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Research Article

Optimal Inequalities among Various Means of Two Arguments

Ming-yu Shi,[1] Yu-ming Chu,[2] and Yue-ping Jiang[1]

[1] *College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*
[2] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*

Correspondence should be addressed to Ming-yu Shi, mingyulj08@163.com

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We establish two optimal inequalities among power mean $M_p(a,b) = (a^p/2 + b^p/2)^{1/p}$, arithmetic mean $A(a,b) = (a+b)/2$, logarithmic mean $L(a,b) = (a-b)/(\log a - \log b)$, and geometric mean $G(a,b) = \sqrt{ab}$.

1. Introduction

For $p \in R$, the power mean $M_p(a,b)$ of order p of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.1}$$

In the recent past, the power mean $M_p(a,b)$ has been the subject of intensive research. In particular, many remarkable inequalities for the mean can be found in literature [1–11]. It is well-known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in R$ for fixed a and b.

If we denote by

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{a^a}{b^b}\right)^{1/(a-b)}, & b \neq a, \\ a, & b = a, \end{cases} \qquad L(a,b) = \begin{cases} \dfrac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a, \end{cases} \tag{1.2}$$

$A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$ and $H(a,b) = 2ab/(a+b)$ the identric mean, logarithmic mean, arithmetic mean, geometric mean and the harmonic mean, of two positive real numbers a and b, respectively, then

$$\begin{aligned}\min\{a,b\} &\le H(a,b) = M_{-1}(a,b) \le G(a,b) = M_0(a,b) \le L(a,b)\\ &\le I(a,b) \le A(a,b) = M_1(a,b) \le \max\{a,b\}\end{aligned} \tag{1.3}$$

with equality if and only if $b = a$ in each inequality.

In [12], Alzer and Janous established the following sharp double inequality (see also [13, page 350]):

$$M_{\log 2/\log 3}(a,b) \le \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \le M_{2/3}(a,b) \tag{1.4}$$

for all real numbers $a, b > 0$.

In [14], Mao proved

$$M_{1/3}(a,b) \le \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b) \le M_{1/2}(a,b) \tag{1.5}$$

for all real numbers $a, b > 0$, and the constant $1/3$ in the left side inequality cannot be improved.

In [15–17], the authors presented the bounds for L and I in terms of A and G as follows:

$$\begin{gathered}G^{2/3}(a,b)A^{1/3}(a,b) < L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b),\\ \frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) < I(a,b)\end{gathered} \tag{1.6}$$

for all $a, b > 0$ with $a \ne b$.

Alzer [18] proved

$$\sqrt{G(a,b)A(a,b)} < \sqrt{L(a,b)I(a,b)} < \frac{1}{2}(L(a,b) + I(a,b)) < \frac{1}{2}(G(a,b) + A(a,b)). \tag{1.7}$$

In [5], Alzer and Qiu established

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{1.8}$$

for $\alpha \le 2/3$, $\beta \ge 2/e = 0.73575\ldots$ and $a, b > 0$ with $a \ne b$.

The main purpose of this paper is to present the optimal bounds for $A^{\alpha}(a,b)L^{1-\alpha}(a,b)$ and $G^{\alpha}(a,b)L^{1-\alpha}(a,b)$ for all $\alpha \in (0,1)$ in terms of the power mean $M_p(a,b)$. Moreover, two optimal inequalities among $A(a,b)$, $G(a,b)$, $L(a,b)$ and $M_p(a,b)$ are proved.

2. Lemmas

In order to establish our main results we need two inequalities, which we present in this section.

Lemma 2.1. *If* $r \in (0,1)$, *then*

$$g(t) = -\left[t^2 + t^{(1+2r)/3} + (1-2r)\left(t^{(1+2r)/3+1} + t\right)\right]\log t + (1-r)\left(t^{(1+2r)/3+2} - t^{(1+2r)/3} + t^2 - 1\right) > 0 \tag{2.1}$$

for $t \in (1, +\infty)$.

Proof. Let $p = (1+2r)/3$, $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g_1'(t)$, $g_3(t) = t^{1-p}g_2'(t)$, $g_4(t) = t^p g_3'(t)$, $g_5(t) = t^{3-p}g_4'(t)$, $g_6(t) = t^p g_5'(t)$, $g_7(t) = t^{1-p}g_6'(t)$, and $g_8(t) = t^p g_7'(t)$, then simple computation yields

$$g(1) = 0, \tag{2.2}$$

$$\begin{aligned} g_1(t) = &-\left[2t^{2-p} + (1-2r)t^{1-p} + (1-2r)(p+1)t + p\right]\log t \\ &+ (1-r)(p+2)t^2 - (1-2r)t + (1-2r)t^{2-p} \\ &- (1-2r)t^{1-p} - p(1-r) - 1, \end{aligned} \tag{2.3}$$

$$g_1(1) = 0,$$

$$\begin{aligned} g_2(t) = &-\left[2(2-p)t + (1-2r)(p+1)t^p + (1-2r)(1-p)\right]\log t \\ &+ 2(1-r)(2+p)t^{1+p} - (1-2r)(p+2)t^p - pt^{p-1} \\ &+ (2rp - 4r - p)t - (1-2r)(2-p), \end{aligned} \tag{2.4}$$

$$g_2(1) = 0,$$

$$\begin{aligned} g_3(t) = &-\left[2(2-p)t^{1-p} + p(p+1)(1-2r)\right]\log t + p(1-p)t^{-1} \\ &+ (2rp - 4r + p - 4)t^{1-p} - (1-2r)(1-p)t^{-p} \\ &+ 2(1-r)(2+p)(1+p)t - (1-2r)\left(p^2 + 3p + 1\right), \end{aligned} \tag{2.5}$$

$$g_3(1) = 6p - 2 - 4r = 0,$$

$$\begin{aligned} g_4(t) = &-2(1-p)(2-p)\log t + 2(1-r)(2+p)(1+p)t^p \\ &- p(1+p)(1-2r)t^{p-1} + p(p-1)t^{p-2} + p(1-p)(1-2r)t^{-1} \\ &- p^2 - 2rp^2 + 6rp - 4r + 7p - 8, \end{aligned} \tag{2.6}$$

$$g_4(1) = 12p - 4 - 8r = 0,$$

$$
\begin{aligned}
g_5(t) &= 2p(1-r)(p+1)(p+2)t^2 + p\left(1-p^2\right)(1-2r)t \\
&\quad - 2(2-p)(1-p)t^{2-p} - p(1-p)(1-2r)t^{1-p} + p(p-1)(p-2), \\
g_5(1) &= 2p^3 + 2(1-4r)p^2 + 4(3-r)p - 4 \\
&= \frac{8}{27}\left(-10r^3 - 15r^2 + 24r + 1\right) > 0,
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
g_6(t) &= 4p(p+1)(p+2)(1-r)t^{p+1} + p\left(1-p^2\right)(1-2r)t^p \\
&\quad - 2(2-p)^2(1-p)t - p(1-p)^2(1-2r), \\
g_6(1) &= 4p^3 + 4(1-4r)p^2 + 8(3-r)p - 8 \\
&= \frac{16}{27}\left(-10r^3 - 15r^2 + 24r + 1\right) > 0,
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
g_7(t) &= 4p(2+p)(1+p)^2(1-r)t - 2(1-p)(2-p)^2t^{1-p} \\
&\quad + p^2\left(1-p^2\right)(1-2r), \\
g_7(1) &= (3-2r)p^4 + 2(9-8r)p^3 + 11(1-2r)p^2 + 8(3-r)p - 8 \\
&= \frac{1}{81}\left(-32r^5 - 400r^4 - 888r^3 - 412r^2 + 1576r + 156\right) > 0,
\end{aligned}
\tag{2.9}
$$

$$
g_8(t) = 4p(1+p)^2(2+p)(1-r)t^p - 2(1-p)^2(2-p)^2, \tag{2.10}
$$

$$
\begin{aligned}
g_8(1) &= 2(1-2r)p^4 + 4(7-4r)p^3 - 2(3+10r)p^2 + 8(4-r)p - 8 \\
&= \frac{8}{81}\left(-8r^5 - 60r^4 - 82r^3 - 79r^2 + 198r + 31\right) > 0.
\end{aligned}
\tag{2.11}
$$

From (2.10) we clearly see that $g_8(t)$ is strictly increasing in $(1, +\infty)$. Therefore, Lemma 2.1 follows from (2.2)–(2.9) and (2.11) together with the monotonicity of $g_8(t)$. □

Lemma 2.2. *If* $r \in (0,1)$, *then*

$$
\begin{aligned}
g(t) = {} & \left[rt^{(1-r)/3+1} + (r-2)t^{(1-r)/3} + (r-2)t + r\right]\log t \\
& + 2(1-r)\left(t^{(1-r)/3+1} - t^{(1-r)/3} + t - 1\right) > 0
\end{aligned}
\tag{2.12}
$$

for $t \in (1, +\infty)$.

Proof. Let $p = (1-r)/3$, $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g_1'(t)$, $g_3(t) = t^{1-p}g_2'(t)$, $g_4(t) = t^{p+2}g_3'(t)$, and $g_5(t) = t^{1-p}g_4'(t)$, then simple computation leads to

$$g(1) = 0, \tag{2.13}$$

$$\begin{aligned} g_1(t) &= \left[r(p+1)t + (r-2)t^{1-p} + p(r-2)\right]\log t - rt^{1-p} + rt^{-p} \\ &\quad + (2+2p-2pr-r)t + 2pr - 2p + r - 2, \end{aligned} \tag{2.14}$$

$$g_1(1) = 0,$$

$$\begin{aligned} g_2(t) &= \left[r(p+1)t^p + (1-p)(r-2)\right]\log t + (2p+2-rp)t^p \\ &\quad + p(r-2)t^{p-1} - rpt^{-1} + pr - 2, \end{aligned} \tag{2.15}$$

$$g_2(1) = 0,$$

$$\begin{aligned} g_3(t) &= pr(p+1)\log t + (r-2)(1-p)t^{-p} + rpt^{-p-1} \\ &\quad + p(p-1)(r-2)t^{-1} + 2p^2 - rp^2 + pr + 2p + r, \end{aligned} \tag{2.16}$$

$$g_3(1) = 6p + 2r - 2 = 0,$$

$$\begin{aligned} g_4(t) &= p(p+1)rt^{p+1} + p(1-p)(r-2)t^p \\ &\quad - p(1-p)(r-2)t - p(p+1)r, \end{aligned} \tag{2.17}$$

$$g_4(1) = 0,$$

$$g_5(t) = rp(1+p)^2 t + p(1-p)(2-r)t^{1-p} + p^2(1-p)(r-2), \tag{2.18}$$

$$\begin{aligned} g_5(1) &= 2p\left(p^2 + 2rp - 2p + 1\right) \\ &= \frac{2(1-r)}{27}\left(-5r^2 + 10r + 4\right) > 0. \end{aligned} \tag{2.19}$$

From (2.18) we clearly see that $g_5(t)$ is strictly increasing in $(1, +\infty)$. Therefore, Lemma 2.2 follows from (2.13)–(2.17) and (2.19) together with the monotonicity of $g_5(t)$. □

3. Main Results

Theorem 3.1. *If $\alpha \in (0,1)$, then*

$$A^{\alpha}(a,b)L^{1-\alpha}(a,b) \le M_{(1+2\alpha)/3}(a,b) \tag{3.1}$$

with equality if and only if $a = b$, and the parameter $(1+2\alpha)/3$ cannot be improved.

Proof. If $a = b$, then we clearly see that

$$a = A^{\alpha}(a,b)L^{1-\alpha}(a,b) = M_{(1+2\alpha)/3}(a,b) = b. \tag{3.2}$$

If $a \neq b$, then without loss of generality we assume that $a > b$ and let $t = a/b > 1$; hence elementary calculation yields

$$M_{(1+2\alpha)/3}(a,b) - A^{\alpha}(a,b)L^{1-\alpha}(a,b) = b\left[\left(\frac{t^{(1+2\alpha)/3}+1}{2}\right)^{3/(1+2\alpha)} - \left(\frac{1+t}{2}\right)^{\alpha}\left(\frac{t-1}{\log t}\right)^{1-\alpha}\right]. \tag{3.3}$$

Let

$$\begin{aligned} f(t) = {} & \frac{3}{1+2\alpha}\log\left(t^{(1+2\alpha)/3}+1\right) - \frac{3}{1+2\alpha}\log 2 - \alpha\log\frac{t+1}{2} + \alpha\log 2 \\ & -(1-\alpha)\log(t-1) + (1-\alpha)\log(\log t), \end{aligned} \tag{3.4}$$

then

$$\lim_{t\to 1} f(t) = 0, \tag{3.5}$$

$$f'(t) = \frac{g(t)}{t(t+1)(t-1)\left(t^{(1+2\alpha)/3}+1\right)\log t}, \tag{3.6}$$

where

$$\begin{aligned} g(t) = {} & -\left[t^2 + t^{(1+2\alpha)/3} + (1-2\alpha)\left(t^{(1+2\alpha)/3+1} + t\right)\right]\log t \\ & + (1-\alpha)\left(t^{(1+2\alpha)/3+2} - t^{(1+2\alpha)/3} + t^2 - 1\right). \end{aligned} \tag{3.7}$$

From Lemma 2.1 and (3.6) we know that

$$f'(t) > 0 \tag{3.8}$$

for $t \in (1, +\infty)$.

Therefore, we get

$$M_{(1+2\alpha)/3}(a,b) > A^{\alpha}(a,b)L^{1-\alpha}(a,b) \tag{3.9}$$

for $a > b$ that follows from (3.3)–(3.5) and (3.8).

Next, we prove that the parameter $(1+2\alpha)/3$ cannot be improved.

For any $0 < \varepsilon < (1+2\alpha)/3$, let $0 < t < 1$, then (1.1) leads to

$$\begin{aligned} &A^{\alpha}(t+1,1)L^{1-\alpha}(t+1,1) - M_{(1+2\alpha)/3-\varepsilon}(t+1,1) \\ &\quad = \left(1+\frac{t}{2}\right)^{\alpha} \cdot \left[\frac{t}{\log(t+1)}\right]^{1-\alpha} - \left[\frac{(t+1)^{(1+2\alpha)/3-\varepsilon}+1}{2}\right]^{1/((1+2\alpha)/3-\varepsilon)} \\ &\quad = \frac{f_1(t)}{[\log(1+t)]^{1-\alpha}}, \end{aligned} \tag{3.10}$$

where

$$f_1(t) = \left(1+\frac{t}{2}\right)^{\alpha} \cdot t^{1-\alpha} - \left[\frac{(t+1)^{(1+2\alpha-3\varepsilon)/3}+1}{2}\right]^{3/(1+2\alpha-3\varepsilon)} [\log(1+t)]^{1-\alpha}. \tag{3.11}$$

Making use of the Taylor expansion we get

$$f_1(t) = \left[\frac{\varepsilon}{8}t^2 + o\left(t^2\right)\right] \cdot t^{1-\alpha}. \tag{3.12}$$

Equations (3.10) and (3.12) imply that for any $\alpha \in (0,1)$ and $0 < \varepsilon < (1+2\alpha)/3$, there exists $0 < \delta_1(\varepsilon,\alpha) < 1$, such that

$$A^{\alpha}(1,1+t)L^{1-\alpha}(1,1+t) > M_{(1+2\alpha)/3-\varepsilon}(1,1+t) \tag{3.13}$$

for $t \in (0,\delta_1)$. □

Remark 3.2. For any $0 < \alpha < 1$ we have

$$M_0(a,b) \leq A^{\alpha}(a,b)L^{1-\alpha}(a,b) \tag{3.14}$$

for all $a,b > 0$, with equality if and only if $a = b$, and the parameter 0 in the lower bound cannot be improved.

In fact, if $a = b$, then we clearly see that $M_0(a,b) = A^{\alpha}(a,b)L^{1-\alpha}(a,b) = a$. If $a \neq b$, then $M_0(a,b) < A^{\alpha}(a,b)L^{1-\alpha}(a,b)$ follows from $M_0(a,b) < L(a,b) < A(a,b)$.

Next, we prove that the parameter 0 in the lower bound cannot be improved.

For any $\varepsilon > 0$, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{M_{\varepsilon}(t,1)}{A^{\alpha}(t,1)L^{1-\alpha}(t,1)} &= \lim_{t\to\infty} \frac{((t^{\varepsilon}+1)/2)^{1/\varepsilon}}{((t+1)/2)^{\alpha}((t-1)/\log t)^{1-\alpha}} \\ &= \lim_{t\to\infty} \frac{((1+t^{-\varepsilon})/2)^{1/\varepsilon}(\log t)^{1-\alpha}}{((1+t^{-1})/2\ )^{\alpha}(1-t^{-1})^{1-\alpha}} \\ &= +\infty. \end{aligned} \tag{3.15}$$

Equation (3.15) implies that for any $\varepsilon > 0$, there exists $T_1 = T_1(\varepsilon) > 1$, such that

$$A^{\alpha}(t,1)L^{1-\alpha}(t,1) < M_{\varepsilon}(t,1) \tag{3.16}$$

for $t \in (T_1, +\infty)$.

Theorem 3.3. *If $\alpha \in (0,1)$, then*

$$G^{\alpha}(a,b)L^{1-\alpha}(a,b) \leq M_{(1-\alpha)/3}(a,b) \tag{3.17}$$

for all $a, b > 0$, with equality if and only if $a = b$, and the parameter $(1-\alpha)/3$ cannot be improved.

Proof. If $a = b$, then we clearly see that

$$a = G^{\alpha}(a,b)L^{1-\alpha}(a,b) = M_{(1-\alpha)/3}(a,b) = b. \tag{3.18}$$

If $a \neq b$, then without loss of generality, we assume that $a > b$, let $t = a/b > 1$; hence simple computation leads to

$$M_{(1-\alpha)/3}(a,b) - G^{\alpha}(a,b)L^{1-\alpha}(a,b) = b\left[\left(\frac{t^{(1-\alpha)/3}+1}{2}\right)^{3/(1-\alpha)} - t^{\alpha/2}\left(\frac{t-1}{\log t}\right)^{1-\alpha}\right]. \tag{3.19}$$

Let

$$\begin{aligned} h(t) = {} & \frac{3}{1-\alpha}\log\left(t^{(1-\alpha)/3}+1\right) - \frac{3}{1-\alpha}\log 2 - \alpha\log\frac{t+1}{2} + \alpha\log 2 \\ & - (1-\alpha)\log(t-1) + (1-\alpha)\log(\log t), \end{aligned} \tag{3.20}$$

then

$$\lim_{t\to 1} h(t) = 0, \tag{3.21}$$

$$h'(t) = \frac{g(t)}{2t(t-1)\left(t^{(1-\alpha)/3}+1\right)\log t}, \tag{3.22}$$

where

$$\begin{aligned} g(t) = {} & \left[\alpha t^{(1-\alpha)/3+1} + (\alpha-2)t^{(1-\alpha)/3} + (\alpha-2)t + \alpha\right]\log t \\ & + 2(1-\alpha)\left(t^{(1-\alpha)/3+1} - t^{(1-\alpha)/3} + t - 1\right). \end{aligned} \tag{3.23}$$

From Lemma 2.2 and (3.22) we know that

$$h'(t) > 0 \tag{3.24}$$

for $t \in (1, +\infty)$.

Therefore, we get

$$M_{(1-\alpha)/3}(a,b) > G^{\alpha}(a,b)L^{1-\alpha}(a,b) \tag{3.25}$$

for $a > b$ that follows from (3.19)–(3.21) and (3.24).

Next, we prove that the constants $(1-\alpha)/3$ cannot be improved.

For any $0 < \varepsilon < (1-\alpha)/3$, let $0 < t < 1$, then (1.1) leads to

$$\begin{aligned} &G^{\alpha}(t+1,1)L^{1-\alpha}(t+1,1) - M_{(1-\alpha)/3-\varepsilon}(t+1,1) \\ &\quad = (1+t)^{\alpha/2}\cdot\left[\frac{t}{\log(t+1)}\right]^{1-\alpha} - \left[\frac{(t+1)^{(1-\alpha)/3-\varepsilon}+1}{2}\right]^{1/((1-\alpha)/3-\varepsilon)} \\ &\quad = \frac{h_1(t)}{[\log(1+t)]^{1-\alpha}}, \end{aligned} \tag{3.26}$$

where

$$h_1(t) = (1+t)^{\alpha/2}\cdot t^{1-\alpha} - \left[\frac{(t+1)^{(1-\alpha-3\varepsilon)/3}+1}{2}\right]^{3/(1-\alpha-3\varepsilon)}[\log(1+t)]^{1-\alpha}. \tag{3.27}$$

Making use of the Taylor expansion we get

$$h_1(t) = \left[\frac{\varepsilon}{8}t^2 + o\left(t^2\right)\right]\cdot t^{1-\alpha}. \tag{3.28}$$

Equations (3.26) and (3.28) imply that for any $\alpha \in (0,1)$ and $0 < \varepsilon < (1-\alpha)/3$, there exists $0 < \delta_2(\varepsilon,\alpha) < 1$, such that

$$A^{\alpha}(1,1+t)L^{1-\alpha}(1,1+t) > M_{(1-\alpha)/3-\varepsilon}(1,1+t) \tag{3.29}$$

for $t \in (0,\delta_2)$. □

Remark 3.4. For any $\varepsilon > 0$, we have

$$\begin{aligned} \lim_{t\to\infty}\frac{M_{\varepsilon}(t,1)}{G^{\alpha}(t,1)L^{1-\alpha}(t,1)} &= \lim_{t\to\infty}\frac{((t^{\varepsilon}+1)/2)^{1/\varepsilon}}{t^{\alpha/2}((t-1)/\log t)^{1-\alpha}} \\ &= \lim_{t\to\infty}\frac{((1+t^{-\varepsilon})/2)^{1/\varepsilon}t^{\alpha/2}(\log t)^{1-\alpha}}{(1-t^{-1})^{1-\alpha}} = +\infty. \end{aligned} \tag{3.30}$$

Therefore, (3.30) implies that inequality

$$M_0(a,b) \le G^{\alpha}(a,b)L^{1-\alpha}(a,b) \tag{3.31}$$

holds for all $\alpha \in (0, 1)$ and $a, b > 0$, with equality if and only if $a = b$ and the parameter 0 in the lower bound cannot be improved.

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