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Research Article

Sharp Power Mean Bounds for the Combination of Seiffert and Geometric Means

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We answer the question: for $\alpha \in (0,1)$, what are the greatest value p and the least value q such that the double inequality $M_p(a,b) < P^{\alpha}(a,b)G^{1-\alpha}(a,b) < M_q(a,b)$ holds for all $a,b>0$ with $a \neq b$. Here, $M_p(a,b)$, $P(a,b)$, and $G(a,b)$ denote the power of order p, Seiffert, and geometric means of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$, the power mean $M_p(a,b)$ of order p and the Seiffert mean $P(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$
$$P(a,b) = \begin{cases} \dfrac{a-b}{4\arctan\left(\sqrt{a/b}\right) - \pi}, & a \neq b, \\ a, & a = b. \end{cases} \tag{1.1}$$

The main properties of the power mean are given in [1]. It is well known that $M_p(a,b)$ is strictly increasing with respect to p for fixed $a,b>0$ with $a \neq b$. Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for the power mean can be found in the literature [2–16].

The Seiffert mean P was introduced by Seiffert in [17], it can be rewritten in the following symmetric form (see [18, (2.4)]):

$$P(a,b)=\begin{cases}\dfrac{a-b}{2\arcsin((a-b)/(a+b))}, & a\neq b,\\ a, & a=b.\end{cases} \tag{1.2}$$

Let $A(a,b)=(1/2)(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=\begin{cases}(b-a)/(\log b-\log a), & b\neq a,\\ a, & b=a,\end{cases}$ $H(a,b)=2ab/(a+b)$ and $I(a,b)=\begin{cases}1/e(b^b/a^a)^{1/(b-a)}, & b\neq a,\\ a, & b=a\end{cases}$ be the arithmetic, geometric, logarithmic, harmonic, and identric means of two positive numbers a and b, respectively. Then it is well known that

$$\begin{aligned}\min\{a,b\}&<H(a,b)=M_{-1}(a,b)<G(a,b)=M_0(a,b)\\ &<L(a,b)<I(a,b)<A(a,b)=M_1(a,b)<\max\{a,b\}\end{aligned} \tag{1.3}$$

for all $a,b>0$ with $a\neq b$.

In [9], Alzer and Janous presented the sharp power mean bounds for the sum $(2/3)A(a,b)+(1/3)G(a,b)$ as follows:

$$M_{\log 2/\log 3}(a,b)<\frac{2}{3}A(a,b)+\frac{1}{3}G(a,b)<M_{2/3}(a,b) \tag{1.4}$$

for all $a,b>0$ with $a\neq b$.

In [17], Seiffert proved that

$$L(a,b)<P(a,b)<I(a,b) \tag{1.5}$$

for all $a,b>0$ with $a\neq b$.

The following power mean bounds for the Seiffert mean was given by Jagers [19]:

$$M_{1/2}(a,b)<P(a,b)<M_{2/3}(a,b) \tag{1.6}$$

for all $a,b>0$ with $a\neq b$.

In [20, 21], the authors presented the bounds for the Seiffert mean P in terms of A and G as follows:

$$\begin{gathered}P(a,b)>\frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)},\\ \frac{1}{2}A(a,b)+\frac{1}{2}G(a,b)<P(a,b)<\frac{2}{3}A(a,b)+\frac{1}{3}G(a,b),\\ P(a,b)>A^{1/3}(a,b)G^{2/3}(a,b)\end{gathered} \tag{1.7}$$

for all $a,b>0$ with $a\neq b$.

The following sharp lower power mean bounds for $(1/3)G(a,b)+(2/3)H(a,b)$, $(2/3)G(a,b)+(1/3)H(a,b)$, and $P(a,b)$ can be found in [4, 6]:

$$\begin{gathered}\frac{1}{3}G(a,b)+\frac{2}{3}H(a,b)>M_{-2/3}(a,b),\\ \frac{2}{3}G(a,b)+\frac{1}{3}H(a,b)>M_{-1/3}(a,b),\\ P(a,b)>M_{\log 2/\log\pi}(a,b)\end{gathered}\tag{1.8}$$

for all $a,b>0$ with $a\neq b$.

The purpose of this paper is to answer the question: for $\alpha\in(0,1)$, what are the greatest value p and the least value q such that the double inequality $M_p(a,b)<P^{\alpha}(a,b)G^{1-\alpha}(a,b)<M_q(a,b)$ holds for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main result, we need several lemmas which we present in this section.

Lemma 2.1. *Let $\lambda\in(0,1/3)$, $x\in[1,\infty)$ and $h(x)=(1-3\lambda)x^{2\lambda+1}-(1+3\lambda)x^{2\lambda}-(1+3\lambda)x+(1-3\lambda)$. Then there exists $x_0>1$ such that $h(x)<0$ for $x\in[1,x_0)$, $h(x)>0$ for $x\in(x_0,\infty)$ and $h(x_0)=0$.*

Proof. Simple computations lead to

$$h(1)=-12\lambda<0,\tag{2.1}$$

$$\lim_{x\to+\infty}h(x)=+\infty,\tag{2.2}$$

$$\begin{gathered}h'(x)=(1-3\lambda)(2\lambda+1)x^{2\lambda}-2\lambda(1+3\lambda)x^{2\lambda-1}-(1+3\lambda),\\ h'(1)=-6\lambda(1+2\lambda)<0,\end{gathered}\tag{2.3}$$

$$\lim_{x\to+\infty}h'(x)=+\infty,\tag{2.4}$$

$$h''(x)=2\lambda x^{2\lambda-2}[(1+2\lambda)(1-3\lambda)x+(1-2\lambda)(1+3\lambda)]>0.\tag{2.5}$$

Inequality (2.5) implies that $h'(x)$ is strictly increasing in $[1,\infty)$. Then (2.3) and (2.4) lead to that there exists $x_1>1$ such that $h'(x)<0$ for $x\in[1,x_1)$ and $h'(x)>0$ for $x\in(x_1,\infty)$. Hence, $h(x)$ is strictly decreasing in $[1,x_1]$ and strictly increasing in $[x_1,\infty)$.

Therefore, Lemma 2.1 follows from (2.1) and (2.2) together with the monotonicity of $h(x)$. □

Lemma 2.2. *If $\lambda\in(1/\sqrt{10},1/3)$, then the following statements are true:*

(1) $300\lambda^4+324\lambda^3+29\lambda^2-45\lambda-8<0$;

(2) $-1176\lambda^5-24\lambda^4+114\lambda^3+10\lambda^2-3\lambda-1<0$;

(3) $-24\lambda^5+72\lambda^4+178\lambda^3+6\lambda^2-25\lambda-3<0$.

Proof. Simple computations lead to

(1) $300\lambda^4+324\lambda^3+29\lambda^2-45\lambda-8 < 300\times(1/3)^4+324\times(1/3)^3+29\times(1/3)^2-45/\sqrt{10}-8 = (590-243\sqrt{10})/54 < 0$;

(2) $-1176\lambda^5-24\lambda^4+114\lambda^3+10\lambda^2-3\lambda-1 < -1176\times(1/\sqrt{10})^5-24\times(1/\sqrt{10})^4+114\times(1/3)^3+10\times(1/3)^2-3/\sqrt{10}-1 = (3070-1107\sqrt{10})/750 < 0$;

(3) $-24\lambda^5+72\lambda^4+178\lambda^3+6\lambda^2-25\lambda-3 < -24\times(1/\sqrt{10})^5+72\times(1/3)^4+178\times(1/3)^3+6\times(1/3)^2-25/\sqrt{10}-3 = (34750-17037\sqrt{10})/6750 < 0$. □

Lemma 2.3. *If $\alpha \in (0, 3/\sqrt{10}]$, then*

$$I^{\alpha}(a,b)G^{1-\alpha}(a,b) < M_{2\alpha/3}(a,b) \tag{2.6}$$

holds for all $a, b > 0$ with $a \neq b$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $\beta = \alpha/3 \in (0, 1/\sqrt{10}]$. Then

$$\begin{aligned}
&\log\left[M_{2\alpha/3}(a,b)\right] - \log\left[I^{\alpha}(a,b)G^{1-\alpha}(a,b)\right] \\
&\quad = \log\left[M_{2\beta}(a,b)\right] - \log\left[I^{3\beta}(a,b)G^{1-3\beta}(a,b)\right] \\
&\quad = \frac{1}{2\beta}\log\frac{1+t^{2\beta}}{2} - \frac{3\beta t}{t-1}\log t - \frac{1-3\beta}{2}\log t + 3\beta.
\end{aligned} \tag{2.7}$$

Let

$$f(t) = \frac{1}{2\beta}\log\frac{1+t^{2\beta}}{2} - \frac{3\beta t}{t-1}\log t - \frac{1-3\beta}{2}\log t + 3\beta. \tag{2.8}$$

Then simple computations lead to

$$\lim_{t\to 1} f(t) = 0, \tag{2.9}$$

$$f'(t) = \frac{g(t)}{(t-1)^2}, \tag{2.10}$$

where $g(t) = (t^{2\beta+1} - 2t^{2\beta} + t^{2\beta-1}) / (1+t^{2\beta}) - 3\beta(t-1) + 3\beta\log t - (1-3\beta)(t-2+1/t)/2$,

$$\begin{aligned}
g(1) &= 0, \\
g'(t) &= \frac{g_1(t)}{2t^2\left(1+t^{2\beta}\right)^2},
\end{aligned} \tag{2.11}$$

where $g_1(t) = (1-3\beta)t^{4\beta+2} + 6\beta t^{4\beta+1} - (1+3\beta)t^{4\beta} - 2\beta t^{2\beta+2} + 4\beta t^{2\beta+1} - 2\beta t^{2\beta} - (1+3\beta)t^2 + 6\beta t + (1-3\beta)$,

$$g_1(1) = 0, \tag{2.12}$$

$$g_1'(1) = 0, \tag{2.13}$$

$$g_1''(1) = 0. \tag{2.14}$$

Let $g_2(t) = g_1'''(t)/(8\beta t^{2\beta-3})$ and $g_3(t) = g_2'''(t)/(2\beta t^{2\beta-3})$. Then

$$\begin{aligned} g_2(t) = {} & (1-3\beta)(2\beta+1)(4\beta+1)t^{2\beta+2} + 3\beta(4\beta+1)(4\beta-1)t^{2\beta+1} \\ & - (1+3\beta)(4\beta-1)(2\beta-1)t^{2\beta} - \beta(\beta+1)(2\beta+1)t^2 \\ & + \beta(2\beta+1)(2\beta-1)t - \beta(\beta-1)(2\beta-1), \end{aligned} \tag{2.15}$$

$$g_2(1) = 0, \tag{2.16}$$

$$g_2'(1) = 2\left(1-\sqrt{10}\beta\right)\left(1+\sqrt{10}\beta\right) \geq 0, \tag{2.17}$$

$$g_2''(1) = 2(6\beta+1)\left(1-\sqrt{10}\beta\right)\left(1+\sqrt{10}\beta\right) \geq 0, \tag{2.18}$$

$$\begin{aligned} g_3(t) = {} & 2(1-3\beta)(2\beta+1)^2(4\beta+1)(\beta+1)t^2 + 3\beta(4\beta+1)(4\beta-1) \\ & \times (2\beta+1)(2\beta-1)t - 2(1+3\beta)(4\beta-1)(2\beta-1)^2(\beta-1), \end{aligned} \tag{2.19}$$

$$g_3(1) = 9\beta\left(3-28\beta^2\right) > 0, \tag{2.20}$$

$$g_3'(t) = (2\beta+1)(4\beta+1)\left[4(1-3\beta)(2\beta+1)(\beta+1)t + 3\beta(4\beta-1)(2\beta-1)\right], \tag{2.21}$$

$$g_3'(1) = (2\beta+1)(4\beta+1)\left(4+3\beta-46\beta^2\right), \tag{2.22}$$

$$\min_{\beta\in[0,1/10]}\left(4+3\beta-46\beta^2\right) = \frac{3\sqrt{10}-6}{10} > 0. \tag{2.23}$$

From (2.21)–(2.23) we clearly see that $g_3'(t) > g_3'(1) \geq (3\sqrt{10}-6)/10(2\beta+1)(4\beta+1) > 0$ for $t \in (1,\infty)$, hence $g_3(t)$ is strictly increasing in $[1,\infty)$. Then (2.20) implies that $g_3(t) > 0$ for $t \in (1,\infty)$, hence $g_2''(t)$ is strictly increasing in $[1,\infty)$.

It follows from (2.18) and the monotonicity of $g_2''(t)$ that $g_2''(t) > 0$ for $t \in (1,\infty)$, hence $g_2'(t)$ is strictly increasing in $[1,\infty)$. Then (2.17) implies that $g_2'(t) > 0$ for $t \in (1,\infty)$, therefore $g_2(t)$ is strictly increasing in $[1,\infty)$.

Equation (2.16) and the monotonicity of $g_2(t)$ lead to that $g_2(t) > 0$ for $t \in (1,\infty)$, so $g_1''(t)$ is strictly increasing in $[1,\infty)$.

From (2.9)–(2.14) and the monotonicity of $g_1''(t)$ we can deduce that

$$f(t) > 0 \tag{2.24}$$

for $t \in (1,\infty)$.

Therefore, Lemma 2.3 follows from (2.7) and (2.8) together with (2.24). □

Remark 2.4. In [22, Theorem 3.1], the authors proved that

$$I^{1/2}(a,b)G^{1/2}(a,b) < M_{1/3}(a,b), \tag{2.25}$$

$$I^{2/3}(a,b)G^{1/3}(a,b) < M_{4/9}(a,b), \tag{2.26}$$

$$I^{1/3}(a,b)G^{2/3}(a,b) < M_{2/9}(a,b) \tag{2.27}$$

for all $a,b > 0$ with $a \neq b$.

Obviously, (2.6) is the generalization of (2.25)–(2.27).

Remark 2.5. If $\alpha \in (3/\sqrt{10}, 1)$, then $\beta = \alpha/3 \in (1/\sqrt{10}, 1/3)$ and (2.17) leads to

$$g_2'(1) = 2\left(1 - \sqrt{10}\beta\right)\left(1 + \sqrt{10}\beta\right) < 0. \tag{2.28}$$

Inequality (2.28) and the continuity of $g_2'(t)$ imply that there exists $\delta > 0$ such that

$$g'_2(t) < 0 \tag{2.29}$$

for $t \in [1, 1+\delta)$.

From (2.29) and (2.9)–(2.16) we can deduce that

$$f(t) < 0 \tag{2.30}$$

for $t \in (1, 1+\delta)$.

Equations (2.7) and (2.8) together with (2.30) lead to

$$I^{\alpha}(a,b)G^{1-\alpha}(a,b) > M_{2\alpha/3}(a,b) \tag{2.31}$$

for all $a,b > 0$ with $a/b \in (1, 1+\delta) \cup (1/(1+\delta), 1)$.

Therefore, $\alpha_0 = 3/\sqrt{10}$ is the largest value in $(0,1)$ such that inequality (2.6) holds for $\alpha \in (0, \alpha_0]$.

3. Main Result

Theorem 3.1. *If $\alpha \in (0,1)$, then*

$$M_0(a,b) < P^{\alpha}(a,b)G^{1-\alpha}(a,b) < M_{2\alpha/3}(a,b) \tag{3.1}$$

holds for all $a,b > 0$ with $a \neq b$, and $M_0(a,b)$ and $M_{2\alpha/3}(a,b)$ are the best possible lower and upper power mean bounds for the product $P^{\alpha}(a,b)G^{1-\alpha}(a,b)$.

Proof. For all $a, b > 0$ with $a \neq b$, from (1.3), (1.5) and Lemma 2.1 we clearly see that

$$P^{\alpha}(a,b)G^{1-\alpha}(a,b) > M_0(a,b) \tag{3.2}$$

for all $\alpha \in (0,1)$, and

$$P^{\alpha}(a,b)G^{1-\alpha}(a,b) < M_{2\alpha/3}(a,b) \tag{3.3}$$

for $\alpha \in (0, 3/\sqrt{10}]$.

Next, we prove that (3.3) is also true for $\alpha \in (3/\sqrt{10}, 1)$ and all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $\beta = \alpha/3 \in (1/\sqrt{10}, 1/3)$. Then (1.1) leads to

$$\begin{aligned} &\log[M_{2\alpha/3}(a,b)] - \log\left[P^{\alpha}(a,b)G^{1-\alpha}(a,b)\right] \\ &\quad = \log\left[M_{2\beta}(a,b)\right] - \log\left[P^{3\beta}(a,b)G^{1-3\beta}(a,b)\right] \\ &\quad = \frac{1}{2\beta}\log\frac{1+t^{4\beta}}{2} - 3\beta\log\frac{t^2-1}{4\arctan t - \pi} - (1-3\beta)\log t. \end{aligned} \tag{3.4}$$

Let

$$F(t) = \frac{1}{2\beta}\log\frac{1+t^{4\beta}}{2} - 3\beta\log\frac{t^2-1}{4\arctan t - \pi} - (1-3\beta)\log t. \tag{3.5}$$

Then simple computations lead to

$$\lim_{t\to 1} F(t) = 0, \tag{3.6}$$

$$F'(t) = \frac{(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)}{t(t^2-1)(t^{4\beta}+1)} + \frac{12\beta}{(t^2+1)(4\arctan t - \pi)}. \tag{3.7}$$

From Lemma 2.1 we know that there exists $\lambda_0 > 1$ such that

$$(1-3\beta)\lambda_0{}^{4\beta+2} - (1+3\beta)\lambda_0{}^{4\beta} - (1+3\beta)\lambda_0{}^2 + (1-3\beta) = 0, \tag{3.8}$$

$$(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta) < 0 \tag{3.9}$$

for $t \in [1, \lambda_0)$, and

$$(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta) > 0 \tag{3.10}$$

for $t \in (\lambda_0, \infty)$.

We divide two cases to prove that

$$F'(t) > 0 \tag{3.11}$$

for $t > 1$. □

Case 1. $t \in [\lambda_0, \infty)$. Then (3.11) follows from (3.7) and (3.8) together with (3.10).

Case 2. $t \in (1, \lambda_0)$. Then (3.7) can be written as

$$F'(t) = \frac{(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)}{t(t^2-1)(t^{4\beta}+1)(4\arctan t - \pi)} F_1(t), \tag{3.12}$$

where

$$F_1(t) = 4\arctan t + \frac{12\beta t(t^2-1)(t^{4\beta}+1)}{(t^2+1)\left[(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)\right]} - \pi. \tag{3.13}$$

Let $x = t^2 \in (1, {\lambda_0}^2)$, then (3.13) leads to

$$\begin{gathered} F_1(1) = 0, \\ F_1'(t) = \frac{4F_2(x)}{(t^2+1)^2\left[(1-3\beta)t^{4\beta+2} - (1+3\beta)t^{4\beta} - (1+3\beta)t^2 + (1-3\beta)\right]^2}, \end{gathered} \tag{3.14}$$

where

$$\begin{aligned} F_2(x) = {} & \left(18\beta^2 - 9\beta + 1\right)x^{4\beta+3} - \left(18\beta^2 - 3\beta + 1\right)x^{4\beta+2} \\ & - \left(18\beta^2 + 3\beta + 1\right)x^{4\beta+1} + \left(18\beta^2 + 9\beta + 1\right)x^{4\beta} \\ & + 2\left(6\beta^2 - 1\right)x^{2\beta+3} - 2\left(6\beta^2 - 1\right)x^{2\beta+2} \\ & - 2\left(6\beta^2 - 1\right)x^{2\beta+1} + 2\left(6\beta^2 - 1\right)x^{2\beta} \\ & + \left(18\beta^2 + 9\beta + 1\right)x^3 - \left(18\beta^2 + 3\beta + 1\right)x^2 \\ & - \left(18\beta^2 - 3\beta + 1\right)x + \left(18\beta^2 - 9\beta + 1\right). \end{aligned} \tag{3.15}$$

Let $F_3(x) = (x^{4-2\beta}/8\beta)F_2{}^{(4)}(x)$, then (3.15) leads to

$$F_2(1) = 0,$$
$$F_2'(1) = 0,$$
$$F_2''(1) = 0, \tag{3.16}$$
$$F_2'''(1) = 0,$$

$$\begin{aligned} F_3(x) = {} & \left(18\beta^2 - 9\beta + 1\right)(4\beta + 1)(4\beta + 3)(2\beta + 1)x^{2\beta+3} \\ & - \left(18\beta^2 - 3\beta + 1\right)(2\beta + 1)(4\beta + 1)(4\beta - 1)x^{2\beta+2} \\ & - \left(18\beta^2 + 3\beta + 1\right)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^{2\beta+1} \\ & + \left(18\beta^2 + 9\beta + 1\right)(4\beta - 1)(2\beta - 1)(4\beta - 3)x^{2\beta} \\ & + \left(6\beta^2 - 1\right)(2\beta + 3)(\beta + 1)(2\beta + 1)x^3 \\ & - \left(6\beta^2 - 1\right)(\beta + 1)(2\beta + 1)(2\beta - 1)x^2 \\ & - \left(6\beta^2 - 1\right)(2\beta + 1)(2\beta - 1)(\beta - 1)x \\ & + \left(6\beta^2 - 1\right)(2\beta - 1)(\beta - 1)(2\beta - 3). \end{aligned} \tag{3.17}$$

From $\beta \in (1/\sqrt{10}, 1/3)$ and $x > 1$, we clearly see that $(18\beta^2-9\beta+1)(4\beta+1)(4\beta+3)(2\beta+1)x^{2\beta+3} < (18\beta^2-9\beta+1)(4\beta+1)(4\beta+3)(2\beta+1)x^3$, $-(18\beta^2-3\beta+1)(2\beta+1)(4\beta+1)(4\beta-1)x^{2\beta+2} < -(18\beta^2 - 3\beta + 1)(2\beta + 1)(4\beta + 1)(4\beta - 1)x^2$, $-(18\beta^2 + 3\beta + 1)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^{2\beta+1} < -(18\beta^2 + 3\beta + 1)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^2$ and $(18\beta^2 + 9\beta + 1)(4\beta - 1)(2\beta - 1)(4\beta - 3)x^{2\beta} < (18\beta^2 + 9\beta + 1)(4\beta - 1)(2\beta - 1)(4\beta - 3)x$. These inequalities and Lemma 2.2 lead to

$$\begin{aligned} F_3(x) < {} & \left(18\beta^2 - 9\beta + 1\right)(4\beta + 1)(4\beta + 3)(2\beta + 1)x^3 \\ & - \left(18\beta^2 - 3\beta + 1\right)(2\beta + 1)(4\beta + 1)(4\beta - 1)x^2 \\ & - \left(18\beta^2 + 3\beta + 1\right)(4\beta + 1)(4\beta - 1)(2\beta - 1)x^2 \\ & + \left(18\beta^2 + 9\beta + 1\right)(4\beta - 1)(2\beta - 1)(4\beta - 3)x \\ & + \left(6\beta^2 - 1\right)(2\beta + 3)(\beta + 1)(2\beta + 1)x^3 \\ & - \left(6\beta^2 - 1\right)(\beta + 1)(2\beta + 1)(2\beta - 1)x^2 \\ & - \left(6\beta^2 - 1\right)(2\beta + 1)(2\beta - 1)(\beta - 1)x \\ & + \left(6\beta^2 - 1\right)(2\beta - 1)(\beta - 1)(2\beta - 3) \end{aligned}$$

$$\begin{aligned}
&= 2\beta\left(300\beta^4 + 324\beta^3 + 29\beta^2 - 45\beta - 8\right)x^3 \\
&\quad + \left(-1176\beta^5 - 24\beta^4 + 114\beta^3 + 10\beta^2 - 3\beta - 1\right)x^2 \\
&\quad + 2\left(276\beta^5 - 276\beta^4 + 3\beta^3 + 43\beta^2 - 3\beta - 1\right)x \\
&\quad + \left(24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3\right) \\
&< 2\beta\left(300\beta^4 + 324\beta^3 + 29\beta^2 - 45\beta - 8\right)x \\
&\quad + \left(-1176\beta^5 - 24\beta^4 + 114\beta^3 + 10\beta^2 - 3\beta - 1\right)x \\
&\quad + 2\left(276\beta^5 - 276\beta^4 + 3\beta^3 + 43\beta^2 - 3\beta - 1\right)x \\
&\quad + 24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3 \\
&= \left(-24\beta^5 + 72\beta^4 + 178\beta^3 + 6\beta^2 - 25\beta - 3\right)x \\
&\quad + 24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3 \\
&< -24\beta^5 + 72\beta^4 + 178\beta^3 + 6\beta^2 - 25\beta - 3 \\
&\quad + 24\beta^5 - 72\beta^4 + 62\beta^3 - 6\beta^2 - 11\beta + 3 \\
&= 12\beta\left(20\beta^2 - 3\right) < 0.
\end{aligned} \tag{3.18}$$

From (3.16)–(3.18) we can deduce that

$$F_2(x) < 0 \tag{3.19}$$

for $x \in (1, \lambda_0{}^2)$.

Equation (3.14) together with (3.19) imply that

$$F_1(t) < 0 \tag{3.20}$$

for $t \in (1, \lambda_0)$.

Therefore, (3.11) follows from (3.9) and (3.12) together with (3.20).

It follows from (3.4)–(3.6) and (3.11) that

$$P^{\alpha}(a,b)G^{1-\alpha}(a,b) < M_{2\alpha/3}(a,b) \tag{3.21}$$

for $\alpha \in (3/\sqrt{10}, 1)$ and all $a, b > 0$ with $a \neq b$.

At last, we prove that $M_0(a,b)$ and $M_{2\alpha/3}(a,b)$ are the best possible lower- and upper-power mean bounds for the product $P^{\alpha}(a,b)G^{1-\alpha}(a,b)$, respectively.

For any $0 < \varepsilon < (2/3)\alpha$ and $x > 0$, from (1.1) one has

$$\lim_{x\to+\infty}\frac{M_\varepsilon(1,x)}{P^\alpha(1,x)G^{1-\alpha}(1,x)} = \lim_{x\to+\infty}\left[\frac{((1+x^{-\varepsilon})/2)^{1/\varepsilon}}{\left((1-x^{-1})/\left(4\arctan\sqrt{x}-\pi\right)\right)^\alpha}x^{(1-\alpha)/2}\right] = +\infty, \tag{3.22}$$

$$\left[P^\alpha(1,1+x)G^{1-\alpha}(1,1+x)\right]^{(2\alpha/3)-\varepsilon} - \left[M_{(2\alpha/3)-\varepsilon}(1,1+x)\right]^{(2\alpha/3)-\varepsilon} = \frac{J(x)}{\left(4\arctan\sqrt{1+x}-\pi\right)^{\alpha(2\alpha-3\varepsilon)/3}}, \tag{3.23}$$

where $J(x) = x^{\alpha(2\alpha-3\varepsilon)/3}(1+x)^{(1-\alpha)(2\alpha-3\varepsilon)/6} - (1/2)\left(4\arctan\sqrt{1+x}-\pi\right)^{\alpha(2\alpha-3\varepsilon)/3}[1+(1+x)^{(2\alpha-3\varepsilon)/3}]$.

Let $x \to 0$, making use of the Taylor expansion we get

$$J(x) = \frac{1}{24}\varepsilon(2\alpha-3\varepsilon)x^{(1/3)\alpha(2\alpha-3\varepsilon)+2} + o\left(x^{(1/3)\alpha(2\alpha-3\varepsilon)+2}\right). \tag{3.24}$$

Equation (3.22) implies that for any $0 < \varepsilon < (2/3)\alpha$ there exists $X = X(\varepsilon,\alpha) > 1$ such that $M_\varepsilon(1,x) > P^\alpha(1,x)G^{1-\alpha}(1,x)$ for $x \in (X,\infty)$.

Equations (3.23) and (3.24) imply that for any $0 < \varepsilon < (2/3)\alpha$ there exists $\delta = \delta(\varepsilon,\alpha) > 0$ such that $P^\alpha(1,1+x)G^{1-\alpha}(1,1+x) > M_{2\alpha/3-\varepsilon}(1,1+x)$ for $x \in (0,\delta)$.

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