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Research Article

A Note on the Multiple Twisted Carlitz's Type q-Bernoulli Polynomials

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We give the twisted Carlitz's type q-Bernoulli polynomials and numbers associated with p-adic q-inetgrals and discuss their properties. Furthermore, we define the multiple twisted Carlitz's type q-Bernoulli polynomials and numbers and obtain the distribution relation for them.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, be the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|1-q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q} \tag{1.1}$$

(cf. [1–20]) for all $x \in \mathbb{Z}_p$. For a fixed odd positive integer d with $(p,d) = 1$, let

$$\begin{gathered} X = X_d = \frac{\varprojlim_n \mathbb{Z}}{dp^n\mathbb{Z}}, \qquad X_1 = \mathbb{Z}_p, \qquad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^n}\}, \end{gathered} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^n$. For any $n \in \mathbb{N}$,

$$\mu_q\left(a + dp^n\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^n\right]_q} \tag{1.3}$$

is known to be a distribution on X (cf. [1–20]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in \mathrm{UD}(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y} \tag{1.4}$$

have a limit $l = f'(a)$ as $(x, y) \to (a, a)$ (cf. [10–13]). The p-adic q-integral of a function $f \in \mathrm{UD}(\mathbb{Z}_p)$ was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{n\to\infty} \frac{1}{\left[p^n\right]_q} \sum_{x=0}^{p^n-1} f(x)q^x. \tag{1.5}$$

By using p-adic q-integrals on $\mathbb{Z}_p$, it is well known that

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_1(x) \frac{t^n}{n!}, \tag{1.6}$$

where $\mu_1(x + p^n\mathbb{Z}_p) = 1/p^n$. Then, we note that the Bernoulli numbers B_n were defined as

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{1.7}$$

and hence, we have

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x) \tag{1.8}$$

for all $n \in \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, the multiple Bernoulli polynomials $B_n^{(k)}(x)$ were defined as

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \tag{1.9}$$

(cf. [2]). We note that

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} \left(x + x_1 + \cdots + x_k\right)^n d\mu_1(x_1) \cdots d\mu_1(x_k). \tag{1.10}$$

From (1.9) and (1.10), we obtain

$$B_n^{(k)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} (x + x_1 + \cdots + x_k)^n d\mu_1(x_1) \cdots d\mu_1(x_k). \tag{1.11}$$

In view of (1.11), the multiple Carlitz's type q-Bernoulli polynomials were defined as

$$\beta_n^{(k,q)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} [x + x_1 + \cdots + x_k]_q^n d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{1.12}$$

In this case, $x = 0$, we write $\beta_n^{(k,q)}(0) = \beta_n^{(k,q)}$, which were called the Carlitz's type q-Bernoulli numbers. By (1.11) and (1.12), we note that

$$\lim_{q \to 1} \beta_n^{(k,q)} = B_n^{(k,1)} = B_n^k. \tag{1.13}$$

In Section 2, we give the twisted Carlitz's type q-Bernoulli polynomials and numbers associated with p-adic q-inetgrals and discuss their properties. In Section 3, we define the multiple twisted Carlitz's type q-Bernoulli polynomials and numbers. We also obtain the distribution relation for them.

2. Twisted Carlitz's type q-Bernoulli polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. By using p-adic q-integral on $\mathbb{Z}_p$, we derive

$$I_q(f_1) = \frac{1}{q} I_q(f) + \left(\frac{q-1}{\log q} f'(0) + (q-1) f(0) \right), \tag{2.1}$$

(cf. [8]), where $f_1(x) = f(x+1)$. From (1.5), we can derive

$$q^n I_q(f_n) = I_q(f) + \frac{q(q-1)}{\log q} \left(\sum_{i=0}^{n-1} f'(i) q^i + \log q \sum_{i=0}^{n-1} f(i) q^i \right), \tag{2.2}$$

(cf. [8]), where $n \in \mathbb{N}$ and $f_n(x) = f(x+n)$.

Let $T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n} = C_{p^\infty}$ be the locally constant space, where $C_{p^n} = \{w \mid w^{p^n} = 1\}$ is the cyclic group of order p^n. For $w \in T_p$, we denote the locally constant function by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$, $x \to w^x$. If we take $f(x) = \phi_w(x) = w^x$, then we have

$$\int_{\mathbb{Z}_p} e^{tx} \phi_w(x) d\mu_q(x) = \left(\frac{\log q + t}{q w e^t - 1} \right) \frac{q(q-1)}{\log q} \equiv F_w^q(t). \tag{2.3}$$

Now we define the twisted q-Bernoulli polynomials as follows:

$$F_w^q(x,t) = \left(\frac{\log q + t}{qwe^t - 1}\right)\frac{q(q-1)}{\log q}e^{xt} = \sum_{n=0}^{\infty} B_{n,w}^q(x)\frac{t^n}{n!}. \tag{2.4}$$

We note that $B_{n,w}^q(0) = B_{n,w}^q$ are called the twisted q-Bernoulli numbers and by substituting $w = 1$, $\lim_{q\to1} B_{n,1}^q = B_n$ are the familiar Bernoulli numbers. By (2.3), we obtain the following Witt's type formula for the twisted q-Bernoulli polynomials and numbers.

Theorem 2.1. *For $n \in \mathbb{N}$ and $w \in T_p$, one has*

$$\int_{\mathbb{Z}_p} (t+x)^n w^t d\mu_q(t) = B_{n,w}^q(x). \tag{2.5}$$

From (2.5), we consider the twisted Carliz's type q-Bernoulli polynomials by using p-adic q-integrals. For $w \in T_p$, we define the twisted Carlitz's type q-Bernoulli polynomials as follows:

$$\beta_{n,w}^q(x) = \frac{1}{1-q}\int_{\mathbb{Z}_p} [t+x]_q^n w^t d\mu_q(t). \tag{2.6}$$

When $x = 0$, we write $\beta_{n,w}^q(0) = \beta_{n,w}^q$ which are called twisted Carlitz's type q-Bernoulli numbers. Note that if $w = 1$, then $\lim_{q\to1}\beta_{n,1}^q = B_n$. From (2.6), we can see that

$$\beta_{n,w}^q(x) = \frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i} q^{ix}(-1)^i \frac{1}{1-q^{i+1}w}. \tag{2.7}$$

From (2.7), we can derive the generating function for the twisted Carlitz's type q-Bernoulli polynomials as follows:

$$\begin{aligned}
G_w^q(x,t) &= \sum_{n=0}^{\infty}\beta_{n,w}^q(x)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i} q^{ix}(-1)^i\frac{1}{1-q^{i+1}w}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i} q^{ix}(-1)^i\sum_{l=0}^{\infty}q^{(i+1)l}w^l\right)\frac{t^n}{n!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{q^l w^l}{(1-q)^n}\sum_{i=0}^{n}\binom{n}{i} q^{(x+l)i}(-1)^i\right)\frac{t^n}{n!} \\
&= \sum_{l=0}^{\infty}q^l w^l\sum_{n=0}^{\infty}\frac{\left(1-q^{x+l}\right)^n}{(1-q)^n}\frac{t^n}{n!} \\
&= \sum_{l=0}^{\infty}q^l w^l e^{[x+l]_q t}.
\end{aligned} \tag{2.8}$$

Then it is easily to see that

$$G_w^q(x,t) = \int_{\mathbb{Z}_p} e^{[t+x]_q t} w^t d\mu_q(t). \tag{2.9}$$

By the kth differentiation on both sides of (2.8) at $t = 0$, we also have

$$\beta_{n,w}^{q}(x) = \frac{d^n}{dt^n} G_w^q(x,t)\bigg|_{t=0} = \sum_{l=0}^{\infty} q^l w^l [x+l]_q^n \tag{2.10}$$

for $n \in \mathbb{N} \cup \{0\}$. We note that

$$\beta_{n,w}^{q} = \beta_{n,w}^{q}(0) = \sum_{l=0}^{\infty} q^l w^l [l]_q^n. \tag{2.11}$$

In view of (2.10), we define twisted Carlitz's type q-zeta function as follows:

$$\zeta_w^q(s,x) = \sum_{l=0}^{\infty} \frac{q^l w^l}{[x+l]_q^s} \tag{2.12}$$

for all $s \in \mathbb{C}$ and $\mathrm{Re}(x) > 0$. We note that $\zeta_w^q(s,x)$ is analytic function in the whole complex s-plane. We also have the following theorem in which twisted Carlitz's type q-zeta functions interpolate twisted Carlitz's type q-Bernoulli numbers and polynomials.

Theorem 2.2. *For $k \in \mathbb{N} \cup \{0\}$ and $w \in T_p$, one has*

$$\begin{aligned} \zeta_w^q(-k,x) &= \beta_{k,w}^{q}(x), \\ \zeta_w^q(-k,0) &= \beta_{k,w}^{q}. \end{aligned} \tag{2.13}$$

From (2.11), we obtain the following distribution relation for the twisted q-Bernoulli polynomials.

Theorem 2.3. *For $r \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, and $w \in T_p$, one has*

$$\beta_{n,w}^{q}(x) = [r]_q^{n-1} \sum_{i=0}^{r-1} w^i q^i \beta_{n,w^r}^{q^r}\left(\frac{i+x}{r}\right). \tag{2.14}$$

Proof. If we put $i + rl = j$ and $i = 1 \cdots r$ and $l = 0, 1, \ldots$, then by (2.11), we have

$$\begin{aligned} \beta_{n,w}^{q}(x) &= \sum_{j=0}^{\infty} w^j q^j [x+j]_q^n \\ &= \sum_{l=0}^{\infty} \sum_{i=0}^{r-1} w^{i+rl} q^{i+rl} [x+i+rl]_q^n \\ &= \left(\frac{1-q^r}{1-q}\right)^n \sum_{i=0}^{r-1} w^i q^i \sum_{l=0}^{\infty} w^{rl} q^{rl} \left(\frac{1-q^{r((i+x)/r+l)}}{1-q^r}\right)^n \\ &= [r]_q^{n-1} \sum_{i=0}^{r-1} w^i q^i \beta_{n,w^r}^{q^r}\left(\frac{i+x}{r}\right). \end{aligned} \tag{2.15}$$

□

3. Multiple twisted Carlitz's type q-Bernoulli polynomials

In this section, we consider the multiple twisted Carlitz's type q-Bernoulli polynomials as follows:

$$\beta_{k,w}^{(h,q)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} [x_1 + \cdots + x_h + x]_q^n w^{x_1+\cdots+x_h} d\mu_q(x_1) \cdots d\mu_q(x_h)$$
$$= \lim_{\varrho\to\infty} \frac{1}{[p^\varrho]_q^h} \sum_{x_1\cdots x_h=0}^{p^\varrho-1} [x + x_1 + \cdots + x_h]_q^n w^{x_1+\cdots+x_h} q^{x_1+\cdots+x_h}, \tag{3.1}$$

where $h \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, and $w \in T_p$. We note that $\beta_{n,w}^{(h,q)}(0) = \beta_{n,w}^{(h,q)}$ are called the multiple twisted Carlitz's type q-Bernoulli numbers. We also obtain the generating function of the multiple twisted Carlitz's type q-Bernoulli polynomials as follows:

$$G_w^{(h,q)}(x,t) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} e^{[x_1+\cdots+x_h+x]_q t} w^{x_1+\cdots+x_h} d\mu_q(x_1) \cdots d\mu_q(x_h)$$
$$= \sum_{l=0}^{\infty} \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} [x_1 + \cdots + x_h + x]_q^l w^{x_1+\cdots+x_h} d\mu_q(x_1) \cdots d\mu_q(x_h) \frac{t^l}{l!} \tag{3.2}$$
$$= \sum_{l=0}^{\infty} \beta_{l,w}^{(h,q)}(x) \frac{t^l}{l!}.$$

Finally, we have the following distribution relation for the multiple twisted q-Bernoulli polynomials.

Theorem 3.1. *For each $w \in T_p$, $h, r \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, and $w \in T_p$,*

$$\beta_{n,w}^{(h,q)}(x) = [r]_q^{n-h} \sum_{j_1,\ldots,j_h=0}^{r-1} w^{j_1+\cdots+j_h} q^{j_1+\cdots+j_h} \beta_{n,w^r}^{(h,q^r)}\left(\frac{x + j_1 + \cdots + j_h}{r}\right). \tag{3.3}$$

Proof. If we put $j_k + rl_k = x_k$, $j_k = 0, 1, \ldots r-1$, and $k = 1 \cdots h$, then by (3.1), we have

$$\beta_{k,w}^{(h,q)}(x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{h\text{-times}} [x_1 + \cdots + x_h + x]_q^n w^{x_1+\cdots+x_h} d\mu_q(x_1) \cdots d\mu_q(x_h)$$

$$= \lim_{\varrho\to\infty} \frac{1}{[rp^\varrho]_q^h} \sum_{x_1\cdots x_h=0}^{rp^\varrho-1} [x + x_1 + \cdots + x_h]_q^n w^{x_1+\cdots+x_h} q^{x_1+\cdots+x_h}$$

$$= \lim_{\varrho\to\infty} [r]_q^{n-h} \frac{1}{[p^\varrho]_{q^r}^h} \sum_{j_1,\ldots,j_h=0}^{r-1} \sum_{l_1\cdots l_h=0}^{p^\varrho-1} [x + j_1 + rl_1 + \cdots + j_h + rl_h]_q^n \cdot w^{j_1+rl_1+\cdots+j_h+rl_h} q^{j_1+rl_1+\cdots+j_h+rl_h}$$

$$= [r]_q^{n-h} \sum_{j_1,\dots,j_h=0}^{r-1} w^{j_1+\cdots+j_h} q^{j_1+\cdots+j_h} \cdot \lim_{\varrho\to\infty} \frac{1}{[p^\varrho]_{q^r}^h} \sum_{l_1\cdots l_h=0}^{p^\varrho-1} \left[\frac{x+j_1+\cdots+j_h}{r} + l_1+\cdots+l_h\right]_{q^r}^n w^{r(l_1+\cdots+l_h)} q^{r(l_1+\cdots+l_h)}$$

$$= [r]_q^{n-h} \sum_{j_1,\dots,j_h=0}^{r-1} w^{j_1+\cdots+j_h} q^{j_1+\cdots+j_h} \beta_{n,w^r}^{(h,q^r)} \left(\frac{x+j_1+\cdots+j_h}{r}\right). \tag{3.4}$$

□

Question 1. Are there the analytic multiple twisted Carlitz's type q-zeta functions which interpolate multiple twisted Carlitz's type q-Bernoulli polynomials?

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