Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 780287, 6 pages
doi:10.1155/2009/780287

Research Article

The Tsirelson Space $\mathcal{T}^{(p)}$ Has a Unique Unconditional Basis up to Permutation for $0 < p < 1$

F. Albiac and C. Leránoz

Departamento de Matemáticas, Universidad Pública de Navarra, Pamplona 31006, Spain

Correspondence should be addressed to F. Albiac, fernando.albiac@unavarra.es

Received 16 July 2009; Accepted 3 December 2009

Recommended by Simeon Reich

We show that the p-convexified Tsirelson space $\mathcal{T}^{(p)}$ for $0 < p < 1$ and all its complemented subspaces with unconditional basis have unique unconditional basis up to permutation. The techniques involved in the proof are different from the methods that have been used in all the other uniqueness results in the nonlocally convex setting.

1. Introduction: The Problem of Uniqueness of Unconditional Basis

If $(X, \|\cdot\|)$ is a quasi-Banach space (in particular a Banach space) with a normalized unconditional basis $(e_n)_{n=1}^{\infty}$ (i.e., $\|e_n\| = 1$ for all $n \in \mathbb{N}$), X is said to have a *unique unconditional basis (up to permutation)* if whenever $(x_n)_{n=1}^{\infty}$ is another normalized unconditional basis of X, then $(x_n)_{n=1}^{\infty}$ is equivalent to $(e_n)_{n=1}^{\infty}$ (after a permutation); that is, there exists an automorphism of X which takes one basis to (a permutation of) the other.

The problem of uniqueness of unconditional basis is classical. It is well known that c_0, ℓ_1, and ℓ_2 have a unique unconditional basis and that any other Banach space with an unconditional basis fails to have this property [1–3].

If an unconditional basis is unique, in particular it must be equivalent to all its permutations and hence must be symmetric. Thus, the obvious modification for spaces whose canonical basis is unconditional but not symmetric is to require uniqueness of unconditional basis via a permutation, which in many ways is a more natural concept for unconditional bases. Classifying those Banach spaces with unique unconditional bases up to permutation, however, has turned out to be a much more difficult task. The first step toward this classification was taken in 1976 by Edelstein and Wojtaszczyk [4], who showed that any finite direct sum of c_0, ℓ_1, and ℓ_2 had that property. After their work, Bourgain, Casazza, Lindenstrauss, and Tzafriri embarked on a comprehensive study aimed at classifying those

Banach spaces with unique uconditional basis up to permutation that culminated in 1985 with their AMS *Memoir* [5]. They considered infinite direct sums of the Banach spaces with unique unconditional basis and showed that the spaces $c_0(\ell_1)$, $c_0(\ell_2)$, $\ell_1(c_0)$, and $\ell_1(\ell_2)$ all have unique unconditional bases up to permutation, while, surprisingly, $\ell_2(\ell_1)$ and $\ell_2(c_0)$ do not.

However, all hopes for a satisfactory classification were shattered when they discovered that a space of a totally different character, a certain variant of Tsirelson space $\mathcal{T}$, also had a unique unconditional basis up to permutation. More recently, further examples of "pathological" spaces with unique unconditional basis up to permutation have been given in [6, 7].

In the context of quasi-Banach spaces which are not Banach spaces, the uniqueness of unconditional basis seems to be the norm rather than the exception. For instance, it was shown in [8] that a wide class of nonlocally convex Orlicz sequence spaces, including the ℓ_p spaces for $0 < p < 1$, have a unique unconditional basis. The same is true in nonlocally convex Lorentz sequence spaces [9, 10] and in the Hardy spaces $H_p(\mathbb{T})$ for $0 < p < 1$ [11].

Analogously, it seemed only natural to translate the question of uniqueness of unconditional basis up to permutation to the setting of nonlocally convex spaces that are infinite direct sums of the classical quasi-Banach spaces with a unique unconditional basis, namely, $\ell_p(\ell_q)$, $\ell_p(c_0)$, $\ell_p(\ell_1)$, $\ell_p(\ell_2)$, $c_0(\ell_p)$, $\ell_1(\ell_p)$, and $\ell_2(\ell_p)$, where $0 < p,\ q < 1$. With the exception of $\ell_2(\ell_p)$ that remains elusive, the uniqueness of unconditional basis has been established for all the other spaces (see [10, 12–14], in chronological order). Although the proofs are very different depending on each case, all of them rely *explicitly* on the corresponding results for their respective Banach envelopes shown in [5] and revolve around the all-pervading "large coefficient" technique ([10, Theorem 2.3]) for establishing the equivalence of basic sequences.

In this paper we change the strategy, and the proofs in Section 2 hinge on Theorem 1.1. This is a general result on complemented unconditional basic sequences in lattice anti-Euclidean quasi-Banach spaces which extends a result from [7]. We recall that a quasi-Banach lattice is called *sufficiently lattice Euclidean* if there is a constant M so that for any $n \in \mathbb{N}$ there are operators $S_n : X \to \ell_2^n$ and $T_n : \ell_2^n \to X$ so that $S_n \circ T_n = I_{\ell_2^n}$, $\|S_n\|\|T_n\| \leq M$, and S_n is a lattice homomorphism. This is equivalent to asking that ℓ_2 is finitely representable as a complemented sublattice of X. We will say that X is *lattice anti-Euclidean* if it is not sufficiently lattice Euclidean. We will also use the term *sequence space* to mean a quasi-Banach space of sequences so that the canonical basis vectors form a 1-unconditional basis.

Theorem 1.1 (see [14, Theorem 3.4]). *Let Y and Z be quasi-Banach sequence spaces. Suppose that Z is p-convex for some $p > 0$ and that Y is isomorphic to a complemented subspace of Z. Suppose that the Banach envelope $\widehat{Y}$ of Y is lattice anti-Euclidean. Then there exists $N \in \mathbb{N}$ and a complemented disjoint positive sequence (v_n) in Z^N that is equivalent to the unit vector basis (u_n) in Y. Furthermore, the projection P of Z^N onto $[v_n]$ may be given in the form*

$$P(z) = \sum_{n=1}^{\infty} v_n^*(z) v_n, \tag{1.1}$$

where $v_n^* \geq 0$ and $\operatorname{supp}(v_n^*) \subseteq \operatorname{supp}(v_n)$ *for all n.*

To help determine whether two unconditional bases are permutatively equivalent we will use the following form of the Cantor-Bernstein principle.

Theorem 1.2 (see [11, Proposition 2.11]). *Suppose $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ are two unconditional basic sequences of a quasi-Banach space X, then (u_n) and (v_n) are equivalent (up to permutation) if and only if (u_n) is equivalent (up to permutation) to a subsequence of (v_n) and (v_n) is equivalent (up to permutation) to a subsequence of (u_n).*

2. Main Result

In this section we settle the question of uniqueness of unconditional basis in the spaces $\mathcal{T}^{(p)}$ for $0 < p < 1$. These spaces are the nonlocally convex counterpart to the p-convexification of Tsirelson space for $p > 1$, introduced by Figiel and Johnson in [15]. Those readers who are unfamiliar with Tsirelson space will find a handy construction and some of its elementary properties in [16]. For a more in-depth approach, the standard reference is [17] (cf. [15]).

Given $0 < p < \infty$, the p-convexification $\mathcal{T}^{(p)}$ of Tsirelson space $\mathcal{T}$ is obtained from $\mathcal{T}$ by putting

$$\|x\|_{\mathcal{T}^{(p)}} = \left\| \left(|a_n|^p\right)_{n=1}^{\infty} \right\|_{\mathcal{T}}^{1/p} \tag{2.1}$$

for those sequences of real numbers $x = (a_n)_{n=1}^{\infty}$ such that $(|a_n|^p)_{n=1}^{\infty} \in \mathcal{T}$. Equation (2.1) defines a norm for $1 \le p$ and a p-norm when $0 < p < 1$. Obviously, the space $(\mathcal{T}^{(1)}, \|\cdot\|_{\mathcal{T}^{(1)}})$ is simply $(\mathcal{T}, \|\cdot\|_{\mathcal{T}})$.

The canonical unit vectors form a 1-unconditional basis of $\mathcal{T}^{(p)}$ for all $0 < p < \infty$. Casazza and Kalton established in [7] the uniqueness of unconditional basis up to permutation of $\mathcal{T}$ and its complemented subspaces with unconditional basis as a byproduct of their study of complemented basic sequences in lattice anti-Euclidean Banach spaces. Their result answered a question by Bourgain et al. in [5], where the authors had proved the uniqueness of unconditional basis up to permutation in the 2-convexification $\mathcal{T}^{(2)}$ of $\mathcal{T}$.

Here we show that for $p < 1$, the space $\mathcal{T}^{(p)}$ and its complemented subspaces with unconditional basis belong to the class of quasi-Banach spaces with a unique unconditional basis, up to permutation. The proof follows the steps of the corresponding uniqueness result for $\mathcal{T}$. This will require first to extend some concepts and results from [7] to quasi-Banach sequence spaces. Despite the fact that the changes are minor, we decided to include them here for reference and for future use.

Definition 2.1. Borrowing the definition from [7, Section 5], if X is a quasi-Banach space with unconditional basis we will say that X is *left-dominant with constant* $\gamma \ge 1$ if whenever $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are two disjoint sequences in c_{00} with $\|u_k\|_X \ge \|v_k\|_X$ and such that $\max\{\text{supp } u_k\} < \min\{\text{supp } v_k\}$ for $1 \le k \le n$, then $\|\sum_{k=1}^{n} v_k\|_X \le \gamma \|\sum_{k=1}^{n} u_k\|_X$. Similarly, we will say that X is *right-dominant with constant* γ if whenever $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ are two disjoint sequences in c_{00} with $\|u_k\|_X \le \|v_k\|_X$ and such that $\max\{\text{supp } u_k\} < \min\{\text{supp } v_k\}$ for $1 \le k \le n$, then $\|\sum_{k=1}^{n} u_k\|_X \le \gamma \|\sum_{k=1}^{n} v_k\|_X$. We will refer to any normalized unconditional basic sequence as being left-or right-dominant if the the associated sequence space is left-or right-dominant respectively.

For example, the ℓ_p spaces are left- and right-dominant when $0 < p < \infty$. In turn, $\mathcal{T}$ is a right-dominant space as shown in [7, Proposition 5.12], and by the Remark thereafter we infer that so is $\mathcal{T}^{(p)}$ for $0 < p < \infty$.

Our next results, Propositions 2.2 and 2.5, are essentially Theorems 5.6 and 5.7 of [7], respectively, slightly modified to suit our purposes.

Proposition 2.2 (see [7, Theorem 5.6]). *Let X be a left- or right-dominant quasi-Banach space X with unconditional basis $(e_n)_{n=1}^{\infty}$. Suppose that $(u_n)_{n\in S}$ ($|S| \leq \infty$) is a complemented normalized disjoint sequence in X, then $(u_n)_{n\in S}$ is permutatively equivalent to a subsequence $(e_{k_n})_{n=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$.*

Proof. The proof in the locally convex case applies almost verbatim to this setting and hence we omit it. □

The next two lemmas will be used in what follows.

Lemma 2.3 (see [7, Proposition 5.4]). *Suppose that $(u_n)_{n=1}^{\infty}$ is a left- (resp, right-) dominant basis of a quasi-Banach space X and that π is a permutation of the natural numbers such that $(u_{\pi(n)})_{n=1}^{\infty}$ is also left- (resp, right-) dominant. then there is a constant C such that for any $(\alpha_k) \in c_{00}$,*

$$\left\| \sum_{k=1}^{\infty} \alpha_k u_{2k} \right\| \leq C \left\| \sum_{k=1}^{\infty} \alpha_k u_{\pi(k)} \right\| \tag{2.2}$$

resp,

$$\left\| \sum_{k=1}^{\infty} \alpha_k u_{2k} \right\| \geq C^{-1} \left\| \sum_{k=1}^{\infty} \alpha_k u_{\pi(k)} \right\|. \tag{2.3}$$

We say that (u_n) is *equivalent to its square* if (u_n) is permutatively equivalent to the basis $\{(u_1, 0), (0, u_1), (u_2, 0), \ldots, \}$ of $[u_n] \oplus [u_n]$.

Lemma 2.4 (see [7, Proposition 5.5]). *Let $(u_n)_{n=1}^{\infty}$ be a left- or right-dominant basis of a quasi-Banach space X. In order that $(u_n)_{n=1}^{\infty}$ be equivalent to its square it is necessary and sufficient that $(u_n)_{n=1}^{\infty}$ be equivalent to $(u_{2n})_{n=1}^{\infty}$.*

Recall from the theory of Schauder bases that if $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are basic sequences in a quasi-Banach space X, then $(x_n)_{n=1}^{\infty}$ *dominates* $(y_n)_{n=1}^{\infty}$ if for all choices of scalars $(a_n)_{n=1}^{\infty}$, whenever $\sum_{n=1}^{\infty} a_n x_n \in [x_n]_{n=1}^{\infty}$ then $\sum_{n=1}^{\infty} a_n y_n \in [y_n]_{n=1}^{\infty}$.

Proposition 2.5 (see [7, Theorem 5.7]). *Suppose that X is a quasi-Banach space with a left- (or right-) dominant unconditional basis $(e_n)_{n=1}^{\infty}$ which induces a p-convex lattice structure on X for some $0 < p < 1$ and such that $(e_n)_{n=1}^{\infty}$ is equivalent to its square. Assume that the Banach envelope $\widehat{X}$ of X is lattice anti-Euclidean. Then,*

(1) *every complemented normalized unconditional basic sequence $(u_n)_{n\in S}$ in X is permutatively equivalent to a subsequence $(e_{k_n})_{n=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$,*

(2) *X has a unique unconditional basis, up to permutation.*

Proof. Consider the left-dominant case. This assumption combined with the fact that $(e_n)_{n=1}^{\infty}$ is equivalent to its square implies by Lemma 2.4 that $(e_n)_{n=1}^{\infty}$ and $(e_{2n})_{n=1}^{\infty}$ are equivalent.

If $(u_n)_{n\in S}$ is a complemented normalized unconditional basic sequence in X, then, by Theorem 1.1, $(u_n)_{n\in S}$ is equivalent to a complemented disjoint sequence in X, which by Proposition 2.2 will in turn be permutatively equivalent to a subsequence $(e_{k_n})_{n=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$. Thus (1) follows.

To show (2), suppose that $(u_n)_{n=1}^{\infty}$ is a normalized unconditional basis of X. Applying Theorem 1.1 again we see that $(e_n)_{n=1}^{\infty}$ is equivalent to a complemented disjoint sequence of the N-fold basis $(u_n)^N$ of X^N for some $N \in \mathbb{N}$. The sequence $(f_n) = (e_{k_n})^N$ written in the obvious order is easily seen to be left-dominant, so $(e_n)_{n=1}^{\infty}$ is permutatively equivalent to a subset of $(e_{k_n})^N$. On the other hand, $(e_{k_n})^N$ is permutatively equivalent to a subset of $(e_n)^N$, which is permutatively equivalent to (e_n). Theorem 1.2 yields that $(e_{k_n})^N$ and (e_n) are permutatively equivalent.

The sequence (f_n) dominates (e_{2n}) by Lemma 2.3, and similarly (e_n) dominates (f_{2n}). Since (e_n) and (e_{4n}) are equivalent we deduce that (f_{2n}) is equivalent to (e_n). Now $(f_{2n-1})_{n=1}^{\infty}$ is dominated by $(f_1, f_2, f_4, \dots, f_{2n-2}, \dots)$ and dominates $(f_1, f_4, f_8, \dots, f_{4n-4}, \dots)$ and so is also equivalent to (f_{2n}). Therefore (f_n) is equivalent to (e_n). Now (f_{Nn}) is equivalent to (e_{Nn}) and hence to (e_n). Since $f_{Nn} = (0, \dots, 0, e_{k_n})$, (e_{k_n}) is equivalent to (e_n) and the proof is complete. □

Theorem 2.6. *If $0 < p < 1$, the p-convexified Tsirelson space $\mathcal{T}^{(p)}$ and all its complemented subspaces with unconditional basis have a unique unconditional basis, up to permutation.*

Proof. For $0 < p < 1$, the space $\mathcal{T}^{(p)}$ is p-convex, right-dominant, and its Banach envelope, ℓ_1, is lattice anti-Euclidean. Besides, a straightforward argument on equivalence of basic sequences combined with the fact that $(e_n)_{n=1}^{\infty}$ and $(e_{2n})_{n=1}^{\infty}$ are equivalent in $\mathcal{T}$ (see [17, page 14]) yields that $(e_n)_{n=1}^{\infty}$ and $(e_{2n})_{n=1}^{\infty}$ are equivalent also in $\mathcal{T}^{(p)}$. An appeal to Lemma 2.4 yields the equivalence of (e_n) and $(e_n)^2$ so that $\mathcal{T}^{(p)}$ is lattice isomorphic to its square. Hence the uniqueness of unconditional basis in $\mathcal{T}^{(p)}$ is a consequence of Proposition 2.5(2).

If Y is a (finite or infinite-dimensional) complemented subspace of $\mathcal{T}^{(p)}$ with normalized unconditional basis $(u_n)_{n\in S}$, then $(u_n)_{n\in S}$ is permutatively equivalent to a subsequence $(e_{k_n})_{n=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$ by Proposition 2.5(1). Clearly, $(e_{k_n})_{n=1}^{\infty}$ is right-dominant and equivalent to $(e_{k_{2n}})_{n=1}^{\infty}$ (see [17, page 14]) and the result follows in the same way as in the preceding paragraph. □

Remark 2.7. For $1 < p \neq 2$ the p-convexification $\mathcal{T}^{(p)}$ of $\mathcal{T}$ does not have a unique unconditional basis up to permutation. Indeed, as Kalton pointed out, this is so because $\mathcal{T}^{(p)}$ can be represented as $(\ell_p^1 \oplus \ell_p^2 \oplus \cdots \ell_p^n \oplus \cdots)_{\mathcal{T}}$, and, in this sum, the factor ℓ_p^n has an unconditional basis containing among its vectors an ℓ_2^k with $k \sim \log n$ (see [18, page 1649]). This also implies that, in this case, $\mathcal{T}^{(p)}$ is sufficiently Euclidean and so the arguments in the proofs of Theorems 2.6 and [7, Theorem 5.7] will not work.

Our work leaves open the following uniqueness questions.

Problem 1. Let $0 < p < 1$. Does the space $\ell_p(\mathcal{T})$ have unique unconditional basis up to a permutation?

Problem 2. It is known that $c_0(\mathcal{T})$ fails to have a unique unconditional basis up to permutation [19]. It would be interesting to know whether the same holds or not in the spaces $c_0(\mathcal{T}^{(p)})$ for $0 < p < 1$.

Problem 3. Determine if $\ell_1(\mathcal{T}^{(p)})$ has a unique unconditional basis up to permutation when $p < 1$ and when $p = 1$.

Acknowledgments

The first-named author acknowledges the support from the Spanish Ministerio de Ciencia e Innovación Research Project *Operadores, retículos, y geometrtía de espacios de Banach*, Reference no. MTM2008-02652/MTM.

References

[1] G. Köthe and O. Toeplitz, "Lineare Raume mit unendlich vielen Koordinaten und Ringen unendlicher Matrizen," *Journal für die Reine und Angewandte Mathematik*, vol. 171, pp. 193–226, 1934.
[2] J. Lindenstrauss and A. Pełczyński, "Absolutely summing operators in L_p-spaces and their applications," *Studia Mathematica*, vol. 29, pp. 275–326, 1968.
[3] J. Lindenstrauss and M. Zippin, "Banach spaces with a unique unconditional basis," *Journal of Functional Analysis*, vol. 3, pp. 115–125, 1969.
[4] I. S. Èdel'šteĭn and P. Wojtaszczyk, "On projections and unconditional bases in direct sums of Banach spaces," *Studia Mathematica*, vol. 56, no. 3, pp. 263–276, 1976.
[5] J. Bourgain, P. G. Casazza, J. Lindenstrauss, and L. Tzafriri, "Banach spaces with a unique unconditional basis, up to permutation," *Memoirs of the American Mathematical Society*, vol. 54, no. 322, 1985.
[6] W. T. Gowers, "A solution to Banach's hyperplane problem," *The Bulletin of the London Mathematical Society*, vol. 26, no. 6, pp. 523–530, 1994.
[7] P. G. Casazza and N. J. Kalton, "Uniqueness of unconditional bases in Banach spaces," *Israel Journal of Mathematics*, vol. 103, pp. 141–175, 1998.
[8] N. J. Kalton, "Orlicz sequence spaces without local convexity," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 81, no. 2, pp. 253–277, 1977.
[9] F. Albiac and C. Leránoz, "Uniqueness of unconditional basis in Lorentz sequence spaces," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1643–1647, 2008.
[10] N. J. Kalton, C. Leránoz, and P. Wojtaszczyk, "Uniqueness of unconditional bases in quasi-Banach spaces with applications to Hardy spaces," *Israel Journal of Mathematics*, vol. 72, no. 3, pp. 299–311, 1990.
[11] P. Wojtaszczyk, "Uniqueness of unconditional bases in quasi-Banach spaces with applications to Hardy spaces. II," *Israel Journal of Mathematics*, vol. 97, pp. 253–280, 1997.
[12] C. Leránoz, "Uniqueness of unconditional bases of $c_0(l_p)$, $0 < p < 1$," *Studia Mathematica*, vol. 102, no. 3, pp. 193–207, 1992.
[13] F. Albiac and C. Leránoz, "Uniqueness of unconditional basis of $l_p(c_0)$ and $l_p(l_2)$, $0 < p < 1$," *Studia Mathematica*, vol. 150, no. 1, pp. 35–52, 2002.
[14] F. Albiac, N. Kalton, and C. Leránoz, "Uniqueness of the unconditional basis of $l_1(l_p)$ and $l_p(l_1)$, $0 < p < 1$," *Positivity*, vol. 8, no. 4, pp. 443–454, 2004.
[15] T. Figiel and W. B. Johnson, "A uniformly convex Banach space which contains no l_p," *Compositio Mathematica*, vol. 29, pp. 179–190, 1974.
[16] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, vol. 233 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2006.
[17] P. G. Casazza and T. J. Shura, *Tsirel'Son's Space*, vol. 1363 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1989, with an appendix by J. Baker, O. Slotterbeck and R. Aro.
[18] L. Tzafriri, "Uniqueness of structure in Banach spaces," in *Handbook of the Geometry of Banach Spaces, Vol. 2*, pp. 1635–1669, North-Holland, Amsterdam, The Netherlands, 2003.
[19] P. G. Casazza and N. J. Kalton, "Uniqueness of unconditional bases in c_0-products," *Studia Mathematica*, vol. 133, no. 3, pp. 275–294, 1999.