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Research Article

On Genocchi Numbers and Polynomials

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The main purpose of this paper is to study the distribution of Genocchi polynomials. Finally, we construct the Genocchi zeta function which interpolates Genocchi polynomials at negative integers.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rationalnumbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks about q-extension, q is variously considered as an indeterminate, a complex, $q \in \mathbb{C}$, or a p-adic number, $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < 1$. The ordinary Genocchi polynomials are defined as the generating function:

$$F(t,x) = \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi. \tag{1.1}$$

For a fixed positive integer d with $(p,d)=1$, set

$$\begin{gathered} X = X_d = \lim_{\overleftarrow{N}} \frac{\mathbb{Z}}{dP^N\mathbb{Z}}, \quad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp,\\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\} \end{gathered} \tag{1.2}$$

(cf. [1–30]), where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. We say that f is uniformly differential function at $a \in \mathbb{Z}_p$ and write $f \in UD(\mathbb{Z}_p)$ if the difference quotients, $F_f(x,y) = (f(x)-f(y))/(x-y)$, have a limit $f'(a)$ as $(x,y) \to (a,a)$. Throughout this paper, we use the following notation:

$$[x]_{-q} = \frac{1-(-q)^x}{1+q}, \qquad [x]_q = \frac{1-q^x}{1-q}. \tag{1.3}$$

For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x; \tag{1.4}$$

see [1–27]. Note that

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x). \tag{1.5}$$

In this paper, we investigate some interesting integral equations related to $I_{-1}(f)$. From these integral equations related to $I_{-1}(f)$, we can derive many interesting properties of Genocchi numbers and polynomials. The main purpose of this paper is to derive the distribution relations of the Genocchi polynomials, and to constructthe Genocchi zeta function which interpolates the Genocchi polynomials at negative integers.

2. Genocchi numbers and polynomials

The Genocchi numbers are defined as

$$G_0 = 0, \quad (G+1)^n + G_n = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{2.1}$$

where G^n is replaced by G_n, symbolically. The Genocchi polynomials are also defined as

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} G_k. \tag{2.2}$$

From (2.1), we note that $G_1 = 1,\ G_2 = -1,\ G_3 = 0,\ G_4 = 1, \ldots, G_{2k+1} = 0$, and $G_{2k} \in \mathbb{Z}$ $(k = 1, 2, \ldots)$. The fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-1}} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{see [1]}. \tag{2.3}$$

Let $f_1(x)$ be translation with $f_1(x) = f(x+1)$. Then we have the following integral equation. Note that $I_{-1}(f_1) + I_{-1}(f) = 2f(0)$. From (2.3), we can derive

$$t\int_{\mathbb{Z}_p} e^{xt}\, d\mu_{-1}(x) = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = t\sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1} \frac{t^n}{n!}, \tag{2.4}$$

$$t\int_{\mathbb{Z}_p} e^{(x+y)t}\, d\mu_{-1}(y) = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = t\sum_{n=0}^{\infty}\frac{G_{n+1}(x)}{n+1}\frac{t^n}{n!}. \tag{2.5}$$

Thus, we obtain

$$\int_{\mathbb{Z}_p} x^n\, d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \qquad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}. \tag{2.6}$$

For $n \in \mathbb{N}$, we have

$$\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) + 2\sum_{\ell=0}^{n-1}(-1)^{n-1+\ell} f(\ell), \quad \text{see [1–27]}. \tag{2.7}$$

By (2.6) and (2.7), if we take $f(x) = x^k (k \in \mathbb{Z}^+)$, we easily see that

$$\int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) - \int_{\mathbb{Z}_p} x^k\, d\mu_{-1}(x) = 2\sum_{\ell=0}^{n-1}(-1)^{\ell-1}\ell^k \quad \text{if } n \equiv 0 (\text{mod } 2). \tag{2.8}$$

Thus, we have

$$\frac{G_{k+1}(n)}{k+1} - \frac{G_{k+1}}{k+1} = 2\sum_{\ell=0}^{n-1}(-1)^{\ell-1}\ell^k \quad \text{if } n \equiv 0 (\text{mod } 2). \tag{2.9}$$

If $n \equiv 1 (\text{mod } 2)$, then we know that

$$\int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^k\, d\mu_{-1}(x) = 2\sum_{\ell=0}^{n-1}(-1)^{\ell}\ell^k \quad \text{if } n \equiv 1 (\text{mod } 2). \tag{2.10}$$

Thus, we get

$$\frac{G_{k+1}(n)}{k+1} + \frac{G_{k+1}}{k+1} = 2\sum_{\ell=0}^{n-1}(-1)^{\ell}\ell^k, \quad \text{see [1–30]}. \tag{2.11}$$

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$, with $d \equiv 1 (\text{mod } 2)$. Then, we consider the generalized Genocchi numbers attached to χ as follows:

$$\frac{G_{n+1,\chi}}{n+1} = \int_X \chi(x)x^n\, d\mu_{-1}(x), \quad G_{0,\chi} = 0, \tag{2.12}$$

where $n \in \mathbb{Z}_+$. From (2.7) and (2.12), we note that

$$t\int_X e^{xt}\chi(x)d\mu_{-1}(x) = \frac{2\sum_{\ell=0}^{d-1}(-1)^{\ell}\chi(\ell)e^{\ell t}}{e^{dt}+1}t = \sum_{n=0}^{\infty}\frac{G_{n,\chi}}{n!}t^n. \tag{2.13}$$

By (2.12) and (2.13), it is not difficult to show that

$$\begin{aligned} \frac{G_{n+1,\chi}}{n+1} &= \int_X \chi(x)x^n\, d\mu_{-1}(x) \\ &= d^n\sum_{a=0}^{d-1}\chi(a)(-1)^a\int_{\mathbb{Z}_p}\left(\frac{a}{d}+x\right)^n d\mu_{-1}(x) \\ &= d^n\sum_{a=0}^{d-1}\chi(a)(-1)^a\frac{G_{n+1}(a/d)}{n+1}, \end{aligned} \tag{2.14}$$

$$\int_X (x+y)^n d\mu_{-1}(y) = d^n\sum_{n=0}^{d-1}(-1)^a\int_{\mathbb{Z}_p}\left(\frac{x+a}{d}+y\right)^n d\mu_{-1}(y). \tag{2.15}$$

By (2.6) and (2.15), we obtain the following theorem.

Theorem 2.1. *Let $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\, 2)$, and let χ be the Dirichlet characterwith conductor d. Then, one has*

$$\frac{G_{n+1}(x)}{n+1} = d^n \sum_{a=0}^{d-1} (-1)^a \frac{G_{n+1}((x+a)/d)}{n+1} \quad (\textit{distribution relation for Genocchi polynomials}), \tag{2.16}$$

$$\frac{G_{n+1,\chi}}{n+1} = d^n \sum_{a=0}^{d-1} \chi(a)(-1)^a \frac{G_{n+1}(a/d)}{n+1}. \tag{2.17}$$

3. Genocchi zeta function

Let $F(t,x)$ be the generating function of $G_k(x)$ in complex plane as follows:

$$\begin{aligned} F(t,x) &= \frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \\ &= t \sum_{n=0}^{\infty} \frac{G_{n+1}(x)}{n+1} \frac{t^n}{n!}, \quad |t| < \pi. \end{aligned} \tag{3.1}$$

Then, we show that

$$F(t,x) = 2t \sum_{n=0}^{\infty} (-1)^n e^{(n+x)t}. \tag{3.2}$$

By (3.1) and (3.2), we easily see that

$$G_k(x) = \left.\frac{d^k F(t,x)}{dt^k}\right|_{t=0} = 2k \sum_{n=0}^{\infty} (-1)^n (n+x)^{k-1}. \tag{3.3}$$

Therefore, we obtain the following proposition.

Proposition 3.1. *For $k \in \mathbb{N}$, one has*

$$\frac{G_k(x)}{k} = 2 \sum_{n=0}^{\infty} (-1)^n (n+x)^{k-1}. \tag{3.4}$$

From Proposition 3.1, we can derive the Genocchi zeta function which interpolates Genocchi polynomials at negative integers.

For $s \in \mathbb{C}$, we define the Hurwitz-type Genocchi zeta function as follows.

Definition 3.2. *For $s \in \mathbb{C}$,*

$$\zeta_G(s,x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \qquad \zeta_G(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}. \tag{3.5}$$

By Proposition 3.1 and Definition 3.2, we obtain the following theorem.

Theorem 3.3. *For $k \in \mathbb{N}$, one has*

$$\zeta_G(1-k,x) = \frac{G_k(x)}{k}, \qquad \zeta_G(1-k) = \frac{G_k}{k}. \tag{3.6}$$

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$, with $d \equiv 1 (\bmod 2)$, and let $F_\chi(t)$ be the generating function in $\mathbb{C}$ of $G_{n,\chi}$. Then, we have

$$F_\chi(t) = 2\frac{\sum_{\ell=0}^{d-1}(-1)^\ell \chi(\ell)e^{\ell t}}{e^{dt}+1}t = \sum_{n=0}^{\infty} G_{n,\chi}\frac{t^n}{n!}, \quad |t| < \frac{\pi}{d}. \tag{3.7}$$

From (3.7), we derive

$$\begin{aligned} F_\chi(t) &= \sum_{n=0}^{\infty} G_{n,\chi}\frac{t^n}{n!} = \sum_{n=1}^{\infty} G_{n,\chi}\frac{t^n}{n!} \\ &= t\sum_{n=0}^{\infty}\frac{G_{n+1,\chi}}{n+1}\frac{t^n}{n!} \\ &= t\sum_{n=0}^{\infty}\left(d^n\sum_{a=0}^{d-1}\chi(a)(-1)^a\frac{G_{n+1}(a/d)}{n+1}\right)\frac{t^n}{n!} \\ &= \sum_{a=0}^{d-1}\chi(a)(-1)^a\left(dt\sum_{n=0}^{\infty}\frac{G_{n+1}(a/d)}{n+1}\frac{d^n t^n}{n!}\right). \end{aligned} \tag{3.8}$$

By (3.1), (3.2), and (3.8), we easily see that

$$\begin{aligned} F_\chi(t) &= \sum_{a=0}^{d-1}\chi(a)(-1)^a\left(2t\sum_{k=0}^{\infty}(-1)^k e^{(k+a/d)dt}\right) \\ &= 2t\sum_{k=0}^{\infty}\sum_{a=0}^{d-1}\chi(a+dk)(-1)^{a+dk}e^{(dk+a)t} \\ &= 2t\sum_{n=0}^{\infty}\chi(n)(-1)^n e^{nt} \\ &= 2t\sum_{n=1}^{\infty}\chi(n)(-1)^n e^{nt}. \end{aligned} \tag{3.9}$$

From (3.9), we can derive

$$G_{k,\chi} = \frac{d^k}{dt^k}F_\chi(t)\bigg|_{t=0} = k\left(2\sum_{n=1}^{\infty}\chi(n)(-1)^n n^{k-1}\right). \tag{3.10}$$

Thus, we have

$$\frac{G_{k},\chi}{k} = 2\sum_{k=0}^{\infty}\chi(n)(-1)^n n^{k-1}. \tag{3.11}$$

Now, we consider the Dirichlet-type Genocchi ℓ-function in complex plane as follows. For $s \in \mathbb{C}$, define

$$\ell_{G,\chi}(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n^s}. \tag{3.12}$$

By (3.11) and (3.12), we obtain the following theorem.

Theorem 3.4. *Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$, with $d \equiv 1 (\bmod 2)$, and let $k \in \mathbb{Z}^+$. Then, one has*

$$\ell_{G,\chi}(1-k) = \frac{G_{k,\chi}}{k}. \tag{3.13}$$

Remark 3.5. In [1], we can observe the value of Genocchi zeta function at positive integers as follows:

$$\zeta_G(2n) = \frac{(-1)^{n-1}\pi^{2n}(2-4^n)}{2(2n)!(1-4^n)} G_{2n}, \quad \text{cf. [1]}. \tag{3.14}$$

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