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Research Article

Slowly Oscillating Continuity

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A function f is continuous if and only if, for each point x_0 in the domain, $\lim_{n\to\infty} f(x_n) = f(x_0)$, whenever $\lim_{n\to\infty} x_n = x_0$. This is equivalent to the statement that $(f(x_n))$ is a convergent sequence whenever (x_n) is convergent. The concept of slowly oscillating continuity is defined in the sense that a function f is slowly oscillating continuous if it transforms slowly oscillating sequences to slowly oscillating sequences, that is, $(f(x_n))$ is slowly oscillating whenever (x_n) is slowly oscillating. A sequence (x_n) of points in $\mathbf{R}$ is slowly oscillating if $\lim_{\lambda\to 1^+} \overline{\lim}_n \max_{n+1\leq k\leq[\lambda n]} |x_k - x_n| = 0$, where $[\lambda n]$ denotes the integer part of λn. Using $\varepsilon > 0$'s and δ's, this is equivalent to the case when, for any given $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1+\delta)n$. A new type compactness is also defined and some new results related to compactness are obtained.

1. Introduction

The notion of statistical convergence was introduced by Fast [1] and has been investigated in [2]. In [3], Zygmund called it "almost convergence" and established a relation between it and strong summability.

Following the idea given in the 1946 American Mathematical Monthly problem [4], a number of authors including Posner [5], Iwiński [6], Srinivasan [7], Antoni [8], Antoni and Šalát [9], Spigel and Krupnik [10] have studied A continuity defined by a regular summability matrix A. Some authors (Öztürk [11], Savaş and Das [12], Borsík and Šalát [13]) have studied A continuity for methods of almost convergence or for related methods.

Recently, Connor and Grosse-Erdman [14] have given sequential definitions of continuity for real functions calling it G continuity instead of A continuity and their results cover the earlier works related to A continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, denoted by c_G, into $\mathbf{R}$

where s is the set of all sequences of points in $\mathbf{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G convergent with $G(\mathbf{x}) = \ell$ then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

The purpose of this paper is to introduce a concept of slowly oscillating continuity which cannot be given by means of any G as in [14] and prove that slowly oscillating continuity implies ordinary continuity and so statistical continuity and is implied by uniform continuity; and introduce several other types of continuities; and introduce some new types of compactness.

2. Slowly oscillating continuity

It is known that a sequence (x_n) of points in $\mathbf{R}$ is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \le k \le [\lambda n]} \left| x_k - x_n \right| = 0 \tag{2.1}$$

in which $[\lambda n]$ denotes the integer part of λn. Using $\varepsilon > 0$'s and δ's, this is equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \ge N(\varepsilon)$ and $n \le m \le (1+\delta)n$. It is well known that a function f is continuous if and only if, for each point x_0 in the domain, $\lim_{n\to\infty} f(x_n) = f(x_0)$ whenever $\lim_{n\to\infty} x_n = x_0$. This is equivalent to the statement that $(f(x_n))$ is a convergent sequence whenever (x_n) is convergent. As a result of completeness, when the domain of the function is all of $\mathbf{R}$, or a complete subset of $\mathbf{R}$, this is also equivalent to the fact that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is Cauchy. This suggests that we might introduce a concept of slowly oscillating continuity as defined in the sense that a function f is slowly oscillating continuous if it transforms slowly oscillating sequences to slowly oscillating sequences, that is, $(f(x_n))$ is slowly oscillating whenever (x_n) is slowly oscillating.

We note that the sum of two slowly oscillating continuous functions is slowly oscillating continuous and that the composite of two slowly oscillating continuous functions is slowly oscillating continuous; but the product of two slowly oscillating continuous functions needs not be slowly oscillating continuous as can be seen by considering product of the slowly oscillating function $f(x) = x$ with itself.

In connection with slowly oscillating sequences and convergent sequences, the problem arises to investigate the following types of continuity of functions on $\mathbf{R}$ where c denotes the set of all convergent sequences and w denotes the set of all slowly oscillating sequences of points in $\mathbf{R}$.

(w) Slowly oscillating continuity of f on $\mathbf{R}$.

(wc) $(x_n) \in w \Rightarrow (f(x_n)) \in c$.

(c) $(x_n) \in c \Rightarrow (f(x_n)) \in c$.

(cp1) $\lim_{n\to\infty} f(x_n) = f(x_0)$ whenever $\lim_{n\to\infty} x_n = x_0$.

(d) $(x_n) \in c \Rightarrow (f(x_n)) \in w$.

(u) Uniform continuity of f on $\mathbf{R}$.

It is clear that (c) is equivalent to (cp1) for each $x_0 \in \mathbf{R}$. It is easy to see that (wc) implies (d); (wc) implies (w); and (w) implies (d). Now we give a proof of the implication (w) implies (c) in the following.

Theorem 2.1. *If f is slowly oscillating continuous on a subset E of $\mathbf{R}$, then it is continuous on E in the ordinary sense.*

Proof. Let (x_n) be any convergent sequence with $\lim_{k\to\infty} x_k = x_0$. Then the sequence $(x_1, x_0, x_2, x_0, \ldots, x_0, x_n, x_0, \ldots)$ also converges to x_0. Then the sequence $(x_1, x_0, x_2, x_0, \ldots, x_0, x_n, x_0, \ldots)$ is slowly oscillating hence, by the hypothesis, the sequence $(f(x_1), f(x_0), f(x_2), f(x_0), \ldots, f(x_0), f(x_n), f(x_0), \ldots)$ is slowly oscillating. It follows from this that $(f(x_1), f(x_0), f(x_2), f(x_0), \ldots, f(x_0), f(x_n), f(x_0), \ldots)$ converges to $f(x_0)$, so the sequence $(f(x_1), f(x_2), \ldots, f(x_n), \ldots)$ also converges to $f(x_0)$. This completes the proof. □

The converse is not always true for the function $f(x) = x^2$ is an example.

Corollary 2.2. *If f is slowly oscillating continuous, then it is statistically continuous.*

Corollary 2.3. *If (x_n) is statistically convergent and slowly oscillating and f is a slowly oscillating continuous function, then $(f(x_n))$ is a convergent sequence.*

Theorem 2.4. *If a function f on a subset E of $\mathbf{R}$ is uniformly continuous, then it is slowly oscillating continuous on E.*

Proof. Let f be a uniformly continuous function and let $\mathbf{x} = (x_n)$ be any slowly oscillating sequence of points in E. To prove that $f((x_n))$ is slowly oscillating, take any $\varepsilon > 0$. Uniform continuity of f implies that there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$, whenever $|x - y| < \delta$. Since (x_n) is slowly oscillating, for this $\delta > 0$, there exist a $\delta_1 > 0$ and $N = N(\delta) = N_1(\varepsilon)$ such that $|x_m - x_n| < \delta$ if $n \geq N(\delta)$ and $n \leq m \leq (1 + \delta_1)n$. Hence $|f(x_m) - f(x_n)| < \varepsilon$ if $n \geq N(\delta)$ and $n \leq m \leq (1 + \delta_1)n$. It follows from this that $(f(x_n))$ is slowly oscillating. □

It is well known that any continuous function on a compact set is also uniformly continuous. It is also true for a regular subsequential method G that any continuous function on a G sequentially compact set is also uniformly continuous (see [15] for the definition of G compactness).

Theorem 2.5. *If (f_n) is a sequence of slowly oscillating continuous functions defined on a subset E of $\mathbf{R}$ and (f_n) is uniformly convergent to a function f, then f is slowly oscillating continuous on E.*

Proof. Let (x_n) be a slowly oscillating sequence and $\varepsilon > 0$. Then there exists a positive integer N such that $|f_n(x) - f(x)| < \varepsilon/3$ for all $x \in E$, whenever $n \geq N$. As f_N is slowly oscillating continuous, there exist a $\delta > 0$ and a positive integer N_1 such that

$$\left|f_N(x_m) - f_N(x_n)\right| < \frac{\varepsilon}{3} \tag{2.2}$$

for $n \geq N_1(\varepsilon)$ and $n \leq m \leq (1 + \delta)n$. Now for $n \geq N_1(\varepsilon)$ and $n \leq m \leq (1 + \delta)n$, we have

$$\left|f(x_m) - f(x_n)\right| \leq \left|f(x_m) - f_N(x_m)\right| + \left|f_N(x_m) - f_N(x_n)\right| + \left|f_N(x_n) - f(x_n)\right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \tag{2.3}$$

This completes the proof of the theorem. □

Now we can give the definition of slowly oscillating compactness of a subset of $\mathbf{R}$.

Definition 2.6. A subset F of $\mathbf{R}$ is called slowly oscillating compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in F there is a slowly oscillating subsequence $\mathbf{z} = (x_{n_k})$ of $\mathbf{x}$.

Any compact subset of $\mathbf{R}$ is slowly oscillating compact. Union of two slowly oscillating compact subsets of $\mathbf{R}$ is slowly oscillating compact. We note that any subset of a slowly oscillating compact set is also slowly oscillating compact, and so intersection of any slowly oscillating compact subsets of $\mathbf{R}$ is slowly oscillating compact.

Theorem 2.7. *For any regular subsequential method G, if a subset F of $\mathbf{R}$ is G sequentially compact, then it is slowly oscillating compact.*

Proof. The proof can be obtained by noticing the regularity and subsequentiality of G (see [15] for the detail of G compactness). □

The converse is not necessarily true, for example, $\{\sum_{k=1}^{n}(1/k) : n \in \mathbf{N}\}$ is slowly oscillating compact, but it is not G sequentially compact when $G(\mathbf{x}) := \lim \mathbf{x}$ on c.

Theorem 2.8. *Slowly oscillating continuous image of any slowly oscillating compact subset of $\mathbf{R}$ is slowly oscillating compact.*

Proof. Let f be a slowly oscillating continuous function on $\mathbf{R}$ and let F be a slowly oscillating compact subset of $\mathbf{R}$. Take any sequence $\mathbf{y} = (y_k)$ of points in $f(F)$. Write $y_k = f(x_k)$ for each $k \in \mathbf{N}$. As F is assumed to be slowly oscillating compact, there exists a subsequence $z = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x} = (x_k)$ for which $\mathbf{z} = (z_k)$ is slowly oscillating. Since f is slowly oscillating continuous, the image of the sequence $\mathbf{z} = (z_k)$, $f(\mathbf{z}) = (f(z_k))$ is slowly oscillating. Since $f(\mathbf{z}) = (f(z_k))$ is a subsequence of the sequence $\mathbf{y}$, the proof is completed. □

We add one more compactness defining as saying that a subset F of $\mathbf{R}$ is called Cauchy compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in F, there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ which is Cauchy; we see that any Cauchy compact subset of $\mathbf{R}$ is also slowly oscillating compact, and a slowly oscillating continuous image of any Cauchy compact subset of $\mathbf{R}$ is Cauchy compact.

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