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Research Article

A Tauberian Theorem with a Generalized One-Sided Condition

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We prove a Tauberian theorem to recover moderate oscillation of a real sequence $u = (u_n)$ out of Abel limitability of the sequence $(V_n^{(1)}(\Delta u))$ and some additional condition on the general control modulo of oscillatory behavior of integer order of $u = (u_n)$.

1. Introduction

Let $u = (u_n)$ be a sequence of real numbers. Throughout this paper the symbols $u_n = o(1)$ and $u_n = O(1)$ mean, respectively, that $u_n \to 0$ as $n \to \infty$ and that (u_n) is bounded for large enough n. Denote by $\omega_n^{(0)}(u) = n\Delta u_n$ the classical control modulo of the oscillatory behavior of (u_n). For each integer $m \geq 1$ and for all nonnegative integer n, define by

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n(\omega^{(m-1)}(u)) \tag{1.1}$$

the general control modulo of the oscillatory behavior of order m. For a sequence $u = (u_n)$,

$$u_n - \sigma_n(u) = V_n^{(0)}(\Delta u), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $\sigma_n(u) = (1/(n+1))\sum_{k=0}^{n} u_k$, $V_n^{(0)}(\Delta u) = (1/(n+1))\sum_{k=0}^{n} k\Delta u_k$, and

$$\Delta u_n = \begin{cases} u_n - u_{n-1}, & n \geq 1, \\ u_0, & n = 0. \end{cases} \tag{1.3}$$

For each integer $m \geq 1$ and for all nonnegative integer n, define

$$V_n^{(m)}(\Delta u) = \sigma_n\big(V^{(m-1)}(\Delta u)\big). \tag{1.4}$$

A sequence $u = (u_n)$ is said to be left one-sidedly bounded if $u_n \geq -C$ for all nonnegative integers n and for some $C \geq 0$. A sequence $u = (u_n)$ is said to be left one-sidedly bounded with respect to sequence (C_n) if $u_n \geq -C_n$ for all nonnegative integers n. A sequence (u_n) is said to be Abel limitable if the limit

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} u_n x^n = A(u) \tag{1.5}$$

exists and is finite. A classical Tauberian theorem of Hardy and Littlewood [1] says that if $(\omega_n^{(0)}(u))$ is left one-sidedly bounded and (1.5) exists, then $\lim_n u_n = A(u)$. Dik [2] improved Hardy and Littlewood's theorem [1] by proving that if $(\omega_n^{(1)}(u))$ is left one-sidedly bounded and (1.5) exists, then $\lim_n u_n = A(u)$.

Č. V. Stanojević and V. B. Stanojević [3] proved the following theorem.

Theorem 1.1. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that*

$$\left(\sum_{k=1}^{n} \frac{M_k}{k}\right) \text{ is slowly oscillating} \tag{1.6}$$

and $(\omega_n^{(2)}(u))$ is left one-sidedly bounded with respect to the sequence (M_n). If

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} V_n^{(1)}(\Delta u)x^n = A\big(V^{(1)}(\Delta u)\big) \tag{1.7}$$

exists, then $u = (u_n)$ is slowly oscillating.

We remind the reader that a sequence (u_n) is slowly oscillating [4] if

$$\lim_{\lambda\to 1^+}\limsup_{n}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| = 0, \tag{1.8}$$

and more generally, it is moderately oscillating [4] if, for $\lambda > 1$,

$$\limsup_{n}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| < \infty, \tag{1.9}$$

where $[\lambda n]$ denotes the integer part of λn.

An equivalent definition of slowly oscillating sequence (u_n) is given by Dik [2] in terms of $(V_n^{(0)}(\Delta u))$. A sequence $u = (u_n)$ is slowly oscillating if and only if $(V_n^{(0)}(\Delta u))$ is slowly oscillating and bounded. Clearly, (1.9) implies that $V_n^{(0)}(\Delta u) = O(1)$.

2. The main theorem

The main goal of this paper is to generalize Č. V. Stanojević and V. B. Stanojević's [3] result for the general control modulo of the oscillatory behavior of the order m, where m is any integer greater than or equal to 1.

THEOREM 2.1. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that*

$$\left(\sum_{k=1}^{n} \frac{M_k}{k}\right) \text{ is moderately oscillating} \tag{2.1}$$

and for some integer $m \geq 1$,

$$(\omega_n^{(m)}(u)) \text{ is left one-sidedly bounded with respect to the sequence } (M_n). \tag{2.2}$$

If (1.7) exists, then $u = (u_n)$ is moderately oscillating.

There would be some cases that for some integer $m \geq 1$, $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6). In this case, we cannot get any information related to the asymptotic behavior of the sequence (u_n) out of (2.2) and (1.5). But for an integer k greater than m, $(\sigma_n(\omega^{(k)}(u)))$ could be left one-sidedly bounded with respect to some nonnegative sequence (M_n) with the property (1.6) as provided in the following example.

Example 2.2. For the sequence (u_n) defined by

$$u_n = \begin{cases} 1, & n = 2^j,\ j = 1,2,3,\ldots, \\ 0, & \text{for other values of } n, \end{cases} \tag{2.3}$$

we have

$$n\Delta u_n = \begin{cases} j, & n = 2^j,\ j = 1,2,3,\ldots, \\ -j, & n = 2^j + 1,\ j = 1,2,3,\ldots, \\ 0, & \text{for other values of } n. \end{cases} \tag{2.4}$$

Since the sequence (u_n) has two subsequences $((u_{2^n})$ and $(u_{2^n+1}))$ converging to different values (1 and 0, resp.), (u_n) does not converge. Consider the series $\sum_{n=1}^{\infty} \Delta u_n x^n$. We may rewrite this series as $f(\Delta u, x) = \sum_{n=1}^{\infty}(x^{2^n} - x^{2^n+1})$. Notice that if $0 \leq x < 1$, then $f(\Delta u, x) \geq 0$. Hence, it follows that

$$\liminf_{x \to 1^-} f(\Delta u, x) \geq 0. \tag{2.5}$$

Also, observe that from the rewritten form of $f(\Delta u, x)$, we have

$$f(\Delta u, x) = (1-x)\sum_{n=1}^{\infty} x^{2^n} \leq (1-x)\left(x^2 + x^4 + x^8 + C\left(\sqrt{\operatorname{In}\left(\frac{1}{x}\right)}\right)^{-1}\right). \tag{2.6}$$

Since $\text{In}(1/x) \sim 1 - x$ as $x \to 1^-$, we have

$$\limsup_{x \to 1^-} f(\Delta u, x) \leq 0. \tag{2.7}$$

From (2.5) and (2.7), it follows that (u_n) is Abel limitable to zero.

It is clear that $(\omega_n^{(0)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6). Indeed, there were such a nonnegative sequence (M_n) with the property (1.6), we would have $-1 = \liminf_n \Delta u_n \geq -\lim_n (M_n/n) = 0$. We also note that for any integer $m \geq 1$, $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6). If $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any sequence (M_n) with the property (1.6) and $A(u)$ exists, then $(\omega_n^{(m+1)}(u))$ is not left one-sidedly bounded with respect to the nonnegative sequence (M_n) with the property (1.6). Suppose that $(\omega_n^{(m+1)}(u))$ is left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6) and $A(u)$ exists. Then by Corollary 2.9, the sequence (u_n) converges and this implies that $(\omega_n^{(m)}(u))$ is left one-sidedly bounded with respect to some nonnegative sequence (M_n) with the property (1.6), which is contrary to the fact that $(\omega_n^{(m)}(u))$ is not left one-sidedly bounded with respect to any nonnegative sequence (M_n) with the property (1.6).

Since $(V_n^{(0)}(\Delta u))$ is bounded, then $V_n^{(0)}(\Delta u) \geq -C$ and

$$\omega_n^{(1)}(\sigma(u))) = n\Delta V_n^{(0)}(\Delta\sigma(u)) = n\Delta V_n^{(1)}(\Delta u) \geq -C \tag{2.8}$$

for some $C \geq 0$. Since $(\sigma_n(u))$ is Abel limitable, by Corollary 2.9, we obtain that $(\sigma_n(u))$ converges.

Remark 2.3. The condition $\omega_n^{(0)}(u) \geq -M_n$ with the properties (1.6) and (2.2) is a Tauberian condition for Abel limitable method, but $\sigma_n(\omega^{(0)}(u)) \geq -M_n$ is not. However,

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n(\omega^{(0)}(u)) \geq -M_n \tag{2.9}$$

is a Tauberian condition for Abel limitable method as proved in Theorem 2.1.

If (u_n) is slowly oscillating or moderately oscillating in the sense of Stanojević [4], then $(V_n^{(0)}(\Delta u))$ is bounded. Hence, for any integer $m \geq 1$, $(\sigma_n(\omega^{(m)}(u)))$ is left one-sidedly bounded with respect to the constant sequence $(M_n) = (C)$. Also, from the definition of slow oscillation, one obtains that the arithmetic means of $(\omega_n^{(m)}(u))$ is slowly oscillating. But, that the converse is not true is provided by example.

We need the following identities and observations for the proof of Theorem 2.1.

Lemma 2.4 [2, 4]. *(i) For $\lambda > 1$,*

$$u_n - \sigma_n(u) = \frac{[\lambda n] + 1}{[\lambda n] - n}(\sigma_{[\lambda n]}(u) - \sigma_n(u)) - \frac{1}{[\lambda n] - n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j, \tag{2.10}$$

where $[\lambda n]$ denotes the integer part of λn.

(i) For $1 < \lambda < 2$,

$$u_n - \sigma_{n-[(\lambda-1)n]-1}(u) = \frac{n+1}{[(\lambda-1)n]+1}\left(\sigma_{n-[(\lambda-1)n]-1}(u) - \sigma_n(u)\right) + \frac{1}{[(\lambda-1)n]+1}\sum_{k=n-[(\lambda-1)n]}^{n}\sum_{j=k+1}^{n}\Delta u_j, \tag{2.11}$$

where $[\lambda n]$ *denotes the integer part of* λn.

Proof. (i) For $\lambda > 1$, define

$$\tau_{n,[\lambda n]}(u) = \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]} u_k. \tag{2.12}$$

The difference $\tau_{n,[\lambda n]}(u) - \sigma_n(u)$ can be written as

$$\begin{aligned}\tau_{n,[\lambda n]}(u) - \sigma_n(u) &= \frac{([\lambda n]+1)\sigma_{[\lambda n]}(u) - (n+1)\sigma_n(u)}{[\lambda n]-n} - \sigma_n(u) \\ &= \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma_{[\lambda n]}(u) - \sigma_n(u)\right).\end{aligned} \tag{2.13}$$

This completes the proof. □

(ii) Proof of Lemma 2.4(ii) is similar to that of Lemma 2.4(i).
For a sequence $u = (u_n)$, we define

$$(n\Delta)_m u_n = (n\Delta)_{m-1}\left((n\Delta)u_n\right) = n\Delta\left((n\Delta)_{m-1}u_n\right), \quad m = 1,2,\ldots, \tag{2.14}$$

where $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n$.

Lemma 2.5. *For each integer* $m \geq 1$,

$$\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u). \tag{2.15}$$

Proof. We do the proof by induction. By definition, for $m = 1$, we have

$$\omega_n^{(1)}(u) = \omega_n^{(0)}(u) - \sigma_n\left(\omega^{(0)}(u)\right) = n\Delta u_n - V_n^{(0)}(\Delta u) = n\Delta V_n^{(0)}(\Delta u). \tag{2.16}$$

Assume the observation is true for $m = k$. That is, assume that

$$\omega_n^{(k)}(u) = (n\Delta)_k V_n^{(k-1)}(\Delta u). \tag{2.17}$$

We must show that the observation is true for $m = k+1$. That is, we must show that

$$\omega_n^{(k+1)}(u) = (n\Delta)_{k+1} V_n^{(k)}(\Delta u). \tag{2.18}$$

Again by definition,

$$\omega_n^{(k+1)}(u) = \omega_n^{(k)}(u) - \sigma_n\left(\omega^{(k)}(u)\right). \tag{2.19}$$

By (2.17),

$$\begin{aligned}\omega_n^{(k+1)}(u) &= (n\Delta)_k V_n^{(k-1)}(\Delta u) - (n\Delta)_k V_n^{(k)}(\Delta u)\\ &= (n\Delta)_k \big(V_n^{(k-1)}(\Delta u) - V_n^{(k)}(\Delta u)\big)\\ &= (n\Delta)_k \big((n\Delta) V_n^{(k)}(\Delta u)\big)\\ &= (n\Delta)_{k+1} V_n^{(k)}(\Delta u).\end{aligned} \tag{2.20}$$

Thus, we conclude that Lemma 2.5 is true for every positive integer m. □

Lemma 2.6. *For each integer $m \geq 1$,*

$$\omega_n^{(m)}(u) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} n\Delta V_n^{(j)}(\Delta u), \tag{2.21}$$

where $\binom{m-1}{j} = (m-1)(m-2)\cdots(m-j+1)/j!$.

Proof. We do the proof by induction. For $m = 1$, we have

$$\begin{aligned}\omega_n^{(1)}(u) &= n\Delta u_n - V_n^{(0)}(\Delta u)\\ &= n\Delta V_n^{(0)}(\Delta u)\\ &= \sum_{j=0}^{0} (-1)^j \binom{0}{j} n\Delta V_n^{(j)}(\Delta u).\end{aligned} \tag{2.22}$$

Assume the observation is true for $m = k$. That is, assume that

$$\omega_n^{(k)}(u) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j)}(\Delta u). \tag{2.23}$$

We must show that the observation is true for $m = k+1$. That is, we must show that

$$\omega_n^{(k+1)}(u) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} n\Delta V_n^{(j)}(\Delta u). \tag{2.24}$$

By definition,

$$\omega_n^{(k+1)}(u) = \omega_n^{(k)}(u) - \sigma_n\big(\omega^{(k)}(u)\big). \tag{2.25}$$

By (2.23),

$$\begin{aligned}\omega_n^{(k+1)}(u) &= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j)}(\Delta u)\\ &\quad - \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} n\Delta V_n^{(j+1)}(\Delta u).\end{aligned} \tag{2.26}$$

Let $j+1=i$ in the second sum. Using this substitution,

$$\begin{aligned}\omega_n^{(k+1)}(u) &= \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}n\Delta V_n^{(j)}(\Delta u)\\ &\quad+\sum_{i=1}^{k}(-1)^i\binom{k-1}{i-1}n\Delta V_n^{(i)}(\Delta u).\end{aligned} \tag{2.27}$$

In the second sum of (2.27), we rename the index of summation j, split the first term off in the first sum and the last term in the second sum of (2.27), we have

$$\begin{aligned}\omega_n^{(k+1)}(u) &= (-1)^0\binom{k-1}{0}n\Delta V_n^{(0)}(\Delta u)+\sum_{j=1}^{k-1}(-1)^j\binom{k-1}{j}n\Delta V_n^{(j)}(\Delta u)\\ &\quad+\sum_{j=1}^{k-1}(-1)^j\binom{k-1}{j-1}n\Delta V_n^{(j)}(\Delta u)+(-1)^k\binom{k-1}{k-1}n\Delta V_n^{(k)}(\Delta u).\end{aligned} \tag{2.28}$$

Rewritten (2.28), we have

$$\begin{aligned}\omega_n^{(k+1)}(u) &= (-1)^0\binom{k-1}{0}n\Delta V_n^{(0)}(\Delta u)\\ &\quad+\sum_{j=1}^{k-1}(-1)^j\left[\binom{k-1}{j}+\binom{k-1}{j-1}\right]n\Delta V_n^{(j)}(\Delta u)\\ &\quad+(-1)^k\binom{k-1}{k-1}n\Delta V_n^{(k)}(\Delta u).\end{aligned} \tag{2.29}$$

Since $\binom{k-1}{j}+\binom{k-1}{j-1}=\binom{k}{j}$, the last identity can be written

$$\begin{aligned}\omega_n^{(k+1)}(u) &= (-1)^0\binom{k-1}{0}n\Delta V_n^{(0)}(\Delta u)\\ &\quad+\sum_{j=1}^{k-1}(-1)^j\binom{k}{j}n\Delta V_n^{(j)}(\Delta u)\\ &\quad+(-1)^k\binom{k-1}{k-1}n\Delta V_n^{(k)}(\Delta u)\\ &= \sum_{j=1}^{k}(-1)^j\binom{k}{j}n\Delta V_n^{(j)}(\Delta u).\end{aligned} \tag{2.30}$$

Thus, we conclude that Lemma 2.6 is true for every positive integer m. □

Corollary 2.7 is an improved version of the main theorem in [3, Theorem 3.1]. Corollaries 2.8 and 2.9 are analogous to classical Tauberian theorems.

Corollary 2.7. *For the real sequence $u=(u_n)$, let there exist a nonnegative sequence $M=(M_n)$ such that $(\sum_{k=1}^{n}(M_k/k))$ is slowly oscillating and condition (2.2) is satisfied. If (1.7) exists, then $u=(u_n)$ is slowly oscillating.*

Proof. Since slow oscillation of $(\sum_{k=1}^{n}(M_k/k))$ implies that $\sigma_n(M)=O(1)$, we have

$$\lim_n V_n^{(1)}(\Delta u)=A\big(V^{(1)}(\Delta u)\big),\qquad n\Delta V_n^{(0)}(\Delta u)\geq -\left(M_n+C\right) \tag{2.31}$$

for some constant C as in Theorem 2.1. Applying Lemma 2.4(i) to $(V_n^{(0)}(\Delta u))$ and noticing that

$$n\Delta V_n^{(0)}(\Delta u)\geq -\left(M_n+C\right) \tag{2.32}$$

for some constant C, we have

$$\begin{aligned} V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u) &\leq \frac{[\lambda n]+1}{[\lambda n]-n}\big(V_{[\lambda n]}^{(1)}(\Delta u)-V_n^{(1)}(\Delta u)\big)\\ &\quad+\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\frac{M_j+C}{j}\\ &\leq \frac{[\lambda n]+1}{[\lambda n]-n}\big(V_{[\lambda n]}^{(1)}(\Delta u)-V_n^{(1)}(\Delta u)\big)\\ &\quad+\frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\frac{M_j}{j}+C_1\log\left(\frac{[\lambda n]}{n}\right) \end{aligned} \tag{2.33}$$

for some constant C_1. From the last inequality, we have

$$\begin{aligned} V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u) &\leq \frac{[\lambda n]+1}{[\lambda n]-n}\big(V_{[\lambda n]}^{(1)}(\Delta u)-V_n^{(1)}(\Delta u)\big)\\ &\quad+\max_{n+1\leq k\leq[\lambda n]}\sum_{j=n+1}^{k}\frac{M_j}{j}+C_1\log\left(\frac{[\lambda n]}{n}\right). \end{aligned} \tag{2.34}$$

Taking lim sup of both sides, we have

$$\begin{aligned} \limsup_n\big(V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u)\big) &\leq \frac{\lambda}{\lambda-1}\limsup_n\big(V_{[\lambda n]}^{(1)}(\Delta u)-V_n^{(1)}(\Delta u)\big)\\ &\quad+\limsup_n\max_{n+1\leq k\leq[\lambda n]}\sum_{j=n+1}^{k}\frac{M_j}{j}+C_1\log\lambda. \end{aligned} \tag{2.35}$$

Since the first term on the right-hand side of the inequality above vanishes,

$$\limsup_n\big(V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u)\big)\leq\limsup_n\max_{n+1\leq k\leq[\lambda n]}\sum_{j=n+1}^{k}\frac{M_j}{j}+C_1\log\lambda. \tag{2.36}$$

Taking the limit of both sides as $\lambda\to 1^+$, we obtain

$$\limsup_n\big(V_n^{(0)}(\Delta u)-V_n^{(1)}(\Delta u)\big)\leq 0. \tag{2.37}$$

In a similar way from Lemma 2.4(ii), we have

$$\liminf_n \left(V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u)\right) \geq 0. \tag{2.38}$$

From (2.37) and (2.38), we have

$$\lim_n V_n^{(0)}(\Delta u) = \lim_n V_n^{(1)}(\Delta u). \tag{2.39}$$

Since

$$\sigma_n(u) = u_0 + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k}, \tag{2.40}$$

from identity (1.2), we can write (u_n) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0. \tag{2.41}$$

Thus, $u = (u_n)$ is slowly oscillating. □

It should be noted that if we take $m = 2$ in Corollary 2.7, we get Č. V. Stanojević and V. B. Stanojević's result.

Corollary 2.8. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that $(\sum_{k=1}^{n}(M_k/k))$ is slowly oscillating and condition (2.2) is satisfied. If $A(\sigma(u))$ exists, then $\lim_n u_n = A(\sigma(u))$.*

Proof. Existence of the limit $A(\sigma(u))$ implies that $A(V^{(1)}(\Delta u)) = 0$. By Corollary 2.7, we have $V_n^{(0)}(\Delta u) = o(1)$ and hence $A(V^{(0)}(\Delta u)) = 0$. From (1.2), it follows that $A(u) = 0$. By Tauber's second theorem [5], $\lim_n u_n = A(\sigma(u))$. □

Corollary 2.9. *For the real sequence $u = (u_n)$, let there exist a nonnegative sequence $M = (M_n)$ such that $(\sum_{k=1}^{n}(M_k/k))$ is slowly oscillating and condition (2.2) is satisfied. If (1.5) exist, then $\lim_n u_n = A(u)$.*

Proof. Since existence of $A(u)$ implies that of $A(\sigma(u))$, proof follows from Corollary 2.8. □

Corollary 2.9 with $m = 3$ follows from Corollary 2.9 with $m = 2$. Indeed, we have for a sequence $u = (u_n)$,

$$\begin{aligned} \omega_n^{(3)}(u) &= (n\Delta)_3 V_n^{(2)}(\Delta u) = (n\Delta)_2 \left(n\Delta V_n^{(2)}(\Delta u)\right) \\ &= (n\Delta)_2 \left(V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u)\right) \\ &= (n\Delta)_2 \left(V_n^{(0)}\left(\Delta V^{(1)}(\Delta u)\right)\right). \end{aligned} \tag{2.42}$$

We note that for a sequence $u = (u_n)$,

$$V_n^{(0)}(\Delta u) - V_n^{(1)}(\Delta u) = V_n^{(0)}\left(\Delta V^{(0)}(\Delta u)\right). \tag{2.43}$$

Taking the arithmetic means of both sides of (2.43), we have

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = V_n^{(1)}(\Delta V^{(0)}(\Delta u)). \tag{2.44}$$

Using (1.2), the identity (2.44) can be expressed as

$$V_n^{(1)}(\Delta u) - V_n^{(2)}(\Delta u) = V_n^{(0)}(\Delta V^{(1)}(\Delta u)). \tag{2.45}$$

From (2.44) and (2.45), we have

$$V_n^{(1)}(\Delta V^{(0)}(\Delta u)) = V_n^{(0)}(\Delta V^{(1)}(\Delta u)). \tag{2.46}$$

We have, by (2.45) and (2.42),

$$\omega_n^{(3)}(u) = (n\Delta)_2(V_n^{(1)}(\Delta V^{(0)}(\Delta u))). \tag{2.47}$$

Existence of the limit $A(u)$ implies that $A(V^{(0)}(\Delta u)) = 0$. By Corollary 2.9 with $m = 2$, we obtain that $V_n^{(0)}(\Delta u) = o(1)$. By Tauber's second theorem [5], $\lim_n u_n = A(u)$.

3. Proof of Theorem 2.1

Proof. From the condition (2.1), it follows that $\sigma_n(M) = O(1)$. Taking the arithmetic mean of both sides of (2.2), we obtain

$$\sigma_n(\omega^{(m)}(u)) = (n\Delta)_m V_n^{(m)}(\Delta u) \geq -\sigma_n(M) \geq -C_0 \tag{3.1}$$

for some constant C_0. By the existence of the limit (1.7),

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(n\Delta)_{m-1}V_n^{(m)}(\Delta u)x^n = 0. \tag{3.2}$$

Since

$$n\Delta((n\Delta)_{m-1}V_n^{(m)}(\Delta u)) \geq -C_0, \tag{3.3}$$

by Hardy and Littlewood's theorem [1],

$$(n\Delta)_{m-1}V_n^{(m)}(\Delta u) = o(1). \tag{3.4}$$

From

$$n\Delta((n\Delta)_{m-1}V_n^{(m)}(\Delta u)) = (n\Delta)_{m-1}V_n^{(m-1)}(\Delta u) - (n\Delta)_{m-1}V_n^{(m)}(\Delta u) \geq -C_0 \tag{3.5}$$

and (3.4), it follows that

$$(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u) \geq -C_1 \tag{3.6}$$

for some constant C_1. The existence of the limit (1.7) implies that

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u)x^n = 0. \tag{3.7}$$

Since

$$n\Delta\left((n\Delta)_{m-2}V_n^{(m-1)}(\Delta u)\right) \geq -C_1, \tag{3.8}$$

again by Hardy and Littlewood's theorem [1],

$$(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u) = o(1). \tag{3.9}$$

Continuing in this way, in $(m-2)$th step we get

$$(n\Delta)_2 V_n^{(2)}(\Delta u) \geq -C_{m-3} \tag{3.10}$$

for some constant C_{m-3}. Since

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} n\Delta V_n^{(2)}(\Delta u)x^n = 0, \tag{3.11}$$

we get

$$n\Delta V_n^{(2)}(\Delta u) = o(1). \tag{3.12}$$

From

$$n\Delta\left(n\Delta V_n^{(2)}(\Delta u)\right) = n\Delta V_n^{(1)}(\Delta u) - n\Delta V_n^{(2)}(\Delta u) \geq -C_{m-3} \tag{3.13}$$

and (3.12) we get

$$n\Delta V_n^{(1)}(\Delta u) \geq -C_{m-2} \tag{3.14}$$

for some constant C_{m-2}. By the existence of the limit (1.7), we obtain that $(V_n^{(1)}(\Delta u))$ converges to $A(V^{(1)}(\Delta u))$. From Lemma 2.6, convergence of $(V_n^{(1)}(\Delta u))$, and condition (2.2), it follows that

$$n\Delta V_n^{(0)}(\Delta u) \geq -\left(M_n + C\right) \tag{3.15}$$

for some constant C. Applying Lemma 2.4(i) and (ii) to $(V_n^{(0)}(\Delta u))$, we have $V_n^{(0)}(\Delta u) = O(1)$. Thus, $u = (u_n)$ is moderately oscillating. □

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