

Here $a = 1$, $m = 0.02$, $k = 1$, $\gamma = 2$, and $\lambda = (1+1)^2/(1-0.02) = 4.0816$. In this case the positive fixed-point $\overline{N} = 1$ and the condition (3.8) is satisfied. Then by Theorem 3.3, every positive solutions oscillates about the positive fixed-point $\overline{N} = 1$, see Figure 5 where we plotted the iterations and the time series in a focus type.

Remark 3.5. (1) We note that the condition (1.8) that has been proposed in [11, Theorem 6.3.1] is not required in the proof of the oscillation results in Theorem 3.3.

Acknowledgment

The author thanks the Research Centre in College of Science in King Saud University for encouragements and supporting this project which has taken the number Math. 2009/33.

References

[1] M. P. Hassell, "Density dependence in single-species populations," *Journal of Animal Ecology*, vol. 44, pp. 283–296, 1975.
[2] P. Cull, "Local and global stability for population models," *Biological Cybernetics*, vol. 54, no. 3, pp. 141–149, 1986.
[3] P. Cull, "Global stability of population models," *Bulletin of Mathematical Biology*, vol. 43, no. 1, pp. 47–58, 1981.
[4] P. Cull, "Stability of discrete one-dimensional population models," *Bulletin of Mathematical Biology*, vol. 50, no. 1, pp. 67–75, 1988.
[5] P. Cull, "Stability in one-dimensional models," *Scientiae Mathematicae Japonicae*, vol. 58, no. 2, pp. 367–375, 2003.
[6] P. Cull, "Population models: stability in one dimension," *Bulletin of Mathematical Biology*, vol. 69, no. 3, pp. 989–1017, 2007.
[7] E. Braverman and S. H. Saker, "Permanence, oscillation and attractivity of the discrete hematopoiesis model with variable coefficients," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 10, pp. 2955–2965, 2007.
[8] E. Braverman and S. H. Saker, "On the Cushing-Henson conjecture, delay difference equations and attenuant cycles," *Journal of Difference Equations and Applications*, vol. 14, no. 3, pp. 275–286, 2008.
[9] E. M. Elabbasy and S. H. Saker, "Periodic solutions and oscillation of discrete non-linear delay population dynamics model with external force," *IMA Journal of Applied Mathematics*, vol. 70, no. 6, pp. 753–767, 2005.
[10] A. F. Ivanov, "On global stability in a nonlinear discrete model," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 23, no. 11, pp. 1383–1389, 1994.
[11] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
[12] R. M. May, "Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos," *Science*, vol. 186, no. 4164, pp. 645–647, 1974.
[13] R. M. May, "Simple mathematical models with very complicated dynamics," *Nature*, vol. 261, pp. 459–467, 1976.
[14] R. M. May, "Nonlinear problems in ecology and resources management," in *Chaotic Behaviour of Deterministic Systems*, G. Toos, R. H. G. Helleman, and R. Stora, Eds., North-Holand, 1983.
[15] S. H. Saker, "Qualitative analysis of discrete nonlinear delay survival red blood cells model," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 2, pp. 471–489, 2008.
[16] S. H. Saker, "Periodic solutions, oscillation and attractivity of discrete nonlinear delay population model," *Mathematical and Computer Modelling*, vol. 47, no. 3-4, pp. 278–297, 2008.
[17] H. R. Thieme, *Mathematics in Population Biology*, Princeton Series in Theoretical and Computational Biology, Princeton University Press, Princeton, NJ, USA, 2003.
[18] A. R. Watkinson, "Density-dependence in single-species populations of plants," *Journal of Theoretical Biology*, vol. 83, no. 2, pp. 345–357, 1980.
[19] V. L. Kocić and G. Ladas, "Global attractivity in a second-order nonlinear difference equation," *Journal of Mathematical Analysis and Applications*, vol. 180, no. 1, pp. 144–150, 1993.
[20] E. C. Pielou, *Population and Community Ecology*, Gordon and Breach, New York, NY, USA, 1974.
[21] I. Győri and M. Pituk, "Asymptotic stability in a linear delay difference equation," pp. 295–299, Gordon and Breach.
[22] L. H. Erbe, H. Xia, and J. S. Yu, "Global stability of a linear nonautonomous delay difference equation," *Journal of Difference Equations and Applications*, vol. 1, no. 2, pp. 151–161, 1995.
[23] I. Kovácsvölgyi, "The asymptotic stability of difference equations," *Applied Mathematics Letters*, vol. 13, no. 1, pp. 1–6, 2000.
[24] J. S. Yu and S. S. Cheng, "A stability criterion for a neutral difference equation with delay," *Applied Mathematics Letters*, vol. 7, no. 6, pp. 75–80, 1994.

[25] S. A. Levin and R. M. May, “A note on difference-delay equations,” *Theoretical Population Biology*, vol. 9, no. 2, pp. 178–187, 1976.
[26] G. Ladas, Ch. G. Philos, and Y. G. Sficas, “Sharp conditions for the oscillation of delay difference equations,” *Journal of Applied Mathematics and Simulation*, vol. 2, no. 2, pp. 101–111, 1989.
[27] S. A. Kuruklis and G. Ladas, “Oscillations and global attractivity in a discrete delay logistic model,” *Quarterly of Applied Mathematics*, vol. 50, no. 2, pp. 227–233, 1992.