Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 847690, 9 pages
doi:10.1155/2009/847690

Research Article

Uniqueness of Entire Functions Sharing Polynomials with Their Derivatives

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Received 1 December 2008; Revised 10 April 2009; Accepted 6 May 2009

Recommended by Stevo Stevic

We use the theory of normal families to prove the following. Let $Q_1(z) = a_1 z^p + a_{1,p-1} z^{p-1} + \cdots + a_{1,0}$ and $Q_2(z) = a_2 z^p + a_{2,p-1} z^{p-1} + \cdots + a_{2,0}$ be two polynomials such that $\deg Q_1 = \deg Q_2 = p$ (where p is a nonnegative integer) and $a_1, a_2 (a_2 \neq 0)$ are two distinct complex numbers. Let $f(z)$ be a transcendental entire function. If $f(z)$ and $f'(z)$ share the polynomial $Q_1(z)$ CM and if $f(z) = Q_2(z)$ whenever $f'(z) = Q_2(z)$, then $f \equiv f'$. This result improves a result due to Li and Yi.

1. Introduction and Main Results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$,and let $P(z)$ be a polynomial or a finite complex number. $\deg P(z)$ denotes the degree of the polynomial $P(z)$. To simplify the statement of our results in this paper, deviating from the common definition, we consider the zero polynomial as a polynomial of degree 0. If $g(z) - P(z) = 0$ whenever $f(z) - P(z) = 0$, we write $f(z) = P(z) \Rightarrow g(z) = P(z)$. If $f(z) = P(z) \Rightarrow g(z) = P(z)$ and $g(z) = P(z) \Rightarrow f(z) = P(z)$, we write $f(z) = P(z) \Leftrightarrow g(z) = P(z)$ and say that $f(z)$ and $g(z)$ share $P(z)$ (IM ignoring multiplicity). If $f(z)-P(z)$ and $g(z)-P(z)$ have the same zeros with the same multiplicities, we write $f(z) = P(z) \rightleftharpoons g(z) = P(z)$ and say that $f(z)$ and $g(z)$ share $P(z)$ (CM counting multiplicity) (see, [1]). In addition, we use notations $\sigma(f)$, $\sigma_2(f)$ to denote the order and the hyperorder of $f(z)$, respectively, where

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \qquad \sigma_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}. \tag{1.1}$$

It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [1, 2].

In 1977, Rubel and Yang [3] proved the well-known theorem.

Theorem A. *Let a and b be two complex numbers such that $b \neq a$, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \rightleftharpoons f'(z) = a$ and $f(z) = b \rightleftharpoons f'(z) = b$, then $f(z) \equiv f'(z)$.*

This result has undergone various extensions and improvements (see, [1]).

In 1979, Mues and Steinmetz [4] proved the following result.

Theorem B. *Let a and b be two complex numbers such that $b \neq a$, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \Leftrightarrow f'(z) = a$ and $f(z) = b \Leftrightarrow f'(z) = b$, then $f(z) \equiv f'(z)$.*

In 2006, Li and Yi [5] proved the following related result.

Theorem C. *Let a and b be two complex numbers such that $b \neq a, 0$, and let $f(z)$ be a nonconstant entire function. If $f(z) = a \rightleftharpoons f'(z) = a$ and $f'(z) = b \Rightarrow f(z) = b$, then $f(z) \equiv f'(z)$.*

Remark 1.1. Meanwhile, Li and Yi [5] give an example to show that $b \neq 0$ cannot be omitted in Theorem C.

In recent years, there have been several papers dealing with entire functions that share a polynomial with their derivatives.

In 2006, Wang [6] proved the following result.

Theorem D. *Let $f(z)$ be a nonconstant entire function, and let $Q(z)$ be a polynomial of degree $q \geq 1$. Let $k \geq q+1$ be an integer. If $f(z) = Q(z) \rightleftharpoons f'(z) = Q(z)$ and if $f^{(k)}(z) = Q(z)$ for every $z \in \mathbb{C}$ with $f(z) = Q(z)$, then $f(z) \equiv f'(z)$.*

In 2007, Li and Yi [7] proved the following result.

Theorem E. *Let $f(z)$ be a nonconstant entire function of hyperorder $\sigma_2(f) < 1/2$, and let $Q(z)$ be a nonconstant polynomial. If $f(z) = Q(z) \rightleftharpoons f'(z) = Q(z)$, then*

$$\frac{f'(z) - Q(z)}{f(z) - Q(z)} \equiv c \tag{1.2}$$

for some constant $c \neq 0$.

In 2008, Grahl and Meng [8] proved the following result.

Theorem F. *Let $f(z)$ be a nonconstant entire function, and let $Q(z)$ be a nonconstant polynomial. Let $k \geq 2$ be an integer. If $f(z) = Q(z) \rightleftharpoons f'(z) = Q(z)$ and if for some positive M we have $|f^{(k)}(z)| \leq M(1 + |Q(z)|)$ for every $z \in \mathbb{C}$ with $f(z) = Q(z)$, then*

$$\frac{f'(z) - Q(z)}{f(z) - Q(z)} \tag{1.3}$$

is constant.

From the ideas of Theorem D to Theorem F, it is natural to ask whether the values a, b in Theorem C can be replaced by two polynomials Q_1, Q_2. The main purpose of this paper is to investigate this problem. We prove the following result.

Theorem 1.2. *Let $Q_1(z) = a_1 z^p + a_{1,p-1} z^{p-1} + \cdots + a_{1,0}$ and $Q_2(z) = a_2 z^p + a_{2,p-1} z^{p-1} + \cdots + a_{2,0}$ be two polynomials such that $\deg Q_1(z) = \deg Q_2(z) = p$ (where p is a nonnegative integer) and a_1, $a_2 (a_2 \neq 0)$ are two distinct complex numbers. Let $f(z)$ be a transcendental entire function. If $f(z) = Q_1(z) \rightleftharpoons f'(z) = Q_1(z)$ and $f'(z) = Q_2 \Rightarrow f(z) = Q_2(z)$, then $f(z) \equiv f'(z)$.*

Remark 1.3. The following example shows the hypothesis that f is transcendental cannot be omitted in Theorem 1.2.

Example 1.4. Let $f(z) = z^3$, $Q_1(z) = 2z^3 - 3z^2$ and $Q_2(z) = z^3$. Then

$$\frac{f'(z) - Q_1(z)}{f(z) - Q_1(z)} = 2, \quad f'(z) = Q_2(z) \Longrightarrow f(z) = Q_2(z). \tag{1.4}$$

While $f(z)$ does not satisfy the result of Theorem 1.2.

Remark 1.5. The case $p = 0$ of Theorem 1.2 yields Theorem C.

It seems that we cannot get the result by the methods used in [4, 5]. In order to prove our theorem, we need the following result which is interesting in its own right.

Theorem 1.6. *Let $Q_1(z) = a_1 z^p + a_{1,p-1} z^{p-1} + \cdots + a_{1,0}$ and $Q_2(z) = a_2 z^p + a_{2,p-1} z^{p-1} + \cdots + a_{2,0}$ be two polynomials such that $\deg Q_1(z) = \deg Q_2(z) = p$ (where p is a nonnegative integer) and a_1, $a_2 (a_2 \neq 0)$ are two distinct complex numbers. Let $f(z)$ be a nonconstant entire function, and $f(z) = Q_1(z) \Rightarrow f'(z) = Q_1(z)$ and $f'(z) = Q_2(z) \Rightarrow f(z) = Q_2(z)$, then $f(z)$ is of finite order.*

2. Some Lemmas

In order to prove our theorems, we need the following lemmas.

Let h be a meromorphic function in $\mathbb{C}$. h is called a normal function if there exists a positive M such that $h^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2} \tag{2.1}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on compact subsets of D; see [9].

Normal families, in particular, of holomorphic functions often appear in operator theory on spaces of analytic functions; for example, see in [10, Lemma 3] and in [11, Lemma 4].

Lemma 2.1 (see [12]). *Let $\mathcal{F}$ be a family of analytic functions in the unit disc Δ with the property that for each $f(z) \in \mathcal{F}$, all zeros of $f(z)$ have multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) \in \mathcal{F}$ and $f(z) = 0$. If $\mathcal{F}$ is not normal in Δ, then for $0 \leq \alpha \leq k$, there exist*

(1) *a number $r \in (0,1)$,*

(2) *a sequence of complex numbers z_n, $|z_n| < r$,*

(3) *a sequence of functions $f_n \in \mathcal{F}$, and*

(4) *a sequence of positive numbers $\rho_n \to 0$*

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ converges locally and uniformly (with respect to the spherical metric) to a nonconstant analytic function $g(\xi)$ on $\mathbb{C}$, and moreover, the zeros of $g(\xi)$ are of multiplicity at least k, $g^{\#}(\xi) \leq g^{\#}(0) = kA + 1$.

Lemma 2.2 (see [13]). *A normal meromorphic function has order at most two. A normal entire function is of exponential type and thus has order at most one.*

Lemma 2.3 (see [9, Marty's criterion]). *A family $\mathcal{F}$ of meromorphic functions on a domain D is normal if and only if, for each compact subset $K \subseteq D$, there exists a constant M such that $f^{\#}(z) \leq M$ for each $f \in \mathcal{F}$ and $z \in K$.*

Lemma 2.4 (see [2]). *Let $f(z)$ be a meromorphic function, and let $a_1(z)$, $a_2(z)$, $a_3(z)$ be three distinct meromorphic functions satisfying $T(r, a_i) = S(r, f)$, $i = 1, 2, 3$. Then*

$$T(r,f) \leq \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_3}\right) + S(r,f). \tag{2.2}$$

Lemma 2.5 (see [5]). *Let $\mathcal{F}$ be a family of functions holomorphic on a domain D, and let a and b be two finite complex numbers such that $b \neq a, 0$. If for each $f \in \mathcal{F}$, $f(z) = a \Rightarrow f'(z) = a$ and $f'(z) = b \Rightarrow f(z) = b$, then $\mathcal{F}$ is normal in D.*

3. Proof of Theorem 1.6

If $Q_1 \equiv 0$, by $\deg Q_1 = \deg Q_2$, we obtain $p = 0$, $a_1 = 0$, $Q_2 \equiv a_2 (a_2 \neq 0)$. From the conditions of Theorem 1.6, we obtain $f(z) = 0 \Rightarrow f'(z) = 0$ and $f'(z) = a_2 \Rightarrow f(z) = a_2$. By Lemmas 2.5 and 2.3 we obtain that f is a normal function in D. By Lemma 2.2 we obtain that f is a finite order function.

If $Q_1 \not\equiv 0$, by $\deg Q_1 = \deg Q_2$ and $a_2 \neq 0$, we obtain $a_1 \neq 0$. Now we consider the function $F = f/Q_1 - 1$, and we distinguish two cases.

Case 1. If there exists a constant M such that $F^{\#}(z) \leq M$, by Lemmas 2.3 and 2.2, then F is of finite order. Hence $f = (F+1)Q_1$ is of finite order as well.

Case 2. If there does not exist a constant M such that $F^{\#}(z) \leq M$, then there exists a sequence $(w_n)_n$ such that $w_n \to \infty$ and $F^{\#}(w_n) \to \infty$ for $n \to \infty$.

Since Q_1 is a polynomial, there exists an r_1 such that

$$\left|\frac{Q_1'(z)}{Q_1(z)}\right| \le \frac{2p}{|z|} \quad \forall z \in \mathbb{C} \text{ satisfying } |z| \ge r_1. \tag{3.1}$$

Obviously, if $z \to \infty$, then $2p/|z| \to 0$. Let $r > r_1$, and $D = \{z : |z| \ge r\}$, then F is analytic in D. Without loss of generality, we may assume $|w_n| \ge r+1$ for all n. We define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{f(w_n + z)}{Q_1(w_n + z)} - 1. \tag{3.2}$$

Let $z \in D_1$ be fixed; from the above equality, if $F(w_n + z) = 0$, then $f(w_n + z) = Q_1(w_n + z)$. Noting that $f = Q_1 \Rightarrow f' = Q_1$, then we obtain the following: if $n \to \infty$, then

$$\begin{aligned} \left|F_n'(z)\right| &= \left|\left(\frac{f(w_n + z)}{Q_1(w_n + z)}\right)'\right| = \left|\frac{f'(w_n + z)}{Q_1(w_n + z)} - \frac{f(w_n + z)}{Q_1(w_n + z)}\frac{Q_1'(w_n + z)}{Q_1(w_n + z)}\right| \\ &\le \left|\frac{f'(w_n + z)}{Q_1(w_n + z)}\right| + \left|\frac{f(w_n + z)}{Q_1(w_n + z)}\right|\left|\frac{Q_1'(w_n + z)}{Q_1(w_n + z)}\right| < 2. \end{aligned} \tag{3.3}$$

Obviously, $F_n(z)$ are analytic in D_1 and $F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. It follows from Lemma 2.3 that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 2.1, with ($\alpha = k = 1$ and $A = 2$). Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$, such that $z_n \to 0$ and $\rho_n \to 0$ and such that the sequence $(g_n)_n$ defined by

$$g_n(\xi) = \rho_n^{-1} F_n(z_n + \rho_n\xi) = \rho_n^{-1}\left\{\frac{f(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)} - 1\right\} \longrightarrow g(\xi) \tag{3.4}$$

converges locally and uniformly in $\mathbb{C}$ where $g(\xi)$ is a nonconstant analytic function and $g^{\#}(\xi) \le g^{\#}(0) = A + 1 = 3$. By lemma 2.2, the order of $g(\xi)$ is at most 1.

First, we will prove that $g = 0 \Rightarrow g' = 1$ on $\mathbb{C}$. Suppose that there exists a point ξ_0 such that $g(\xi_0) = 0$. Then by Hurwitz's theorem, there exist ξ_n, $\xi_n \to \xi_0$ as $n \to \infty$ such that for n sufficiently large

$$g_n(\xi_n) = \rho_n^{-1}\left\{\frac{f(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} - 1\right\} = 0. \tag{3.5}$$

This implies $f(w_n + z_n + \rho_n\xi_n) = Q_1(w_n + z_n + \rho_n\xi_n)$. From the above, we obtain

$$g_n'(\xi) = \frac{f'(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)} - \frac{f(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)}\frac{Q_1'(w_n + z_n + \rho_n\xi)}{Q_1(w_n + z_n + \rho_n\xi)}. \tag{3.6}$$

Let $G_n(\xi) = f'(w_n + z_n + \rho_n\xi)/Q_1(w_n + z_n + \rho_n\xi)$, by (3.1), (3.3) and (3.4), it is easy to obtain $\lim_{n\to\infty}G_n(\xi) = \lim_{n\to\infty}g_n'(\xi) = g'(\xi)$. Noting that $f = Q_1 \Rightarrow f' = Q_1$, we have

$$G_n(\xi_n) = \frac{f'(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} = 1 \quad (n \to \infty) \tag{3.7}$$

Thus

$$g'(\xi_0) = \lim_{n\to\infty} G_n(\xi_n) = 1. \tag{3.8}$$

This shows that $g = 0 \Rightarrow g' = 1$.

Next we will prove that $g'(\xi) \neq a_2/a_1$ on $\mathbb{C}$. Suppose that there exists a point ξ_0 such that $g'(\xi_0) = a_2/a_1$. If $g'(\xi) \equiv a_2/a_1$, then $g(\xi) = a_2/a_1\xi + c$, where c is a constant, together with the fact that $g = 0 \Rightarrow g' = 1$ gives $a_2/a_1 = 1$, which contradicts to the assumptions. Thus $g'(\xi) \not\equiv a_2/a_1$. Since $G_n(\xi) - Q_2(w_n + z_n + \rho_n\xi)/Q_1(w_n + z_n + \rho_n\xi) \to g'(\xi) - a_2/a_1$ as $n \to \infty$ and $g'(\xi_0) = a_2/a_1$, by Hurwitz's theorem, there exist $\xi_n \to \xi_0$ as $n \to \infty$ such that for n sufficiently large

$$\begin{aligned} G_n(\xi_n) - \frac{Q_2(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} &= 0 \\ \Longrightarrow f'(w_n + z_n + \rho_n\xi_n) &= Q_2(w_n + z_n + \rho_n\xi_n). \end{aligned} \tag{3.9}$$

Noting that $f' = Q_2 \Rightarrow f = Q_2$, from (3.4) and (3.9) (for n sufficiently large), we have

$$g_n(\xi_n) = \rho_n^{-1}\left\{\frac{f(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} - 1\right\} = \rho_n^{-1}\left\{\frac{Q_2(w_n + z_n + \rho_n\xi_n)}{Q_1(w_n + z_n + \rho_n\xi_n)} - 1\right\}. \tag{3.10}$$

Since $a_2 \neq a_1 (a_1 \neq 0)$, $\deg Q_1 = \deg Q_2 = p$ and $\rho_n \to 0$, by (3.10), we get

$$g(\xi_0) = \lim_{n\to\infty} g_n(\xi_n) = \infty, \tag{3.11}$$

which contradicts $g'(\xi_0) = a_2/a_1$. This shows that $g'(\xi) \neq a_2/a_1$ on $\mathbb{C}$.

Since g is of order at most one, so is g', it follows that

$$g'(\xi) = \frac{a_2}{a_1} + e^{b_0 + b_1\xi}, \tag{3.12}$$

where b_0, b_1 are two finite constants. We divide this case into two subcases.

Subcase 1. If $b_1 = 0$, from (3.12), we have

$$g(\xi) = \left(\frac{a_2}{a_1} + e^{b_0}\right)\xi + c_0, \tag{3.13}$$

where c_0 is a constant. Since $g = 0 \Rightarrow g' = 1$, from (3.13) we have $a_2/a_1 + e^{b_0} = 1$. By a simple calculation, we have $g^{\#}(0) = 1/(1 + |c_0|^2)$, which contradicts $g^{\#}(0) = 3$.

Subcase 2. If $b_1 \neq 0$, by

$$g'(\xi) = \frac{a_2}{a_1} + e^{b_0 + b_1 \xi}, \tag{3.14}$$

we obtain

$$g(\xi) = \frac{a_2}{a_1}\xi + \frac{1}{b_1} e^{b_0 + b_1 \xi} + B, \tag{3.15}$$

where B is a constant. Obviously, $g(\xi) = 0$ has infinitely many solutions. Suppose that there exists a point ξ_0 such that $g(\xi_0) = 0$. By (3.14), (3.15), and $g = 0 \Rightarrow g' = 1$, we get a unique $\xi_0 = (a_2 - a_1 - b_1 B a_1)/b_1 a_2$. Which is a contradiction.

Thus f is of finite order. This completes the proof of the theorem.

4. Proof of Theorem 1.2

Now we distinguish two cases.

Case 1. If $p = 0$, by $\deg Q_1 = \deg Q_2 = 0$, we deduce $Q_1 \equiv a_1$ and $Q_2 \equiv a_2 (a_2 \neq a_1, 0)$. By Theorem C, we obtain $f \equiv f'$.

Case 2. If $p \geq 1$, by $\deg Q_1 = \deg Q_2 = p$ and $a_2 \neq 0$, we deduce $a_1 \neq 0$. So Q_1 is a nonconstant polynomial. By Theorem 1.6, we know that f is of finite order. Thus, the hyperorder $\sigma_2(f) = 0$. Then, by Theorem E, we have

$$\lambda = \frac{f' - Q_1}{f - Q_1}, \tag{4.1}$$

where λ is a nonzero constant. We rewrite it as

$$f' = \lambda f + (1 - \lambda) Q_1. \tag{4.2}$$

If $\lambda = 1$, we obtain $f \equiv f'$.

Now, we assume that $\lambda \neq 1$. Solving (4.2), we obtain

$$f(z) = A e^{\lambda z} + P(z), \tag{4.3}$$

where A is a nonzero constant, and $P(z)$ is a polynomial. Thus, we have

$$f'(z) = A \lambda e^{\lambda z} + P'(z). \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2), we get

$$(\lambda - 1)Q_1 - (\lambda P - P') \equiv 0. \tag{4.5}$$

Next, we will prove that $P'(z) \equiv Q_2(z)$. Suppose that $P'(z) \not\equiv Q_2(z)$, by (4.4) we obtain

$$\overline{N}\left(r, \frac{1}{f'(z) - Q_2(z)}\right) = \overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right). \tag{4.6}$$

Since $f(z)$ is a transcendental entire function and $P'(z) - Q_2(z)$ is a polynomial, we deduce $T(r, P'(z) - Q_2(z)) = S(r, f)$. It is well known that 0 and ∞ are the Picard values of $e^{\lambda z}$. By Lemma 2.4, we obtain

$$T\left(r, \lambda A e^{\lambda z}\right) \leq \overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right) + S(r, f). \tag{4.7}$$

By the Nevanlinna First Fundamental Theorem, we immediately obtain

$$\overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right) \leq T\left(r, \lambda A e^{\lambda z}\right) + S(r, f). \tag{4.8}$$

If we combine (4.7) and (4.8), we obtain

$$\overline{N}\left(r, \frac{1}{A\lambda e^{\lambda z} + P'(z) - Q_2(z)}\right) = T\left(r, \lambda A e^{\lambda z}\right) + S(r, f) \neq S(r, f). \tag{4.9}$$

Since $P'(z) \not\equiv Q_2(z)$, we suppose z_0 is a zero of $f' - Q_2$. By the assumption $f'(z) = Q_2(z) \Rightarrow f(z) = Q_2(z)$, we have $f(z_0) = Q_2(z_0)$. Substituting z_0 into (4.3) and (4.4), we have

$$(\lambda - 1)Q_2(z_0) = \lambda P(z_0) - P'(z_0). \tag{4.10}$$

If $(\lambda - 1)Q_2 - (\lambda P - P') \not\equiv 0$, noting that $(\lambda - 1)Q_2 - (\lambda P - P')$ is a polynomial, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f' - Q_2}\right) &\leq \overline{N}\left(r, \frac{1}{(\lambda - 1)Q_2 - (\lambda P - P')}\right) \\ &\leq T(r, (\lambda - 1)Q_2 - (\lambda P - P')) = S(r, f), \end{aligned} \tag{4.11}$$

which contradicts with (4.9). Hence,

$$(\lambda - 1)Q_2 - (\lambda P - P') \equiv 0. \tag{4.12}$$

Comparing the above equality to (4.5), we have $Q_1 \equiv Q_2$, a contradiction. Thus, we prove $P'(z) \equiv Q_2(z)$. It is easy to see $\deg Q_2 = \deg P'$. By (4.5) we obtain $\deg Q_1 = \deg P$. Finally we deduce $\deg Q_1 \neq \deg Q_2$. This is a contradiction. So $\lambda \neq 1$ is impossible. This completes the proof of Theorem 1.2.

Acknowledgment

The authors are grateful to the referee for his(her) valuable suggestions and comments. The authors would like to express their hearty thanks to Professor Hongxun Yi for his valuable advice and helpful information. The work was supported by the NNSF of China (no. 10371065), the NSF of Shang-dong Province, China (no. Z2002A01), and the NSFC-RFBR.

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