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Research Article

q-Genocchi Numbers and Polynomials Associated with Fermionic p-Adic Invariant Integrals on $\mathbb{Z}_p$

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The main purpose of this paper is to present a systemic study of some families of multiple Genocchi numbers and polynomials. In particular, by using the fermionic p-adic invariant integral on $\mathbb{Z}_p$, we construct p-adic Genocchi numbers and polynomials of higher order. Finally, we derive the following interesting formula: $G^{(k)}_{n+k,q}(x) = 2^k k! \binom{n+k}{k} \sum_{l=0}^{\infty} \sum_{d_0+d_1+\cdots+d_k=k-1, d_i\in\mathbb{N}} (-1)^l (l+x)^n$, where $G^{(k)}_{n+k,q}(x)$ are the q-Genocchi polynomials of order k.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{\nu_p(p)} = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|1-q|_p < 1$, see [1–6].

In $\mathbb{C}$, the ordinary Euler polynomials are defined as

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

In the case $x = 0$, $E_n(0) = E_n$ are called Euler numbers, see [1–13]. Let $\delta_{0,n}$ be the Kronecker symbol. From (1.1) we derive the following relation:

$$E_0 = 1, \ (E+1)^n + E_n = 2\delta_{0,n}, \quad n \in \mathbb{N}, \tag{1.2}$$

(cf. [7–13]). Here, we use the technique method notation by replacing E^n by E_n $(n \geq 0)$, symbolically. The first few are $1, -1/2, 0, 1/4, \ldots$, and $E_{2k} = 0$ for $k = 1, 2, \ldots$. A sequence consisting of the Genocchi numbers G_n satisfies the following relations:

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (|t| < \pi), \tag{1.3}$$

see [11, 12]. It satisfies $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = G_{2k+1} = 0$, $k = 1, 2, 3, \ldots$, and even coefficients are given by

$$G_{2n} = 2\left(1 - 2^{2n}\right) B_{2n} = 2n E_{2n-1}(0), \tag{1.4}$$

where B_n is Bernoulli numbers. The first few Genocchi numbers for even integers are $-1, 1, -3, 17, -155, 2073, \ldots$. The first few prime Genocchi numbers are -3 and 17, which occur at $n = 6$ and 8. There are no others with $n < 10^5$. We now define the Genocchi polynomials as follows:

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.5}$$

Thus, we note that

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \tag{1.6}$$

In this paper, we use the following notations: $[x]_q = (1 - q^x)/(1 - q)$ and $[x]_{-q} = (1 + (-q)^x)/(1 + q)$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-deformed fermionic integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N - 1} f(x)(-q)^x, \tag{1.7}$$

see [1–4]. The fermionic p-adic invariant integral on $\mathbb{Z}_p$ can be obtained as $q \to 1$. That is,

$$I_{-1}(f) = \lim_{q \to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x). \tag{1.8}$$

From (1.8), we easily derive the following integral equation related to fermionic invariant p-adic integral on $\mathbb{Z}_p$:

$$I_{-1}\left(f_1\right) + I_{-1}(f) = 2f(0), \tag{1.9}$$

where $f_1(x) = f(x + 1)$, see [5].

The purpose of this paper is to present a systemic study of some families of multiple Genocchi numbers and polynomials by using the fermionic multivariate p-adic invariant integral on $\mathbb{Z}_p$. In addition, we will investigate some interesting identities related to Genocchi numbers and polynomials.

2. Genocchi numbers associated with fermionic p-adic invariant integral on $\mathbb{Z}_p$

From (1.9) we can derive

$$t\int_{\mathbb{Z}_p} e^{xt}d\mu_{-1}(x) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n\frac{t^n}{n!},$$
$$t\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \tag{2.1}$$

where $G_n(x)$ are Genocchi polynomials. It is easy to check that

$$t\int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_{-1}(x) = \frac{2t}{e^t+e^{-t}} = t \text{ sech } t = \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}2^l G_l\right)\frac{t^n}{n!}. \tag{2.2}$$

By comparing the coefficient on both sides in (2.1), we easily see that

$$\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(x) = \frac{G_{n+1}(x)}{n+1}. \tag{2.3}$$

Therefore, we obtain the following proposition.

Proposition 2.1. *For $k \in \mathbb{Z}_+$,*

(i) $\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(x) = G_{n+1}(x)/(n+1)$ *(Witt's formula for Genocchi polynomials);*

(ii) $\int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_{-1}(x) = (1/(n+1))((1/2)\sum_{l=0}^{n+1}\binom{n+1}{l}2^l G_l)$, *where* $\binom{n}{l} = (n(n-1)\cdots(n-l+1))/l!$.

Let $\vartheta_{\mathbb{C}_p} = \{x \in \mathbb{C}_p \mid |x|_p \le 1\}$ be the integer ring of $\mathbb{C}_p$. We note that $i = (-1)^{1/2} \in \vartheta_{\mathbb{C}_p}$. By using Taylor expansion, we see that

$$e^{ix} = \sum_{n=0}^{\infty} i^n\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}. \tag{2.4}$$

In the p-adic number field, $\sin x$ and $\cos x$ are defined as

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{2.5}$$

From (2.4) and (2.5), we derive

$$e^{ix} = \cos x + i\sin x. \tag{2.6}$$

This is equivalent to

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}. \tag{2.7}$$

By (2.7), we easily see that

$$\sec t = \frac{2}{e^{it} + e^{-it}} = \int_{\mathbb{Z}_p} e^{(2x+1)it} d\mu_{-1}(x) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (2x+1)^n d\mu_{-1}(x) \frac{i^n t^n}{n!}. \tag{2.8}$$

It is not difficult to show that $\int_{\mathbb{Z}_p} (2x+1)^{2n+1} d\mu_{-1}(x) = 0$ for $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. From (2.8), we note that

$$\sec t = \sum_{n=0}^{\infty} i^n \int_{\mathbb{Z}_p} (2x+1)^n d\mu_{-1}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \int_{\mathbb{Z}_p} (2x+1)^{2n} d\mu_{-1}(x) \frac{t^{2n}}{(2n)!}. \tag{2.9}$$

Thus, we have

$$t \sec t = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\sum_{l=0}^{2n+1} \binom{2n+1}{l} 2^l G_l \right) \frac{t^{2n+1}}{(2n+1)!}. \tag{2.10}$$

Now we consider the fermionic multivariate p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$t^k \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} e^{(x_1+x_2+\cdots+x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = 2^k \underbrace{\frac{t^k}{(e^t+1) \cdots (e^t+1)}}_{k\text{-times}} = \sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!}, \tag{2.11}$$

where $G_n^{(k)}$ are the nth Genocchi number of order k. By comparing the coefficient on both sides in (2.11), we see that $G_0^{(k)} = G_1^{(k)} = \cdots = G_n^{(k)} = 0$, and

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) (n+k)_k = G_{n+k}^{(k)}, \tag{2.12}$$

where $(n+k)_k$ is the Jordan factor which is defined by $(n+k)_k = (n+k) \cdots (n+1)$. Thus, we note that

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{G_{n+k}^{(k)}}{\binom{n+k}{n} n!}, \tag{2.13}$$

for $k \in \mathbb{N}$, $n \in \mathbb{Z}_+$.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$,*

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} (x_1 + x_2 + \cdots + x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \frac{G_{n+k}^{(k)}}{\binom{n+k}{n} n!}. \tag{2.14}$$

The multinomial coefficient is well known as

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k>0}} \binom{n}{l_1,\ldots,l_k} x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k}. \tag{2.15}$$

Therefore, we obtain the following corollary.

Corollary 2.3. *For $n \in \mathbb{Z}_+$, $k \in \mathbb{N}$,*

$$\sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k>0}} \binom{n}{l_1,\ldots,l_k} \left(\frac{G_{l_1+1}}{l_1+1}\right)\left(\frac{G_{l_2+1}}{l_2+1}\right)\cdots\left(\frac{G_{l_k+1}}{l_k+1}\right) = \frac{G_{n+k}^{(k)}}{\binom{n+k}{n} n!}. \tag{2.16}$$

For $q \in \mathbb{C}_p$ with $|1-q| < 1$, it is not difficult to show that

$$t\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = \frac{2t}{qe^t + 1}. \tag{2.17}$$

Now, we define the q-extension of the Genocchi numbers as follows:

$$\frac{2t}{qe^t + 1} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}. \tag{2.18}$$

By (2.17) and (2.18), we easily see that

$$\frac{G_{n+1,q}}{n+1} = \int_{\mathbb{Z}_p} q^x x^n d\mu_{-1}(x). \tag{2.19}$$

With the same motivation to construct the Genocchi polynomials of higher order, we can consider the q-extension of higher-order Genocchi numbers as follows:

$$\begin{aligned} t^k \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} & q^{x_2+2x_3+\cdots+(k-1)x_k} e^{(x+x_1+x_2+\cdots+x_k)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\ &= \frac{t^k 2^k}{\underbrace{(e^t+1)(qe^t+1)\cdots(q^{k-1}e^t+1)}_{k\text{-times}}} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}^{(k)} \frac{t^n}{n!}, \end{aligned} \tag{2.20}$$

where $G_{n,q}^{(k)}$ are the q-Genocchi polynomials of order k. The basic q-natural numbers are defined as

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}. \tag{2.21}$$

The q-factorial of n is defined as

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q = (1 + q + \cdots + q^{n-1}) \cdots (1+q) \cdot 1. \tag{2.22}$$

The q-binomial coefficient is also defined as

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}. \tag{2.23}$$

Note that $\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = (n(n-1)\cdots(n-k+1))/k!$. The q-binomial coefficient satisfies the following recurrsion formula:

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q = q^{n-k}\binom{n}{k-1}_q + \binom{n}{k}_q. \tag{2.24}$$

From this recurrsion formula, we can derive

$$\binom{n}{k}_q = \sum_{\substack{d_0+d_1+\cdots+d_k=k-1\\ d_i\in\mathbb{N}}} q^{0\cdot d_0+1\cdot d_1+\cdots+k\cdot d_k}. \tag{2.25}$$

The q-binomial expansion is given by

$$\begin{gathered}\prod_{i=1}^{n}\left(a+bq^{i-1}\right) = \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}a^{n-k}b^k,\\ \prod_{i=1}^{n}\left(1-bq^{i-1}\right)^{-1} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}_q b^k.\end{gathered} \tag{2.26}$$

By (2.20) and (2.26), we see that

$$\begin{aligned}&t^k\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{-times}} q^{x_2+2x_3+\cdots+(k-1)x_k}e^{(x+x_1+x_2+\cdots+x_k)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\ &\quad= t^k2^k\prod_{i=1}^{k}\left(1-\left(-q^{i-1}\right)e^t\right)^{-1}e^{xt}\\ &\quad= t^k2^k\sum_{l=0}^{\infty}\binom{k+l-1}{l}_q(-1)^l e^{(l+x)t}\\ &\quad= t^k\sum_{n=0}^{\infty}\left(2^k\sum_{l=0}^{\infty}\binom{k+l-1}{l}_q(-1)^l(l+x)^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.27}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $n\in\mathbb{Z}_+$, $k\in\mathbb{N}$, we have*

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{-times}} q^{x_2+2x_3+\cdots+(k-1)x_k}\left(x_1+x_2+\cdots+x_k\right)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\ &\quad= 2^k\sum_{l=0}^{\infty}\binom{k+l-1}{l}_q(-1)^l(l+x)^n.\end{aligned} \tag{2.28}$$

By (2.20), it is not difficult to show that

$$\begin{aligned} G^{(k)}_{n+k,q}(x) &= k!\binom{n+k}{k}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{-times}} q^{x_2+2x_3+\cdots+(k-1)x_k}\times\left(x+x_1+x_2+\cdots+x_k\right)^k \\ &\quad \times d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)(n+k)_k, \\ G^{(k)}_{0,q} &= G^{(k)}_{1,q}=\cdots=G^{(k)}_{n,q}=0, \end{aligned} \tag{2.29}$$

where $n=0,1,2,\ldots$. Therefore, we obtain the following corollary.

Corollary 2.5. *For* $n\in\mathbb{Z}_+,\ k\in\mathbb{N}$,

$$\frac{G^{(k)}_{n+k,q}(x)}{k!\binom{n+k}{k}} = 2^k\sum_{l=0}^{\infty}\binom{k+l-1}{l}_q(-1)^l(l+x)^n. \tag{2.30}$$

Corollary 2.6. *For* $n\in\mathbb{Z}_+,\ k\in\mathbb{N}$,

$$G^{(k)}_{n+k,q}(x) = 2^k k!\binom{n+k}{k}\sum_{l=0}^{\infty}\sum_{\substack{d_0+d_1+\cdots+d_k=k-1\\ d_i\in\mathbb{N}}} q^{0\cdot d_0+1\cdot d_1+\cdots+k\cdot d_k}(-1)^l(l+x)^n. \tag{2.31}$$

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References

[1] T. Kim, "q-Euler numbers and polynomials associated with p-adic q-integrals," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 15–27, 2007.
[2] T. Kim, "q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.
[3] T. Kim, "q-Bernoulli numbers associated with q-stirling numbers," *Advances in Difference Equations*, vol. 2008, Article ID 743295, 10 pages, 2008.
[4] T. Kim, "The modified q-Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 16, pp. 161–170, 2008.
[5] T. Kim, "Euler numbers and polynomials associated with zeta functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 581582, 13 pages, 2008.
[6] T. Ernst, "Examples of a q-umbral calculus," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 1, pp. 1–22, 2008.
[7] H. Ozden, I. N. Cangul, and Y. Simsek, "Multivariate interpolation functions of higher order q-Euler numbers and their applications," *Abstract and Applied Analysis*, vol. 2008, Article ID 390857, 16 pages, 2008.
[8] H. Ozden, Y. Simsek, S.-H. Rim, and I. N. Cangul, "A note on p-adic q-Euler measure," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 2, pp. 233–239, 2007.
[9] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on sum of products of (h,q)-twisted Euler polynomials and numbers," *Journal of Inequalities and Applications*, vol. 2008, Article ID 816129, 8 pages, 2008.

[10] Y. Simsek, V. Kurt, and D. Kim, "New approach to the complete sum of products of the twisted (h, q)-Bernoulli numbers and polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 1, pp. 44–56, 2007.

[11] M. Cenkci, M. Can, and V. Kurt, "q-extensions of Genocchi numbers," *Journal of the Korean Mathematical Society*, vol. 43, no. 1, pp. 183–198, 2006.

[12] M. Cenkci, Y. Simsek, and V. Kurt, "Further remarks on multiple p-adic q-L-function of two variables," *Advanced Studies in Contemporary Mathematics*, vol. 14, no. 1, pp. 49–68, 2007.

[13] T. Machide, "Sums of products of Kronecker's double series," *Journal of Number Theory*, vol. 128, no. 4, pp. 820–834, 2008.