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Research Article

On Convexity of Composition and Multiplication Operators on Weighted Hardy Spaces

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A bounded linear operator T on a Hilbert space $\mathscr{H}$, satisfying $\|T^2h\|^2 + \|h\|^2 \geq 2\|Th\|^2$ for every $h \in \mathscr{H}$, is called a convex operator. In this paper, we give necessary and sufficient conditions under which a convex composition operator on a large class of weighted Hardy spaces is an isometry. Also, we discuss convexity of multiplication operators.

1. Introduction and Preliminaries

We denote by $B(\mathscr{H})$ the space of all bounded linear operators on a Hilbert space $\mathscr{H}$. An operator $T \in B(\mathscr{H})$ is said to be *convex*, if

$$\left\|T^2h\right\|^2 + \|h\|^2 \geq 2\|Th\|^2 \tag{1.1}$$

for each $h \in \mathscr{H}$. Note that if T is a convex operator then the sequence $(\|T^nh\|^2)_{n\in\mathbf{N}}$ forms a convex sequence for every $h \in \mathscr{H}$. Taking $\Delta_T = T^*T - I$, it is easily seen that T is a convex operator if and only if $T^*\Delta_T T \geq \Delta_T$.

A weighted Hardy space is a Hilbert space of analytic functions on the open unit disc $\mathbf{D}$ for which the sequence $(z^j)_{j=0}^{\infty}$ forms a complete orthogonal set of nonzero vectors. It is usually assumed that $\|1\| = 1$. Writing $\beta(j) = \|z^j\|$, this space is denoted by $H^2(\beta)$ and its norm is given by

$$\left\|\sum_{j=0}^{\infty} a_j z^j\right\|^2 = \sum_{j=0}^{\infty} \left|a_j\right|^2 \beta(j)^2. \tag{1.2}$$

Let φ be an analytic map of the open unit disc $\mathbf{D}$ into itself, and define $C_\varphi(f) = f \circ \varphi$ whenever f is analytic on $\mathbf{D}$. The function φ is called the symbol of the composition operator. For a positive integer n, the nth iterate of φ, denoted by φ_n, is the function obtained by composing φ with itself n times; also φ_0 is defined to be the identity function. Denote the reproducing kernel at $z \in \mathbf{D}$, for the space $H^2(\beta)$, by K_z. Then $\langle f, K_z \rangle = f(z)$ for every $f \in H^2(\beta)$. It is known that $C_\varphi^*(K_z) = K_{\varphi(z)}$ for all z in $\mathbf{D}$. The generating function for $H^2(\beta)$ is the function given by

$$k(z) = \sum_{j=0}^{\infty} \frac{z^j}{\beta(j)^2}. \tag{1.3}$$

This function is analytic on $\mathbf{D}$. Moreover, if $w \in \mathbf{D}$ then $K_w(z) = k(\overline{w}z)$ and $\|K_w\|^2 = k(|w|^2)$ (see [1]).

Recently, there has been a great interest in studying operator theoretic properties of composition and weighted composition operators, see, for example, monographs [1, 2], papers [3–16], as well as the reference therein.

Isometric operators on weighted Hardy spaces, especially those that are composition operators are discussed by many authors. Isometries of the Hardy space H^2 among composition operators are characterized in [17, page 444], [18] and [12, page 66]. Indeed, it is shown that the only composition operators on H^2 that are isometries are the ones induced by inner functions vanishing at the origin. Bayart [5] generalized this result and showed that every composition operator on H^2 which is similar to an isometry is induced by an inner function with a fixed point in the unit disc. The surjective isometries of H^p, $1 \leq p < \infty$ that are weighted composition operators have been described by Forelli [19]. Carswell and Hammond [6] proved that the isometric composition operators of the weighted Bergman space A^2_α are the rotations. Cima and Wogen [20] have characterized all surjective isometries of the Bloch space. Furthermore, the identification of all isometric composition operators on the Bloch space is due to Colonna [8]. Some related results can be found also in [3, 4, 6, 21–25].

Herein, we are interested in studying the convexity of composition and multiplication operators acting on a weighted Hardy space $H^2(\beta)$. First, we give some preliminary facts on convex operators. Next, we will offer necessary and sufficient conditions under which a convex composition operator may be isometry on a large class of weighted Hardy spaces containing Hardy, Bergman, and Dirichlet spaces. We also discuss on convexity of the adjoint of a composition operator. Finally, we will obtain similar results for multiplication operators and their adjoints. For a good reference on isometric multiplication operators the reader can see [3].

Throughout this paper, T is assumed to be a bounded linear operator on a Hilbert space $\mathscr{H}$. It is easy to see that for every convex operator T, the sequence $(T^{*^n}\Delta_T T^n)_n$ forms an increasing sequence. We use this fact to prove the following theorem.

Theorem 1.1. *If T is a convex operator then so is every nonnegative integer power of T.*

Proof. We argue by using mathematical induction. The convexity of T implies that the result holds for $k = 1$. Suppose that $T^{*^n}\Delta_{T^n}T^n \geq \Delta_{T^n}$, then

$$\begin{aligned} T^{*^{n+1}}\Delta_{T^{n+1}}T^{n+1} - \Delta_{T^{n+1}} &= T^{*^{n+1}}(T^*\Delta_{T^n}T + \Delta_T)T^{n+1} - \Delta_{T^{n+1}} \\ &= T^{*^2}\left(T^{*^n}\Delta_{T^n}T^n\right)T^2 + T^{*^{n+1}}\Delta_T T^{n+1} - \Delta_{T^{n+1}} \end{aligned}$$

$$\begin{aligned}
&\geq T^{*^2}\Delta_{T^n}T^2 + T^{*^n}\Delta_T T^n - \Delta_{T^{n+1}} \\
&= T^{*^2}\left(T^{*^n}T^n - I\right)T^2 + T^{*^n}\Delta_T T^n - T^*\left(T^{*^n}T^n - I\right)T - \Delta_T \\
&= T^{*^n}\left(T^{*^2}T^2\right)T^n - T^{*^2}T^2 + T^{*^n}\Delta_T T^n - T^{*^n}(T^*T)T^n + T^*T - \Delta_T \\
&= T^{*^n}\left(T^{*^2}T^2 - I\right)T^n - T^{*^2}T^2 + I \\
&\geq 2T^{*^n}\Delta_T T^n - T^{*^2}T^2 + I \\
&\geq 2T^*\Delta_T T - T^{*^2}T^2 + I \\
&= T^*\Delta_T T - \Delta_T \geq 0.
\end{aligned} \tag{1.4}$$

So the result holds for $k = n + 1$. □

Proposition 1.2. *If T is a convex operator, then for every nonnegative integer n,*

$$T^{*^n}T^n \geq n\Delta_T + I. \tag{1.5}$$

Proof. We give the assertion by using mathematical induction on n. The result is clearly true for $n = 1$. Suppose that $T^{*^n}T^n \geq n\Delta_T + I$. Thus,

$$\begin{aligned}
T^{*^{n+1}}T^{n+1} &= T^*\left(T^{*^n}T^n\right)T \\
&\geq T^*(n\Delta_T + I)T \\
&= nT^*\Delta_T T + T^*T \\
&= n\left(T^{*^2}T^2 - 2T^*T + I\right) + nT^*T + T^*T - nI \\
&\geq (n+1)T^*T - nI \\
&= (n+1)\Delta_T + I.
\end{aligned} \tag{1.6}$$

So the result holds for $k = n + 1$. □

Proposition 1.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a convex operator and let $h \in \mathcal{H}$ be such that $\sup_{k\geq 0}\|T^k h\| < \infty$. If $\Delta_T \geq 0$, then $\|Th\| = \|h\|$.*

Proof. By applying Proposition 1.2, we observe that for every nonnegative integer n,

$$n\langle \Delta_T h, h\rangle + \|h\|^2 \leq \|T^n h\|^2 \leq \sup_{k\geq 0}\left\|T^k h\right\|^2 < \infty. \tag{1.7}$$

Letting $n \to \infty$, the positivity of Δ_T implies that $\Delta_T h = 0$; hence, $\|Th\| = \|h\|$. □

Proposition 1.4. *Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$ be a convex operator satisfying $\Delta_T \geq 0$. Suppose that there is a nonnegative integer i and a scalar α_i with $0 < |\alpha_i| \leq 1$ so that $Te_i = \alpha_i e_i$, then $\mathcal{M} = \vee_{n \neq i}\{e_n\}$ is an invariant subspace for T.*

Proof. Using Proposition 1.2, we see that

$$\|e_i\|^2 \geq \left\|\alpha_i^n e_i\right\|^2 = \|T^n e_i\|^2 = \langle T^{*n} T^n e_i, e_i \rangle \geq n\langle \Delta_T e_i, e_i \rangle + \|e_i\|^2 \tag{1.8}$$

for every $n \geq 0$. Let $n \to \infty$. Since Δ_T is a positive operator, we conclude that $\Delta_T e_i = 0$. Consequently, $T^* e_i = (1/\alpha_i)T^*Te_i = (1/\alpha_i)e_i$. Now, if $f \in \mathcal{M}$ then $\langle Tf, e_i \rangle = 0$; hence, $Tf \in \mathcal{M}$. □

2. Composition Operators

Our purpose in this section is to discuss on convex composition operators on a weighted Hardy space. Recall that an operator T in $B(\mathcal{H})$ is an isometry, if $\Delta_T = 0$. At first, we give an example of a nonisometric composition operator T on a weighted Hardy space such that $T^*\Delta_T T \geq \Delta_T \geq 0$. For simplicity of notation, Δ_{C_φ} is denoted by Δ_φ.

Example 2.1. Consider the weighted Hardy space $H^2(\beta)$ with weight sequence $(\beta(n))_n$ given by $\beta(n) = n + 1$. Define $\varphi : \mathbf{D} \to \mathbf{D}$ by $\varphi(z) = z^2$. It is easily seen that $C_\varphi(H^2(\beta)) \subseteq H^2(\beta)$, and an application of the closed graph theorem shows that C_φ is bounded. Now, a simple calculation shows that

$$\left\langle \left(C_\varphi^* \Delta_\varphi C_\varphi - \Delta_\varphi\right)\left(z^k\right), z^k \right\rangle = \left\|C_{\varphi 2} z^k\right\|^2 - 2\left\|C_\varphi z^k\right\|^2 + \left\|z^k\right\|^2 > 0 \tag{2.1}$$

for all $k \geq 0$; besides

$$\left\langle \Delta_\varphi z^k, z^k \right\rangle = \left\|C_\varphi z^k\right\|^2 - \left\|z^k\right\|^2 \tag{2.2}$$

which is positive for all $k \geq 1$, and zero whenever $k = 0$. It follows that $C_\varphi^* \Delta_\varphi C_\varphi \geq \Delta_\varphi \geq 0$, but C_φ is not an isometry.

Proposition 2.2. *Suppose that $T : H^2(\beta) \to H^2(\beta)$ is a convex operator satisfying $T1 = 1$ and $\Delta_T \geq 0$, then*

$$M = \left\{f \in H^2(\beta) : f(0) = 0\right\} \tag{2.3}$$

is a nontrivial invariant subspace of T.

Proof. Clearly M is a nontrivial closed subspace of T. To show that M is invariant for T, apply Proposition 1.4 for the Hilbert space $\mathcal{H} = H^2(\beta)$, the orthonormal basis $\{e_n\}_n$ given by $e_n = z^n/\beta(n)$, $i = 0$ and $\alpha_0 = 1$. □

Example 2.3. Consider the Bergman space $A^2(\mathbf{D})$ consisting of all analytic functions f on the open unit disc $\mathbf{D}$, for which

$$\|f\|^2 = \int_{\mathbf{D}} |f(z)|^2 dA(z) < \infty, \tag{2.4}$$

where $dA(z)$ is the normalized Lebesgue area measure on $\mathbf{D}$. If $f \in A^2(\mathbf{D})$ is represented by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}. \tag{2.5}$$

Also, $\{z^k\}_k$ forms an orthogonal basis for $A^2(\mathbf{D})$. Fix nonnegative integers k and n, and observe that

$$\left\|C_\varphi^n z^k\right\|^2 = \|\varphi_k^n\|^2 = \int_{\mathbf{D}} |\varphi_k^n(z)|^2 dA(z) \le \int_{\mathbf{D}} dA(z) = 1. \tag{2.6}$$

Thus, Proposition 1.3 implies that $C_\varphi^* \Delta_\varphi C_\varphi \ge \Delta_\varphi \ge 0$ if and only if C_φ is an isometry. In this case, taking $T = C_\varphi$ and $f(z) = z$ in Proposition 2.2, we conclude that $\varphi(0) = 0$; thus, the Schwarz lemma implies that $|\varphi(z)| \le |z|$ for all $z \in \mathbf{D}$. On the other hand, if $f(z) = z$ then

$$\int_{\mathbf{D}} |\varphi(z)|^2 dA(z) = \|C_\varphi f\|^2 = \|f\|^2 = \int_{\mathbf{D}} |z|^2 dA(z), \tag{2.7}$$

and so $|\varphi(z)| = |z|$ almost everywhere with respect to the area measure. Hence, $\varphi(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$.

Example 2.4. Consider the Hardy space $H^2(\mathbf{D})$. If φ is an analytic self-map of the unit disc, then φ induces a bounded composition operator, and $\|C_\varphi^n z^k\| \le 1$ for all nonnegative integers n and k. Thus, by Proposition 1.3, $C_\varphi^* \Delta_\varphi C_\varphi \ge \Delta_\varphi \ge 0$ if and only if C_φ is an isometry.

Recall that the Dirichlet space $\mathfrak{D}$ is the set of all functions analytic on $\mathbf{D}$ whose derivatives lie in the Bergman space $A^2(\mathbf{D})$. The Dirichlet norm is defined by

$$\|f\|_{\mathfrak{D}}^2 = |f(0)|^2 + \int_{\mathbf{D}} |f'(z)|^2 dA(z). \tag{2.8}$$

If φ is a univalent self-map of $\mathbf{D}$, then C_φ is bounded on $\mathfrak{D}$ [2, page 18]. Also, the area formula [1, page 36], shows that

$$\|C_\varphi f\|_{\mathfrak{D}}^2 = |(f o \varphi)(0)|^2 + \int_{\mathbf{D}} |f'(z)|^2 n_\varphi(z) dA(z), \tag{2.9}$$

where $n_\varphi(z)$ is, as usual, the counting function defined as the cardinality of the set $\{w \in \mathbf{D} : \varphi(w) = z\}$.

In the next theorem, we characterize all convex composition operators C_φ on $\mathfrak{D}$ satisfying $\Delta_\varphi \geq 0$. Note that we cannot use Proposition 1.3 for the Dirichlet space, thanks to the fact that in general the positive powers of C_φ are not uniformly bounded on the z^i's.

Theorem 2.5. *If C_φ is convex on the Dirichlet space $\mathfrak{D}$, then $\Delta_\varphi \geq 0$ if and only if C_φ is an isometry.*

Proof. One implication is clear. Suppose that Δ_φ is a positive operator, and take $T = C_\varphi$ in Proposition 2.2. Since the identity function is in the subspace $M = \{f \in \mathfrak{D} : f(0) = 0\}$, we conclude that $\varphi(0) = 0$. Thus, in light of (2.9), to show that C_φ is an isometry it is sufficient to prove that

$$\int_{\mathbf{D}} |f'(z)| (1 - n_\varphi)(z) dA(z) = 0, \quad \forall f \in \mathfrak{D}. \tag{2.10}$$

Let f be any function in the Dirichlet space $\mathfrak{D}$. Then

$$0 \leq \left\langle \left(C_\varphi^* \Delta_\varphi C_\varphi - \Delta_\varphi\right)(f), f \right\rangle = \int_{\mathbf{D}} |f'(z)|^2 \left(n_{\varphi_2} - 2n_\varphi + 1\right)(z) dA(z). \tag{2.11}$$

Furthermore,

$$0 \leq \langle \Delta_\varphi f, f \rangle = \int_{\mathbf{D}} |f'(z)|^2 \left(n_\varphi - 1\right)(z) dA(z). \tag{2.12}$$

By summing up these two relations we get

$$\int_{\mathbf{D}} |f'(z)|^2 \left(n_{\varphi_2} - n_\varphi\right)(z) dA(z) \geq 0. \tag{2.13}$$

But $n_{\varphi_2}(z) \leq n_\varphi(z)$, and so

$$\int_{\mathbf{D}} |f'(z)|^2 \left(n_{\varphi_2} - n_\varphi\right)(z) dA(z) = 0, \quad \forall f \in \mathfrak{D}. \tag{2.14}$$

This, in turn, implies that $n_{\varphi_2}(z) = n_\varphi(z)$ almost everywhere. Substituting this in (2.11), and then considering (2.12) the assertion will be completed. □

Observe that if $\varphi(0) = 0$, $n_{\varphi_2} - 2n_\varphi + 1 \geq 0$ almost everywhere, and C_φ is bounded on $\mathfrak{D}$ then it is convex. Indeed,

$$\left\langle \left(C_\varphi^* \Delta_\varphi C_\varphi - \Delta_\varphi\right) f, f \right\rangle = \int_{\mathbf{D}} |f'(z)|^2 \left(n_{\varphi_2} - 2n_\varphi + 1\right)(z) dA(z) \geq 0. \tag{2.15}$$

In the next theorem, we turn to the adjoint of a composition operator and give necessary and sufficient conditions under which a convex operator C_φ^* is an isometry.

Theorem 2.6. *Let φ be an analytic self-map of* **D** *with $\varphi(0) = 0$. If C_φ^* is a convex operator on $H^2(\beta)$, then it is an isometry if and only if $\Delta_{C_\varphi^*} \geq 0$.*

Proof. Suppose that $\Delta_{C_\varphi^*} \geq 0$, and assume that φ is not the identity or an elliptic automorphism. By the Denjoy-Wolff theorem φ_n converges uniformly to zero on compact subsets of **D** [1], and so for every $z \in \mathbf{D}$,

$$\lim_{n\to\infty} \left\| K_{\varphi_n(z)} \right\| = \|K_0\|. \tag{2.16}$$

Proposition 1.2 coupled with the fact that $C_\varphi^{*^n} K_z = K_{\varphi_n(z)}$ implies that for all $z \in \mathbf{D}$ and all nonnegative integers n,

$$\left\| K_{\varphi_n(z)} \right\|^2 \geq n\left(\left\| K_{\varphi(z)} \right\|^2 - \|K_z\|^2 \right) + \|K_z\|^2. \tag{2.17}$$

Furthermore, the positivity of Δ_T shows that $\|K_{\varphi(z)}\| \geq \|K_z\|$. Thus, in light of (2.16) and (2.17) we conclude that $\|K_z\| = \|K_{\varphi(z)}\|$ for all $z \in \mathbf{D}$, and so $\|K_z\| = \|K_{\varphi_n(z)}\|$ for every positive integer n. Consequently, $\|K_z\| = \|K_0\|$ for all $z \in \mathbf{D}$. It follows that

$$1 = \|K_0\|^2 = \|K_z\|^2 = k\left(|z|^2\right) = 1 + \sum_{j=1}^{\infty} \frac{\left(|z|^2\right)^j}{\beta(j)^2}, \quad \text{for } z \in \mathbf{D}. \tag{2.18}$$

This contradiction shows that φ is the identity or an elliptic automorphism. Thus, there is a $\theta \in [0, 2\pi)$ so that $\varphi(z) = e^{i\theta} z$ for all $z \in \mathbf{D}$. Now, if $\omega \in \mathbf{D}$ then

$$C_\varphi^* K_\omega(z) = K_{\varphi(\omega)}(z) = k\left(\overline{\varphi(\omega)} z\right) = K_\omega\left(e^{-i\theta} z\right) = K_\omega\left(\varphi^{-1}(z)\right) = C_{\varphi^{-1}} K_\omega(z). \tag{2.19}$$

It follows that $C_\varphi^* = C_{\varphi^{-1}}$. But it is easily seen that $\|C_{\varphi^{-1}} f\| = \|f\|$ for every $f \in H^2(\beta)$. Hence, C_φ^* is an isometry. The converse is obvious. □

3. Multiplication Operators

This section deals with convex multiplication operators on a weighted Hardy space. Recall that a multiplier of $H^2(\beta)$ is an analytic function φ on **D** such that $\varphi H^2(\beta) \subseteq H^2(\beta)$. The set of all multipliers of $H^2(\beta)$ is denoted by $M(H^2(\beta))$. It is known that $M(H^2(\beta)) \subseteq H^\infty$. In fact, if $\varphi \in M(H^2(\beta))$ and f is the constant function 1 then for every positive integer n and for every $z \in \mathbf{D}$ we have

$$|\varphi(z)| = \left| \left\langle M_\varphi^n f, K_z \right\rangle \right|^{1/n} \leq \left\| M_\varphi^n f \right\|^{1/n} \|K_z\|^{1/n} \leq \|M_\varphi\| \|K_z\|^{1/n}. \tag{3.1}$$

Now, letting $n \to \infty$, we conclude that φ is bounded. This coupled with the fact that $\varphi \in H^2(\beta)$ implies that $\varphi \in H^\infty$. If φ is a multiplier, then the multiplication operator M_φ, defined by $M_\varphi f = \varphi f$, is bounded on $H^2(\beta)$. Also note that for each $\lambda \in \mathbf{D}$, $M_\varphi^* K_\lambda = \overline{\varphi(\lambda)} K_\lambda$.

In what follows, the operator M_φ is assumed to be convex. First, we present an example of a nonisometric convex multiplication operator T with $\Delta_T \geq 0$.

Example 3.1. Consider the weighted Hardy space $H^2(\beta)$ with weight sequence $(\beta(n))_n$ given by $\beta(n) = n + 1$. Define the mapping φ on $\mathbf{D}$ by $\varphi(z) = z^2$. Obviously, M_φ is bounded. Furthermore, it is easy to see that for every nonnegative integer k,

$$\left\| M_\varphi^2 z^k \right\|^2 - 2\left\| M_\varphi z^k \right\|^2 + \left\| z^k \right\|^2 > 0,$$
$$\left\| M_\varphi z^k \right\| > \left\| z^k \right\|. \tag{3.2}$$

Consequently, M_φ is convex but not an isometry. Besides, Δ_{M_φ} is a positive operator.

Theorem 3.2. *Let H^∞ consist of all multipliers of $H^2(\beta)$, and let $\varphi \in H^\infty$ be such that $\|\varphi\|_\infty \leq 1$. If $T = M_\varphi$ or $T = M_\varphi^*$ then $T^*\Delta_T T \geq \Delta_T \geq 0$ if and only if T is an isometry.*

Proof. Suppose that T is M_φ or M_φ^* and $T^*\Delta_T T \geq \Delta_T \geq 0$. Define the linear mapping $S : H^\infty \to \mathcal{B}(H^2(\beta))$ by $S(\psi) = M_\psi$. An application of the closed graph theorem implies that S is bounded. Therefore, there is $c > 0$ such that for all $\psi \in H^\infty$,

$$\left\| M_\psi \right\| \leq c\left\| \psi \right\|_\infty. \tag{3.3}$$

It follows that for every $f \in H^2(\beta)$ and every nonnegative integer n,

$$\left\| M_\varphi^n f \right\| \leq c\left\| \varphi^n \right\|_\infty \left\| f \right\| \leq c\left\| f \right\|. \tag{3.4}$$

Thus, $\sup_{n\geq 0}\|M_\varphi^n f\| < \infty$ for every $f \in H^2(\beta)$. Since $\|M_\psi^*\| = \|M_\psi\|$ for all $\psi \in H^\infty$, by a similar method one can show that $\sup_{n\geq 0}\|M_\varphi^{*^n} f\| < \infty$ for all $f \in H^2(\beta)$. Therefore, the result follows from Proposition 1.3. □

Example 3.3. Let $\mathcal{H}$ be the Bergman space or the Hardy space and let T be M_φ or its adjoint on $\mathcal{H}$. It is well known that $M(\mathcal{H}) = H^\infty$. So if φ is a multiplier with $\|\varphi\|_\infty \leq 1$, then by applying the preceding theorem, we observe that $T^*\Delta_T T \geq \Delta_T \geq 0$ if and only if T is an isometry.

We remark herein that if $\varphi(z) = z$ and $T = M_\varphi$ on the Dirichlet space $\mathfrak{D}$, then it is easily seen that $T^*\Delta_T T \geq \Delta_T \geq 0$ but T is not an isometry.

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