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Research Article

q-Bernstein Polynomials Associated with q-Stirling Numbers and Carlitz's q-Bernoulli Numbers

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Recently, Kim (2011) introduced q-Bernstein polynomials which are different q-Bernstein polynomials of Phillips (1997). In this paper, we give a p-adic q-integral representation for q-Bernstein type polynomials and investigate some interesting identities of q-Bernstein type polynomials associated with q-extensions of the binomial distribution, q-Stirling numbers, and Carlitz's q-Bernoulli numbers.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$.

When one talks of q-extensions, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ then one normally assumes $|q| < 1$, and if $q \in \mathbb{C}_p$ then one normally assumes $|1-q|_p < 1$.

The q-bosonic natural numbers are defined by $[n]_q = (1-q^n)/(1-q) = 1+q+q^2+\cdots+q^{n-1}$ for $n \in \mathbb{N}$, and the q-factorial is defined by $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$ (see [1–3]). For the q-extension of binomial coefficients, we use the following notation in the form of

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}. \tag{1.1}$$

Let $C[0,1]$ denote the set of continuous functions on the real interval $[0,1]$. The Bernstein operator for $f \in C[0,1]$ is defined by

$$\mathbb{B}_n(f \mid x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \tag{1.2}$$

where $n, k \in \mathbb{Z}_+$. The polynomials $B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ are called Bernstein polynomials of degree n (see [4–8]). For $f \in C[0,1]$, q-Bernstein type operator of order n for f is defined by

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} [x]_q^k [1-x]_{1/q}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x,q), \tag{1.3}$$

where $n, k \in \mathbb{Z}_+$. Here $B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{1/q}^{n-k}$ are called q-Bernstein type polynomials of degree n (see [9]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and write $f \in \mathrm{UD}(\mathbb{Z}_p)$, if the difference quotient $F_f(x,y) = (f(x)-f(y))/(x-y)$ has a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \tag{1.4}$$

(see [10]). Carlitz's q-Bernoulli numbers can be represented by a p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x]_q^n\, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} [x]_q^n q^x = \beta_{n,q} \tag{1.5}$$

(see [10, 11]). The kth order factorial of $[x]_q$ is defined by

$$[x]_{k,q} = [x]_q [x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)^k} \tag{1.6}$$

and is called the q-factorial of x of order k (see [10]).

In this paper, we give a p-adic q-integral representation for q-Bernstein type polynomials and derive some interesting identities for the q-Bernstein type polynomials associated with the q-extension of binomial distributions, q-Stirling numbers, and Carlitz's q-Bernoulli numbers.

2. q-Bernstein Polynomials

In this section, we assume that $0 < q < 1$. Let $\mathbb{P}_{n,q} = \{\sum_i a_i [x]_q^i \mid a_i \in \mathbb{R}\}$ be the space of q-polynomials of degree less than or equal to n.

We claim that the q-Bernstein type polynomials of degree n defined by (1.3) are a basis for $\mathbb{P}_{n,q}$.

First, we see that the q-Bernstein type polynomials of degree n span the space of q-polynomials. That is, any q-polynomials of degree less than or equal to n can be written as a linear combination of the q-Bernstein type polynomials of degree n.

For $n, k \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$B_{k,n}(x,q) = \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} (-1)^{l-k} [x]_q^l \tag{2.1}$$

(see [9]). If there exist constants $C_0, C_1, \ldots, C_n$ such that $C_0 B_{0,n}(x,q) + C_1 B_{1,n}(x,q) + \cdots + C_n B_{n,n}(x,q) = 0$ holds for all x, then we can derive the following equation from (2.1):

$$\begin{aligned} 0 &= C_0 B_{0,n}(x,q) + C_1 B_{1,n}(x,q) + \cdots + C_n B_{n,n}(x,q) \\ &= C_0 \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i}{0} [x]_q^i + C_1 \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \binom{i}{1} [x]_q^i \\ &\quad + \cdots + C_n \sum_{i=n}^{n} (-1)^{i-n} \binom{n}{i} \binom{i}{n} [x]_q^i \\ &= C_0 + \left\{ \sum_{i=0}^{1} C_i (-1)^{i-1} \binom{n}{1} \binom{1}{i} \right\} [x]_q + \cdots + \left\{ \sum_{i=0}^{n} C_i (-1)^{i-n} \binom{n}{n} \binom{n}{i} \right\} [x]_q^n. \end{aligned} \tag{2.2}$$

Since the power basis is a linearly independent set, it follows that

$$\begin{aligned} C_0 &= 0, \\ \sum_{i=0}^{1} C_i (-1)^{i-1} \binom{n}{1} \binom{1}{i} &= 0, \\ \vdots \qquad &\quad \vdots \\ \sum_{i=0}^{n} C_i (-1)^{i-n} \binom{n}{n} \binom{n}{i} &= 0, \end{aligned} \tag{2.3}$$

which implies that $C_0 = C_1 = \cdots = C_n = 0$ (C_0 is clearly zero, substituting this in the second equation gives $C_1 = 0$, substituting these two into the third equation gives $C_2 = 0$, and so on). Hence, we have the following theorem.

Theorem 2.1. *The q-Bernstein type polynomials of degree n are a basis for $\mathbb{P}_{n,q}$.*

Let us consider a q-polynomial $P_q(x) \in \mathbb{P}_{n,q}$ as a linear combination of q-Bernstein type basis functions as follows:

$$P_q(x) = C_0 B_{0,n}(x,q) + C_1 B_{1,n}(x,q) + \cdots + C_n B_{n,n}(x,q). \tag{2.4}$$

We can write (2.4) as a dot product of two values:

$$P_q(x) = \left(B_{0,n}(x,q), B_{1,n}(x,q), \ldots, B_{n,n}(x,q)\right)\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix}. \tag{2.5}$$

From (2.5), we can derive the following equation:

$$P_q(x) = \left(1, [x]_q, \ldots, [x]_q^n\right)\begin{pmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix}, \tag{2.6}$$

where the b_{ij} are the coefficients of the power basis that are used to determine the respective q-Bernstein type polynomials.

From (1.3) and (2.1), we note that

$$\begin{aligned} B_{0,2}(x,q) &= [1-x]_{1/q}^2 = \sum_{l=0}^{2}\binom{2}{l}(-1)^l [x]_q^l = 1 - 2[x]_q + [x]_q^2, \\ B_{1,2}(x,q) &= \binom{2}{1}[x]_q[1-x]_{1/q} = 2[x]_q - 2[x]_q^2, \\ B_{2,2}(x,q) &= \binom{2}{2}[x]_q^2 = [x]_q^2. \end{aligned} \tag{2.7}$$

In the quadratic case ($n = 2$), the matrix representation is

$$P_q(x) = \left(1, [x]_q, [x]_q^2\right)\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}. \tag{2.8}$$

In the cubic case ($n = 3$), the matrix representation is

$$P_q(x) = \left(1, [x]_q, [x]_q^2, [x]_q^3\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}. \tag{2.9}$$

In many applications of q-Bernstein polynomials, a matrix formulation for the q-Bernstein type polynomials seems to be useful.

Remark 2.2 (see [12]). All results of this section for $q = 1$ are well known in classical case (see Bernstein Polynomials by Joy).

3. q-Bernstein Polynomials, q-Stirling Numbers, and q-Bernoulli Numbers

In this section, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$.

For $f \in \mathrm{UD}(\mathbb{Z}_p)$, let us consider the p-adic analogue of q-Bernstein type operator of order n on $\mathbb{Z}_p$ as follows:

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}[x]_q^k[1-x]_{1/q}^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}(x,q). \tag{3.1}$$

Here $B_{k,n}(x,q)$ is the q-Bernstein type polynomials of degree n on $\mathbb{Z}_p$ defined by

$$B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{1/q}^{n-k}, \tag{3.2}$$

for $n, k \in \mathbb{Z}_+$ and $x \in \mathbb{Z}_p$.

Let $(Eh)(x) = h(x+1)$ be the shift operator. Then the q-difference operator is defined by

$$\Delta_q^n := (E - I)_q^n = \prod_{i=1}^{n}\left(E - q^{i-1}I\right), \tag{3.3}$$

where $(Ih)(x) = h(x)$. From (3.3), we derive the following equation:

$$\Delta_q^n f(0) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(n-k). \tag{3.4}$$

By (3.4), we easily see that

$$f(x)=\sum_{n\geq 0}\binom{x}{n}_q \Delta_q^n f(0) \tag{3.5}$$

(see [10, 11]).

The q-Stirling number of the first kind is defined by

$$\prod_{k=1}^{n}\left(1+[k]_q z\right)=\sum_{k=0}^{n} S_{1,q}(n,k)z^k, \tag{3.6}$$

and the q-Stirling number of the second kind is also defined by

$$\prod_{k=1}^{n}\left(\frac{1}{1+[k]_q z}\right)=\sum_{k=0}^{n} S_{2,q}(n,k)z^k. \tag{3.7}$$

By (3.3), (3.4), (3.6), and (3.7), we get

$$S_{2,q}(n,k)=\frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\binom{k}{j}_q [k-j]_q^n=\frac{q^{-\binom{k}{2}}}{[k]_q!}\Delta_q^k 0^n, \tag{3.8}$$

for $n,k\in\mathbb{Z}_+$ (see [10, 13]).

From the definition of q-Bernstein type polynomials of degree n on $\mathbb{Z}_p$, we easily see that

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)\,d\mu_q(x)=\sum_{l=0}^{n-k}\binom{n-k}{l}\binom{n}{k}(-1)^l\int_{\mathbb{Z}_p}[x]_q^{l+k}\,d\mu_q(x). \tag{3.9}$$

By (1.5) and (3.9), we obtain the following proposition.

Proposition 3.1. *For $n,k\in\mathbb{Z}_+$, one has*

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)\,d\mu_q(x)=\sum_{l=0}^{n-k}\binom{n-k}{l}\binom{n}{k}(-1)^l\beta_{l+k,q}, \tag{3.10}$$

where $\beta_{l+k,q}$ are the $(l+k)$th Carlitz's q-Bernoulli numbers.

From the definition of q-Bernstein polynomial, we note that

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x,q)=\sum_{k=0}^{i} q^{\binom{k}{2}}\binom{x}{k}_q [k]_q! S_{2,q}(k,i-k), \tag{3.11}$$

where $i \in \mathbb{N}$. From the definition of q-binomial coefficient, we have

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k}\binom{n}{k-1}_q + \binom{n}{k}_q. \tag{3.12}$$

By (3.12), we see that

$$\int_{\mathbb{Z}_p} \binom{x}{n}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{(n+1)-\binom{n+1}{2}} \tag{3.13}$$

(see [10, 11]). From (1.5), (3.11), and (3.13), we obtain the following theorem.

Theorem 3.2. *For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, one has*

$$\sum_{k=i}^{n}\sum_{l=0}^{n-k} \frac{\binom{k}{i}}{\binom{n}{i}} \binom{n-k}{l}\binom{n}{k}(-1)^l \beta_{l+k,q} = \sum_{k=0}^{i} q^{\binom{k}{2}} [k]_q! S_{2,q}(k, i-k) \frac{(-1)^k}{[k+1]_q} q^{(k+1)-\binom{k+1}{2}}. \tag{3.14}$$

It is easy to see that, for $i \in \mathbb{N}$,

$$\sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x, q) = [x]_q^i. \tag{3.15}$$

By (3.11) and (3.15), we easily get

$$[x]_q^i = \sum_{k=0}^{i} q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! S_{2,q}(k, i-k) \tag{3.16}$$

(see [10]). Thus, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} [x]_q^i d\mu_q(x) &= \sum_{k=0}^{i} q^{\binom{k}{2}} [k]_q! S_{2,q}(k, i-k) \int_{\mathbb{Z}_p} \binom{x}{k}_q d\mu_q(x) \\ &= q\sum_{k=0}^{i} [k]_q! S_{2,q}(k, i-k) \frac{(-1)^k}{[k+1]_q}.\end{aligned} \tag{3.17}$$

By (1.5) and (3.17), we obtain the following corollary.

Corollary 3.3. *For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, one has*

$$\beta_{i,q} = q\sum_{k=0}^{i} [k]_q! S_{2,q}(k, i-k) \frac{(-1)^k}{[k+1]_q}. \tag{3.18}$$

It is known that

$$S_{2,q}(n,k)=\frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k+n}{k-j}\binom{j+n}{j}_q \tag{3.19}$$

(see [10]) and

$$\binom{n}{k}_q=\sum_{j=0}^{n}\binom{n}{j}(q-1)^{j-k}S_{2,q}(k,j-k). \tag{3.20}$$

By a simple calculation, we have that

$$\begin{aligned} q^{nx}&=\sum_{k=0}^{n}(q-1)^k q^{\binom{k}{2}}\binom{n}{k}_q[x]_{k,q}\\ &=\sum_{m=0}^{n}\left\{\sum_{k=m}^{n}(q-1)^k\binom{n}{k}_q S_{1,q}(k,m)\right\}[x]_q^m,\\ q^{nx}&=\sum_{m=0}^{n}\binom{n}{m}(q-1)^m[x]_q^m. \end{aligned} \tag{3.21}$$

From (3.21), we note that

$$\binom{n}{m}=\sum_{k=m}^{n}(q-1)^{-m+k}\binom{n}{k}_q S_{1,q}(k,m) \tag{3.22}$$

(see [10]).

Thus, we obtain the following proposition.

Proposition 3.4. *For $n,k\in\mathbb{Z}_+$, one has*

$$B_{k,n}(x,q)=\binom{n}{k}[x]_q^k[1-x]_{1/q}^{n-k}=\sum_{m=k}^{n}(q-1)^{-k+m}\binom{n}{m}_q S_{1,q}(m,k)[x]_q^k[1-x]_{1/q}^{n-k}. \tag{3.23}$$

From the definition of the q-Stirling numbers of the first kind, we get

$$q^{\binom{n}{2}}\binom{x}{n}_q[n]_q!=[x]_{n,q}q^{\binom{n}{2}}=\sum_{k=0}^{n}S_{1,q}(n,k)[x]_q^k. \tag{3.24}$$

By (3.11) and (3.24), we obtain the following theorem.

Theorem 3.5. *For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, one has*

$$\sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x,q) = \sum_{k=0}^{i}\sum_{l=0}^{k} S_{1,q}(k,l) S_{2,q}(k,i-k)[x]_q^l. \tag{3.25}$$

By (3.15) and Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *For $i \in \mathbb{Z}_+$, one has*

$$\beta_{i,q} = \sum_{k=0}^{i}\sum_{l=0}^{k} S_{1,q}(k,l) S_{2,q}(k,i-k)\beta_{l.q}. \tag{3.26}$$

The q-Bernoulli polynomials of order $k \in \mathbb{Z}_+$ are defined by

$$\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i}(-1)^i q^{ix}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{l=1}^{k}(k-l+i)x_l} d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{3.27}$$

Thus, we have

$$\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n}\sum_{i=0}^{n}\frac{(-1)^i\binom{n}{i}(i+k)\cdots(i+1)}{[i+k]_q\cdots[i+1]_q} q^{ix} \tag{3.28}$$

(see [10]). The inverse q-Bernoulli polynomials of order k are defined by

$$\beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n}\sum_{i=0}^{n}\frac{(-1)^i\binom{n}{i}q^{ix}}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{l=1}^{k}(k-l+i)x_1} d\mu_q(x_1)\cdots d\mu_q(x_k)}. \tag{3.29}$$

In the special case $x = 0$, $\beta_{n,q}^{(k)}(0) = \beta_{n,q}^{(k)}$ are called the nth q-Bernoulli numbers of order k, and $\beta_{n,q}^{(-k)}(0) = \beta_{n,q}^{(-k)}$ are also called the inverse q-Bernoulli numbers of order k (see [10]).

From (3.29), we have

$$\begin{aligned}\beta_{k,q}^{(-n)} &= \frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{[j+n]_q\cdots[j+1]_q}{(j+n)\cdots(j+1)} \\ &= \frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^j\frac{\binom{k+n}{n-j}}{\binom{k+n}{n}}\binom{j+n}{n}_q\frac{[n]_q!}{n!} \\ &= \frac{[n]_q!}{\binom{k+n}{n}n!}\left\{\frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^j\binom{k+n}{n-j}\binom{j+n}{n}_q\right\}.\end{aligned} \tag{3.30}$$

By (3.19) and (3.30), we get

$$\frac{n!}{[n]_q!}\binom{k+n}{n}\beta_{k,q}^{(-n)} = S_{2,q}(n,k). \tag{3.31}$$

Therefore, by (3.11) and (3.31), we obtain the following theorem.

Theorem 3.7. *For $i, n, k \in \mathbb{Z}_+$, one has*

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x,q) = \sum_{k=0}^{i} q^{\binom{k}{2}} k! \binom{i}{k}\binom{x}{k}_q \beta_{i-k,q}^{(-k)}. \tag{3.32}$$

It is easy to show that

$$q^{\binom{n}{2}}\binom{x}{n}_q = \frac{1}{[n]_q!}\prod_{k=0}^{n-1}\left([x]_q - [k]_q\right) = \frac{1}{[n]_q!}\sum_{k=0}^{n}(-1)^k [x]_q^{n-k} S_{1,q}(n-1,k). \tag{3.33}$$

Thus, we have that

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x,q) = \sum_{k=0}^{i}\sum_{j=0}^{k}(-1)^j [x]_q^{k-j} S_{1,q}(k-1,j)\frac{k!}{[k]_q!}\binom{i}{k}\beta_{i-k,q}^{(-k)}, \tag{3.34}$$

where $n, k, i \in \mathbb{Z}_+$.

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