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Research Article

Modulus of Convexity, the Coeffcient $R(1, X)$, and Normal Structure in Banach Spaces

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Let $\delta_X(\epsilon)$ and $R(1, X)$ be the modulus of convexity and the Domínguez-Benavides coefficient, respectively. According to these two geometric parameters, we obtain a sufficient condition for normal structure, that is, a Banach space X has normal structure if $2\delta_X(1+\epsilon) > \max\{(R(1,x)-1)\epsilon, 1-(1-\epsilon/R(1,X)-1)\}$ for some $\epsilon \in [0,1]$ which generalizes the known result by Gao and Prus.

1. Introduction

Let X be a Banach space. Throughout the paper, denote by S_X, B_X the unit sphere and unit ball of X, respectively. Recall that a Banach space X is said to have normal structure (resp., weak, normal, structure) if for every closed bounded (resp., weakly compact) convex subset C of X with diam $C > 0$, there exists $x \in C$ such that $\sup\{\|x-y\| : y \in C\} < \operatorname{diam} C$, where diam $C = \sup\{\|x-y\| : x, y \in C\}$. For a reflexive Banach space, the normal structure and weak normal structure are the same. Recently a good deal of investigations have focused on finding the sufficient conditions with various geometrical constants for a Banach space to have normal structure (see, e.g., [1–5]). The geometric condition sufficient for normal structure in terms of the modulus of convexity is given by Goebel [6], who proved that X has normal structure provided that $\delta_X(1) > 0$. Here the function $\delta_X(\epsilon) : [0,2] \to [0,1]$, defined by Clarkson [7] as

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X,\ \|x-y\| \geq \epsilon\right\}, \tag{1.1}$$

is called the modulus of convexity of X. Later Gao and Prus generalized the above results as the following (see [2, 8]).

Theorem 1.1. *A Banach space X has normal structure provided that $\delta_X(1+\epsilon) > \epsilon/2$ for some $\epsilon \in [0,1]$.*

In this paper, we obtain a class of Banach spaces with normal structure, which involves the coefficient $R(1,X)$. This coefficient is defined by Domínguez Benavides [9] as

$$R(1,X) = \sup\left\{\liminf_{n\to\infty}\|x + x_n\|\right\}, \tag{1.2}$$

where the supremum is taken over all $x \in X$ with $\|x\| \le 1$ and all weakly null sequence (x_n) in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty}\left(\limsup_{m\to\infty}\|x_n - x_m\|\right) \le 1. \tag{1.3}$$

Obviously, $1 \le R(1,X) \le 2$.

2. Main results

Let us begin this section with a sufficient condition for a Banach space X having weak normal structure and the idea in the following proof is due to [5, Lemma 5].

Lemma 2.1. *Let X be a Banach space for which B_{X^*} is w^*-sequentially compact. If X does not have weak normal structure, then for any $\eta > 0$, there exist $x_1, x_2 \in S_X$ and $f_1, f_2 \in S_{X^*}$, such that*

(1) $|\|x_1 - x_2\| - 1| < \eta$;

(2) $|f_i(x_j)| < \eta$ *for* $i \ne j$ *and* $f_i(x_i) = 1,\ i,j = 1,2$;

(3) $\|x_1 + x_2\| \le R(1,X)(1+\eta)$.

Proof. Assume that X does not have weak normal structure. It is well known that (see, e.g., [10]) there exists a sequence $\{x_n\}$ in X satisfying

(1) x_n is weakly convergent to 0;

(2) diam $(\{x_n\}_{n=1}^{\infty}) = 1 = \lim_n\|x_n - x\|$ for all $x \in \operatorname{clco}\{x_n\}_{n=1}^{\infty}$.

Since B_{X^*} is w^*-sequentially compact, we can find $\{f_n\}$ in S_{X^*} satisfying

(3) $f_n(x_n) = \|x_n\|$ for all n;

(4) $f_n \xrightarrow{w^*} f$ for some $f \in B_{X^*}$.

Let $\eta \in (0,1)$ sufficiently small and $\epsilon = \eta/3$. Then, by the properties of the sequence (x_n), we can choose $n_1 \in \mathbb{N}$ such that

$$\left|f\left(x_{n_1}\right)\right| < \frac{\epsilon}{2}, \qquad 1 - \epsilon \le \left\|x_{n_1}\right\| \le 1. \tag{2.1}$$

Note that the sequence $\{x_n\}$ is weakly null and verifies $D[\{x_n\}] = 1$. It follows from the definition of $R(1,X)$ that

$$\liminf_n\left\|x_n + x_{n_1}\right\| \le R(1,X). \tag{2.2}$$

The rest of the proof is similar to that of [5, Lemma 5]. □

Theorem 2.2. *A Banach space X has normal structure provided that $\delta_X(1+\epsilon) > f(\epsilon)$ for some $\epsilon \in [0,1]$, where the function $f(\epsilon)$ is defined as*

$$f(\epsilon) := \begin{cases} (R(1,X)-1)\dfrac{\epsilon}{2}, & 0 \le \epsilon \le \dfrac{1}{R(1,X)}, \\ \dfrac{1}{2}\left(1 - \dfrac{1-\epsilon}{R(1,X)-1}\right), & \dfrac{1}{R(1,X)} < \epsilon \le 1. \end{cases} \tag{2.3}$$

Proof. Observe that X is uniformly nonsquare [11] and then reflexive. Therefore normal structure and weak normal structure coincide.

Assume first that X fails to have weak normal structure. Fix $\eta > 0$ sufficiently small and $\epsilon \in [0,1]$. It follows that there exist $x_1, x_2 \in S_X$ and $f_1, f_2 \in S_{X^*}$, satisfying the condition in Lemma 2.1. Next, denote by $R := R(1,X)$ and consider two cases for $\epsilon \in [0,1]$.

Case 1 ($\epsilon \in [0, 1/R]$). Now let us put

$$x = \frac{x_1 - x_2}{1+\eta}, \qquad y = \frac{(1-(R-1)\epsilon)x_1 + \epsilon x_2}{1+\eta}, \tag{2.4}$$

and so $x \in B_X$,

$$\|y\| = \left\| \frac{\epsilon}{1+\eta}(x_1 + x_2) + \frac{1-R\epsilon}{1+\eta}x_1 \right\| \le R\epsilon + (1 - R\epsilon) = 1, \tag{2.5}$$

and also that

$$\begin{aligned} \|x - y\| &= \left\| \frac{(R-1)\epsilon}{1+\eta}x_1 - \frac{1+\epsilon}{1+\eta}x_2 \right\| \ge \frac{1+\epsilon}{1+\eta}f_2(x_2) - \frac{(R-1)\epsilon}{1+\eta}f_2(x_1) \ge \frac{1+\epsilon-\eta}{1+\eta}, \\ \|x + y\| &= \left\| \frac{(2-(R-1))\epsilon}{1+\eta}x_1 - \frac{1-\epsilon}{1+\eta}x_2 \right\| \ge \frac{(2-(R-1))\epsilon}{1+\eta}f_1(x_1) - \frac{1-\epsilon}{1+\eta}f_1(x_2) \\ &\ge \left(1 - \frac{2\eta}{1+\eta}\right)(2 - (R-1)\epsilon). \end{aligned} \tag{2.6}$$

By the definition of modulus of convexity,

$$\left(1 - \frac{2\eta}{1+\eta}\right)(2-(R-1)\epsilon) \le \|x+y\| \le 2(1 - \delta_X(\|x-y\|)) \le 2\left(1 - \delta_X\left(\frac{1+\epsilon-\eta}{1+\eta}\right)\right), \tag{2.7}$$

or equivalently,

$$(2-(R-1)\epsilon) \le 2\left(1 - \delta_X\left(\frac{1+\epsilon-\eta}{1+\eta}\right)\right)\left(1 + \frac{2\eta}{1-\eta}\right). \tag{2.8}$$

Letting $\eta \to 0$, we have

$$2\delta_X(1+\epsilon) \le (R-1)\epsilon, \tag{2.9}$$

which contradicts our hypothesis.

Case 2 ($\epsilon \in (1/R, 1]$). In this case $R > 1$, otherwise $\epsilon > 1$. Let

$$x' = \frac{x_2 - x_1}{1+\eta}, \qquad y' = \frac{(1-(R-1)\epsilon')x_1 + \epsilon' x_2}{1+\eta}, \tag{2.10}$$

where $\epsilon' = 1 - (R-1)\epsilon \in [0, 1/R)$. It follows from Case 1 that $x, y \in B_X$,

$$\|x - y\| \geq \left(1 - \frac{2\eta}{1+\eta}\right)\left(2 - (R-1)\epsilon'\right), \qquad \|x + y\| \geq \frac{1+\epsilon'-\eta}{1+\eta}. \tag{2.11}$$

This implies that

$$\delta_X(2 - (R-1)\epsilon') \leq \frac{1}{2}(1-\epsilon'), \tag{2.12}$$

which is equivalent to

$$\delta_X(1+\epsilon) \leq \frac{1}{2}\left(1 - \frac{1-\epsilon}{R-1}\right). \tag{2.13}$$

This is a contradiction. □

Remark 2.3. (1) It is readily seen that $f(\epsilon) \leq \epsilon/2$ for any $\epsilon \in [0, 1]$ and Theorem 2.2 is therefore a generalization of Theorem 1.1. Moreover this generalization is strict whenever X is the space with $R(1, X) < 2$.

(2) Consider the space $X = \mathbb{R}^2$ with the norm $\|(x, y)\| := \max(|x|, |y|, |x - y|)$. It is known that $\delta_X(\epsilon) = \max\{0, (\epsilon - 1)/2\}$ [8] and $R(1, X) = 1$, then X has normal structure from Theorem 2.2, but lies out of the scope of Theorem 1.1.

Corollary 2.4. *Let X be a Banach space with $R(1, X) = 1$ and $\delta_X(1+\epsilon) > 0$ for some $\epsilon \in [0, 1]$, then X has normal structure.*

Corollary 2.5. *If X is a Banach space with*

$$\delta_X\left(1 + \frac{1}{R(1,X)}\right) > \frac{1}{2}\left(1 - \frac{1}{R(1,X)}\right), \tag{2.14}$$

then X has normal structure.

Remark 2.6. Corollary 2.5 is equivalent to [4, Corollary 24].

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