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Research Article

On the Stability of Quadratic Functional Equations

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Let X, Y be vector spaces and k a fixed positive integer. It is shown that a mapping $f(kx+y) + f(kx-y) = 2k^2f(x) + 2f(y)$ for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Furthermore, the Hyers-Ulam-Rassias stability of the above functional equation in Banach spaces is proven.

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mapping and by Th. M. Rassias [4] for linear mappings by considering an *unbounded Cauchy difference.* The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we now call *Hyers-Ulam-Rassias stability of functional equations*. Th. M. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [6], following the same approach as in [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6] as well as by Rassias and Šemrl [7] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. J. M. Rassias [8], following the spirit of the innovative approach of Th. M. Rassias [4] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A Hyers-Ulam-Rassias stability problem for the quadratic functional equation was proved by Skof [9] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [11], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated in [12–17].

Throughout this paper, assume that k is a fixed positive integer.

In this paper, we solve the functional equation

$$f(kx+y)+f(kx-y)=2k^2f(x)+2f(y) \tag{1.2}$$

and prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) in Banach spaces.

2. Hyers-Ulam-Rassias stability of the quadratic functional equation

Proposition 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$f(kx+y)+f(kx-y)=2k^2f(x)+2f(y) \tag{2.1}$$

for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.2}$$

for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $f(0) = 0$.

Letting $y = 0$ in (2.1), we get $f(kx) = k^2f(x)$ for all $x \in X$.

Letting $x = 0$ in (2.1), we get $f(-y) = f(y)$ for all $y \in X$.

It follows from (2.1) that

$$f(kx+y)+f(kx-y)=2k^2f(x)+2f(y)=2f(kx)+2f(y) \tag{2.3}$$

for all $x, y \in X$. So the mapping $f : X \to Y$ satisfies

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.4}$$

for all $x, y \in X$.

Assume that $f : X \to Y$ satisfies $f(x+y)+f(x-y)=2f(x)+2f(y)$ for all $x, y \in X$.

We prove (2.1) for $k = j$ by induction on j.

For the case $j = 1$, (2.1) holds by the assumption.

For the case $j = 2$, since

$$\begin{aligned} f(2x+y)+f(2x-y) &= f(x+y+x)+f(x-y+x) \\ &= 2f(x+y)+2f(x)-f(y)+2f(x-y)+2f(x)-f(-y) \\ &= 2f(x+y)+2f(x-y)+4f(x)-2f(y) \\ &= 4f(x)+4f(y)+4f(x)-2f(y) \\ &= 8f(x)+2f(y) \end{aligned} \tag{2.5}$$

for all $x, y \in X$, then (2.1) holds.

Assume that (2.1) holds for $j = n-2$ and $j = n-1$ $(2 < n \le k)$. By the assumption,

$$\begin{aligned} f(nx+y)+f(nx-y) &= f\big((n-1)x+y+x\big)+f\big((n-1)x-y+x\big) \\ &= 2f\big((n-1)x+y\big)+2f(x)-f\big((n-2)x+y\big) \\ &\quad +2f\big((n-1)x-y\big)+2f(x)-f\big((n-2)x-y\big) \\ &= 4(n-1)^2f(x)+4f(y)+4f(x)-2(n-2)^2f(x)-2f(y) \\ &= 2n^2f(x)+2f(y) \end{aligned} \tag{2.6}$$

for all $x, y \in X$, (2.1) holds for $j = n$. Hence the mapping $f : X \to Y$ satisfies (2.1) for $j = k$. □

From now on, assume that X is a normed vector space with norm $\|\cdot\|$ and that Y is a Banach space with norm $\|\cdot\|$.

For a given mapping $f : X \to Y$, we define

$$Df(x,y) := f(kx+y)+f(kx-y)-2k^2f(x)-2f(y) \tag{2.7}$$

for all $x, y \in X$.

Now we prove the Hyers-Ulam-Rassias stability of the quadratic functional equation $Df(x,y) = 0$.

Theorem 2.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty}\frac{1}{k^{2j}}\varphi(k^jx,k^jy) < \infty, \tag{2.8}$$

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \le \frac{1}{2k^2}\widetilde{\varphi}(x,0) \tag{2.10}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.9), we get

$$\|2f(kx)-2k^2f(x)\| \le \varphi(x,0) \tag{2.11}$$

for all $x \in X$. So

$$\left\|f(x)-\frac{1}{k^2}f(kx)\right\| \le \frac{1}{2k^2}\varphi(x,0) \tag{2.12}$$

for all $x \in X$. Hence

$$\left\|\frac{1}{k^{2l}}f(k^lx)-\frac{1}{k^{2m}}f(k^mx)\right\| \le \sum_{j=l}^{m-1}\frac{1}{2k^{2j+2}}\varphi(k^jx,0) \tag{2.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.13) that the sequence $\{(1/k^{2n})f(k^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(1/k^{2n})f(k^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{k^{2n}} f(k^n x) \tag{2.14}$$

for all $x \in X$.

By (2.8),

$$\|DQ(x,y)\| = \lim_{n\to\infty} \frac{1}{k^{2n}} \|Df(k^n x, k^n y)\| \le \lim_{n\to\infty} \frac{1}{k^{2n}} \varphi(k^n x, k^n y) = 0 \tag{2.15}$$

for all $x, y \in X$. So $DQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.10).

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.1) and (2.10). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \frac{1}{k^{2n}} \|Q(k^n x) - T(k^n x)\| \\ &\le \frac{1}{k^{2n}} \left(\|Q(k^n x) - f(k^n x)\| + \|T(k^n x) - f(k^n x)\|\right) \\ &\le \frac{1}{k^{2n+2}} \widetilde{\varphi}(k^n x, 0),\end{aligned} \tag{2.16}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. So there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.10). □

Corollary 2.3. *Let $p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Df(x,y)\| \le \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.17}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{\theta}{8 - 2^{p+1}} \|x\|^p \tag{2.18}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x,y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.19}$$

for all $x, y \in A$. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ satisfying (2.9) such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} k^{2j} \varphi\left(\frac{x}{k^j}, \frac{y}{k^j}\right) < \infty \tag{2.20}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2} \widetilde{\varphi}\left(\frac{x}{k}, 0\right) \tag{2.21}$$

for all $x \in X$.

Proof. It follows from (2.11) that

$$\left\| f(x) - k^2 f\left(\frac{x}{k}\right) \right\| \le \frac{1}{2}\varphi\left(\frac{x}{k}, 0\right) \tag{2.22}$$

for all $x \in X$. Hence

$$\left\| k^{2l} f\left(\frac{x}{k^l}\right) - k^{2m} f\left(\frac{x}{k^m}\right) \right\| \le \sum_{j=l}^{m-1} \frac{k^{2j}}{2} \varphi\left(\frac{x}{k^{j+1}}, 0\right) \tag{2.23}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.23) that the sequence $\{k^{2n} f(x/k^n)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{k^{2n} f(x/k^n)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} k^{2n} f\left(\frac{x}{k^n}\right) \tag{2.24}$$

for all $x \in X$.

By (2.20),

$$\|DQ(x,y)\| = \lim_{n\to\infty} k^{2n} \left\| Df\left(\frac{x}{k^n}, \frac{y}{k^n}\right) \right\| \le \lim_{n\to\infty} k^{2n} \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0 \tag{2.25}$$

for all $x, y \in X$. So $DQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.23), we get (2.21).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.17). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{\theta}{2^{p+1} - 8} \|x\|^p \tag{2.26}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x,y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.27}$$

for all $x, y \in A$. □

From now on, assume that $k = 2$.

Theorem 2.6. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.9) *such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} \frac{1}{9^j} \varphi(3^j x, 3^j y) < \infty \tag{2.28}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{9} \widetilde{\varphi}(x,x) \tag{2.29}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.9), we get

$$\|f(3x) - 9f(x)\| \leq \varphi(x,x) \tag{2.30}$$

for all $x \in X$. So

$$\left\|f(x) - \frac{1}{9}f(3x)\right\| \leq \frac{1}{9}\varphi(x,x) \tag{2.31}$$

for all $x \in X$. Hence

$$\left\|\frac{1}{9^l}f(3^l x) - \frac{1}{9^m}f(3^m x)\right\| \leq \sum_{j=l}^{m-1}\frac{1}{9^{j+1}}\varphi(3^j x, 3^j x) \tag{2.32}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.32) that the sequence $\{(1/9^n)f(3^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{(1/9^n)f(3^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{9^n}f(3^n x) \tag{2.33}$$

for all $x \in X$.

By (2.28),

$$\|DQ(x,y)\| = \lim_{n\to\infty}\frac{1}{9^n}\|Df(3^n x, 3^n y)\| \leq \lim_{n\to\infty}\frac{1}{9^n}\varphi(3^n x, 3^n y) = 0 \tag{2.34}$$

for all $x, y \in X$. So $DQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.32), we get (2.29).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.7. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Df(x,y)\| \leq \theta \cdot \|x\|^p \cdot \|y\|^p \tag{2.35}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{9 - 9^p}\|x\|^{2p} \tag{2.36}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.6 by taking

$$\varphi(x,y) := \theta \cdot \|x\|^p \cdot \|y\|^p \tag{2.37}$$

for all $x, y \in A$. □

Theorem 2.8. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.9) *such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 9^j \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}\right) < \infty \tag{2.38}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \widetilde{\varphi}\left(\frac{x}{3}, \frac{x}{3}\right) \tag{2.39}$$

for all $x \in X$.

Proof. It follows from (2.30) that

$$\left\| f(x) - 9f\left(\frac{x}{3}\right)\right\| \le \varphi\left(\frac{x}{3}, \frac{x}{3}\right) \tag{2.40}$$

for all $x \in X$. Hence

$$\left\| 9^l f\left(\frac{x}{3^l}\right) - 9^m f\left(\frac{x}{3^m}\right)\right\| \le \sum_{j=l}^{m-1} 9^j \varphi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}\right) \tag{2.41}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.41) that the sequence $\{9^n f(x/3^n)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{9^n f(x/3^n)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right) \tag{2.42}$$

for all $x \in X$.

By (2.38),

$$\|DQ(x, y)\| = \lim_{n\to\infty} \frac{1}{9^n} \|Df(3^n x, 3^n y)\| \le \lim_{n\to\infty} \frac{1}{9^n} \varphi(3^n x, 3^n y) = 0 \tag{2.43}$$

for all $x, y \in X$. So $DQ(x, y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.41), we get (2.39).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.9. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.35). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{\theta}{9^p - 9} ||x||^{2p} \tag{2.44}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.8 by taking

$$\varphi(x, y) := \theta \cdot ||x||^p \cdot ||y||^p \tag{2.45}$$

for all $x, y \in A$. □

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