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Research Article

Degenerate Differential Operators with Parameters

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The nonlocal boundary value problems for regular degenerate differential-operator equations with the parameter are studied. The principal parts of the appropriate generated differential operators are non-self-adjoint. Several conditions for the maximal regularity uniformly with respect to the parameter and the Fredholmness in Banach-valued L_p- spaces of these problems are given. In applications, the nonlocal boundary value problems for degenerate elliptic partial differential equations and for systems of elliptic equations with parameters on cylindrical domain are studied.

1. Introduction, notations, and background

Boundary value problems (BVPs) for differential-operator equations (DOEs) in H-valued (Hilbert space-valued) function spaces have been studied extensively by many researchers (see [1–13] and the references therein). BVPs for DOE on E-valued (Banach space valued) function spaces are studied in [1, 14–17]. The main aim of the present paper is to discuss the BVPs for regular degenerate DOE with the parameter on E-valued function spaces. The maximal regularity and Fredholmness of these problems in Banach-valued L_p-spaces are established. In applications, the nonlocal BVPs for degenerate elliptic partial differential equations and for systems of elliptic equations with parameters on cylindrical domain are studied.

Let E be a Banach space and let $\gamma = \gamma(x)$, $x = (x_1, x_2, \dots, x_n)$, be a positive measurable function on a domain $\Omega \subset R^n$. Let $L_{p,\gamma}(\Omega; E)$ denote the space of strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x)\,dx\right)^{1/p}, \quad 1 \le p < \infty. \tag{1.1}$$

For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega;E)$ will be denoted by $L_p = L_p(\Omega;E)$. $L_{p_1,p_2}(\Omega)$ and $W^l_{p_1,p_2}(\Omega)$ will denote a scalar-valued (p_1,p_2)-integrable function space and Sobolev space with mixed norms, respectively, [18]. Let $B^s_{p,q}$ denote a Besov space (see, e.g., [18, Section 2.3]).

A Banach space E is called the UMD-space (see, e.g., [19, 20]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \tag{1.2}$$

is bounded in the space $L_p(R,E)$, $p \in (1,\infty)$. UMD spaces include, for example, L_p, l_p spaces and Lorentz spaces L_{pq}, $p,q \in (1,\infty)$.

Let $\mathbb{C}$ be a set of complex numbers and

$$S_\varphi = \{\xi;\xi \in \mathbb{C}, \ |\arg\xi - \pi| \le \pi - \varphi\} \cup \{0\}, \quad 0 < \varphi \le \pi. \tag{1.3}$$

A linear operator A is said to be positive in a Banach space E, with bound M if $D(A)$ is dense on E and

$$\|(A - \xi I)^{-1}\|_{B(E)} \le M(1 + |\xi|)^{-1} \tag{1.4}$$

with $\xi \in S_\varphi, \varphi \in (0,\pi]$, where M is a positive constant and I is an identity operator in E, where $L(E)$ is the space of bounded linear operators acting in E. Sometimes instead of $A + \xi I$ will be written $A + \xi$ and denoted by A_ξ. It is known [33, Section 1.15.1] there exist fractional powers A^θ of the positive operator A. By the definition of the positive operator A for all $\xi \in S(\varphi)$,

$$\|\xi(A - \xi I)^{-1}\|_{B(E)} \le M. \tag{1.5}$$

The operator $A(t)$ is said to be positive in the Banach space E uniformly with respect to t if $D(A(t))$ is independent of t, $D(A(t))$ is dense in E, and

$$\|(A(t) - \lambda I)^{-1}\| \le \frac{M}{1 + |\lambda|} \tag{1.6}$$

for all $\lambda \in S(\varphi)$, $\varphi \in (0,\pi]$.

Let $E(A^\theta)$ denote the space $D(A^\theta)$ with graphical norm defined as

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^{1/p}, \quad 1 \le p < \infty, \ -\infty < \theta < \infty. \tag{1.7}$$

Let E_1 and E_2 be two Banach spaces. By $(E_1,E_2)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$, will be denoted an interpolation space for $\{E_1,E_2\}$ by the K-method [21, Section 1.3.1].

We denote by $D(R^n;E)$ the space of E-valued C^∞-functions with compact support, equipped with the usual inductive limit topology and $S = S(R^n;E)$ denotes the E-valued Schwartz space of rapidly decreasing, smooth functions. For $E = \mathbb{C}$ we simply write $D(R^n)$ and $S(R^n)$, respectively. $D'(R^n;E) = L(D(R^n),E)$ denote the space of E-valued distributions and $S'(E) = S'(R^n;E)$ is a space of linear continued mapping from $S(R^n)$ into

E. Let E_1 and E_2 be two Banach spaces. The Fourier transform for $u \in S'(R^n;E)$ is defined by

$$F(u)(\varphi) = u(F(\varphi)), \quad \varphi \in S(R^n). \tag{1.8}$$

Let γ such that $S(R^n;E_1)$ is dense in $L_{p,\gamma}(R^n;E_1)$ (see, e.g., Lemma 2.1). A function $\Psi \in C(R^n;L(E_1,E_2))$ is called a Fourier multiplier from $L_{p,\gamma}(R^n;E_1)$ to $L_{q,\gamma}(R^n;E_2)$ if the map $u \to \Phi u = F^{-1}\Psi(\xi)Fu$, $u \in S(R^n;E_1)$ is well defined and extends to a bounded linear operator

$$\Phi : L_{p,\gamma}(R^n;E_1) \longrightarrow L_{q,\gamma}(R^n;E_2). \tag{1.9}$$

We denote the set of all multipliers from $L_{p,\gamma}(R^n;E_1)$ to $L_{q,\gamma}(R^n;E_2)$ by $M_{p,\gamma}^{q,\gamma}(E_1,E_2)$. For $E_1 = E_2 = E$, we denote the $M_{p,\gamma}^{q,\gamma}(E_1,E_2)$ by $M_{p,\gamma}^{q,\gamma}(E)$. Let $M(h) = \{\Psi_h \in M_{p,\gamma}^{q,\gamma}(E_1,E_2), h \in H\}$ be a collection of multipliers in $M_{p,\gamma}^{q,\gamma}(E_1,E_2)$. A family of sets $M(h) \subset B(E_1,E_2)$ depending on $h \in H$ is called a uniformly collection of multipliers with respect to h if there exists a positive constant C independent on $h \in H$ such that

$$||F^{-1}\Psi_h Fu||_{L_{q,\gamma}(R^n,E_2)} \leq C\|u\|_{L_{p,\gamma}(R^n,E_1)} \tag{1.10}$$

for all $h \in H$ and $u \in S(R^n;E_1)$.

The exposition of the theory of L_p-multipliers of the Fourier transformation, and some related references, can be found in [33, Sections 2.2.1–2.2.4]. In vector-valued function spaces, Fourier multipliers have been studied in [14, 22, 23, 25–27, 29].

A set $K \subset B(E_1,E_2)$ is called R-bounded (see, e.g., [14, 22, 28]) if there is a positive constant C such that for all $T_1,T_2,\dots,T_m \in K$ and $u_1,u_2,\dots,u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y)T_j u_j \right\|_{E_2} dy \leq C\int_0^1 \left\| \sum_{j=1}^m r_j(y)u_j \right\|_{E_1} dy, \tag{1.11}$$

where $\{r_j\}$ is a sequence of independent symmetric $[-1,1]$-valued random variables on $[0,1]$ and $\mathbb{N}$ denotes the set of natural numbers. The smallest such constant C is called the R-bound of K and is denoted by $R(K)$.

A family of sets $K(h) \subset B(E_1,E_2)$ depending on parameter $h \in H$ is called uniformly R-bounded with respect to h if there is a positive constant C such that for all $T_1,T_2,\dots,T_m \in K(h)$ and $u_1,u_2,\dots,u_m \in E_1$, $m \in N$,

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y)T_j(h)u_j \right\|_{E_2} dy \leq C\int_0^1 \left\| \sum_{j=1}^m r_j(y)u_j \right\|_{E_1} dy, \tag{1.12}$$

where the constant C is independent on parameter h (i.e., $\sup_{h\in H} R(K(h)) < \infty$).

Let $W_h = \{\Psi_h \in M_p^q(E_1,E_2),\ h \in H\}$ be a collection of multipliers in $M_p^q(E_1,E_2)$. We say that W_h is a uniform collection of multipliers if there exists a constant $M > 0$ independent on $h \in H$ such that

$$||F^{-1}\Psi_h Fu||_{L_q(R^n;E_2)} \leq M\|u\|_{L_p(R^n;E_1)} \tag{1.13}$$

for all $h \in H$ and $u \in S(R^n;E_1)$.

Let

$$U_n = \{\beta = (\beta_1,\beta_2,\ldots,\beta_n), |\beta| \le n\}, \qquad \xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2},\ldots,\xi_n^{\beta_n}. \tag{1.14}$$

Definition 1.1. The Banach space E is said to be a space satisfying a multiplier condition with respect to $p \in (1,\infty)$ and weight function γ, when for every $\Psi \in C^{(n)}(R^n/0;B(E))$ if the set

$$\{\xi^\beta D_\xi^\beta \Psi(\xi) : \xi \in R^n/0,\ \beta \in U_n\} \tag{1.15}$$

is R-bounded, then $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$.

A Banach space E is said to be a space satisfying a uniform multiplier condition, when for $\Psi_h \in C^{(n)}(R^n;B(E))$ if

$$\sup_{h\in H} R(\{\xi^\beta D_\xi^\beta \Psi_h(\xi) : \xi \in V_n,\ \beta \in U_n\}) < \infty, \tag{1.16}$$

then Ψ_h is a uniform collection of multipliers in $M_p^p(E)$ for $p \in (1,\infty)$.

A Banach space E has a property (α) (see, e.g., [22, 29]) if there exists a constant α such that

$$\left\|\sum_{i,j=1}^N \alpha_{ij}\varepsilon_i\varepsilon_j' x_{ij}\right\|_{L_2(\Omega\times\Omega';E)} \le \alpha \left\|\sum_{i,j=1}^N \varepsilon_i\varepsilon_j' x_{ij}\right\|_{L_2(\Omega\times\Omega';E)} \tag{1.17}$$

for all $N \in \mathbb{N}$, $x_{i,j} \in E$, $\alpha_{ij} \in \{0,1\}$, $i,j = 1,2,\ldots,N$, and all choices of independent, symmetric, $\{-1,1\}$-valued random variables $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N$, $\varepsilon_1',\varepsilon_2',\ldots,\varepsilon_N'$ on probability spaces Ω,Ω'. For example, the spaces $L_p(\Omega)$, $1 \le p < \infty$, have the property (α).

Remark 1.2. The result [21] implies that the space l_p, $p \in (1,\infty)$, satisfies multiplier condition with respect to p and the weight functions

$$\gamma = |x|^\alpha,\ -1 < \alpha < p-1, \quad \gamma = \prod_{k=1}^N \left(1 + \sum_{j=1}^n |x_j|^{\alpha_{jk}}\right)^{\beta_k}, \quad \alpha_{jk} \ge 0,\ N \in \mathbb{N},\ \beta_k \in R. \tag{1.18}$$

Moreover, the UMD spaces with (α) properties satisfy the multiplier condition with respect to $p \in (1,\infty)$ and the weighted function $\gamma = \prod_{k=1}^n |x_k|^{\gamma_k}$, $0 \le \gamma_k < p-1$ (see Theorem 2.2).

It is well known (see [25, 26]) that any Hilbert space satisfies the multiplier condition. There are, however, Banach spaces which are not Hilbert spaces but satisfy the multiplier condition, for example, UMD spaces (see [14, 17, 22, 27]).

Definition 1.3. A positive operator A is said to be R-positive in the Banach space E if there exists $\varphi \in (0,\pi]$ such that the set

$$L_A = \{(1+|\xi|)(A-\xi I)^{-1} : \xi \in S_\varphi\} \tag{1.19}$$

is R-bounded.

Note that in a Hilbert space every norm bounded set is R-bounded. Therefore, in a Hilbert space, all positive operators are R-positive. If A is a generator of a contraction semigroup on $L_q, 1 \leq q \leq \infty$ [30], A has bounded imaginary powers with $\|(-A^{it})\|_{B(E)} \leq Ce^{\nu|t|}$, $\nu < \pi/2$, [31] or if A is generator of a semigroup with Gaussian bound [23] in $E \in$ UMD, then those operators are R-positive.

$\sigma_\infty(E)$ will denote the space of compact operators in E. Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded into E. Let Ω be a domain on R^n and $l = (l_1, l_2, \dots, l_n)$. $W^l_{p,\gamma}(\Omega; E_0, E)$ denotes a space that consists of functions $u \in L_{p,\gamma}(\Omega; E_0)$ such that it has the generalized derivatives $D^{l_k}_k u = (\partial^{l_k}/\partial x^{l_k}_k)u \in L_{p,\gamma}(\Omega; E)$ with norm

$$\|u\|_{W^l_{p,\gamma}(\Omega;E_0,E)} = \|u\|_{L_{p,\gamma}(\Omega;E_0)} + \sum_{k=1}^{n} \left\|D^{l_k}_k u\right\|_{L_{p,\gamma}(\Omega;E)} < \infty. \tag{1.20}$$

Let $t = (t_1, t_2, \dots, t_n)$, where $t_j > 0$ are parameters. We define in this space a norm

$$\|u\|_{W^l_{p,\gamma,t}(\Omega;E_0,E)} = \|u\|_{L_{p,\gamma}(\Omega;E_0)} + \sum_{k=1}^{n} \left\|t_k D^{l_k}_k u\right\|_{L_{p,\gamma}(\Omega;E)}. \tag{1.21}$$

For $E_0 = E$ the space $W^l_{p,\gamma}(\Omega; E_0, E)$ will be denoted by $W^l_{p,\gamma}(\Omega; E)$.

The weight γ is said to satisfy an A_p condition, that is, $\gamma \in A_p$, $1 < p < \infty$, if there is a positive constant C such that

$$\left(\frac{1}{|Q|}\int_Q \gamma(x)dx\right)\left(\frac{1}{|Q|}\int_Q \gamma^{-1/(p-1)}(x)dx\right)^{p-1} \leq C \tag{1.22}$$

for all compacts $Q \subset R^n$.

Condition 1.4. Let $\gamma = \prod_{k=1}^n \gamma_k(x_k)$, where $\gamma_k \in A_p$ and there exist constants C_1, C_2 such that

$$\begin{gathered}
\gamma_k(y_1) \leq C_k \gamma_k(y_2), \qquad \gamma_k^{1/p}(y_2)\gamma_k^{-1/p}(y_1) \leq M_k \left|y_2\right|^{\nu_k/p}\left|y_1\right|^{-\nu_k/p}, \\
\left|\gamma_k^{1/p}(y_2)\gamma_k^{-1/p}(y_1) - 1\right| \leq D_k \left|\left|y_2\right|^{\nu_k/p}\left|y_1\right|^{-\nu_k/p} - 1\right|, \quad y_1, y_2 \in R\backslash\{0\}, \\
0 \leq \nu_k < p-1, \quad k = 1, 2, \dots, n.
\end{gathered} \tag{1.23}$$

2. Background materials

Embedding theorems for vector-valued Sobolev spaces played important role in the present investigation. Embedding theorems in Hilbert-valued function spaces have been studied, for example, in [11–13, 32]. This section is concentrated on weighted anisotropic Banach-valued Sobolev spaces $W^l_{p,\gamma}(\Omega; E_0, E)$ associated with Banach spaces E_0, E. Several conditions are found that ensure the continuity and compactness of embedding operators that are optimal regular in these spaces in terms of interpolations of E_0 and E. In particular, the most regular class of interpolation spaces E_α between E_0, E, depending on α and order of spaces are found that mixed derivatives D^α are bounded and compact from $W^l_{p,\gamma}(\Omega; E_0, E)$ to $L_{p,\gamma}(\Omega; E_\alpha)$. These results generalize and improve the results [11–13, 32]. Multiplier theorems in the operator-valued L_p spaces are important tools in the theory of embedding of function spaces and in BVPs. Since our consideration take place in weighted case with parameterized estimates, so we have to generalize multiplier theorems [22] for the case of $L_{p,\gamma}$ and for multipliers depending on parameters. Lets first show the following needed lemma.

Lemma 2.1. *Let E be a Banach space, $1 \leq p < \infty$, and γ a positive measurable function on an open subset Ω of R^n, essentially bounded on compact subsets of Ω. Then the space $D(\Omega;E)$ is dense in $L_{p,\gamma}(\Omega;E)$.*

Proof. For $u \in L_{p,\gamma}(\Omega;E)$ and $n \in \mathbb{N}$ let $u_n : \Omega \to E$ such that

$$u_n = \begin{cases} u(x) & \text{if } \|u(x)\| \leq n, \\ 0 & \text{if } \|u(x)\| > n. \end{cases} \tag{2.1}$$

By the dominated convergence theorem $\lim_{n\to\infty} \|u - u_n\|_{L_{p,\gamma}(\Omega;E)} = 0$, hence a compactly supported function can be approximated with bounded compactly supported functions, that is, with compactly supported function belonging to $L_p(\Omega;E)$. From the standard proof of the denseness theorem in case of spaces without weight, it follows that if u is a compactly supported function belonging to $L_p(\Omega;E)$, then there exists a compact subset $K \subset \Omega$, with $\operatorname{supp} u \subseteq K$, and a sequence of functions $u_n \in D(\Omega;E)$, with $\operatorname{supp} u_n \subseteq K$ such that $\lim_{n\to\infty} \|u - u_n\|_{L_p(\Omega;E)} = 0$; since

$$\|u - u_n\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_K \|u(x) - u_n(x)\|^p \gamma(x)dx\right)^{1/p} \leq \left(\sup_{x\in K} \gamma(x)\right)^{1/p} \|u - u_n\|_{L_p(\Omega;E)}, \tag{2.2}$$

we have

$$\lim_{n\to\infty} \|u - u_n\|_{L_{p,\gamma}(\Omega;E)} = 0. \tag{2.3}$$

From [15] we have the proof. □

Theorem 2.2. *Let the following conditions hold:*

(1) *the weighted function γ satisfies Condition 1.4;*

(2) *Banach spaces E_1 and E_2 are UMD space with property (α) and let $\Psi \in C^{(n)}(R^n/0; B(E_1,E_2))$.*

If

$$\sup_{h\in H} R\left(\left\{\xi^\beta D_\xi^\beta \Psi_h(\xi) : \xi \in R^n/0,\ \beta \in U_n\right\}\right) < \infty, \tag{2.4}$$

then $\Psi_h(\xi)$ is a uniformal collection of multipliers in $M_{p,\gamma}^{p,\gamma}(E_1,E_2)$.

If $n = 1$, then the result remains true for all $E_1, E_2 \in$ UMD spaces.

In a similar way as [11–13, 15] we obtain the following theorem.

THEOREM 2.3. *Suppose the following conditions are satisfied:*

(1) *E is a Banach space that satisfies the multiplier condition with respect to p and weighted function $\gamma(x)$ and A is an R-positive operator in E for φ with $0 < \varphi \leq \pi$;*

(2) *$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $l = (l_1, l_2, \ldots, l_n)$ are n-tuples of nonnegative integer numbers such that $\varkappa = |\alpha : l| = \sum_{k=1}^{n}(\alpha_k/l_k) \leq 1$, $1 < p < \infty$, $0 < \mu \leq 1 - \varkappa$, and $t = (t_1, t_2, \ldots, t_n)$, $0 < t_k \leq t_0 < \infty$;*

(3) *$\Omega \in R^n$ is a region such that there exists a bounded linear extension operator from $W^l_{p,\gamma}(\Omega; E(A), E)$ to $W^l_{p,\gamma}(R^n; E(A), E)$. Then the following embedding:*

$$D^\alpha W^l_{p,\gamma}(\Omega; E(A), E) \subset L_{p,\gamma}(\Omega; E(A^{1-\varkappa-\mu})) \tag{2.5}$$

is continuous and there exists a positive constant C_μ such that

$$\prod_{k=1}^{n} t_k^{\alpha_k/l_k} \| D^\alpha u \|_{L_{p,\gamma}(\Omega; E(A^{1-\varkappa-\mu}))} \leq C_\mu \big[h^\mu \|u\|_{W^l_{p,\gamma,t}(\Omega; E(A), E)} + h^{-(1-\mu)} \|u\|_{L_{p,\gamma}(\Omega; E)} \big] \tag{2.6}$$

for all $u \in W^l_{p,\gamma}(\Omega; E(A), E)$, and h with $0 < h \leq h_0 < \infty$.

Proof. It is sufficient to prove the estimate (2.6). At first we prove the estimate (2.6) for $\Omega = R^n$. Really, it is easy to see that

$$\| D^\alpha u \|_{L_{p,\gamma}(R^n; E(A^{1-\varkappa-\mu}))} \sim \| F^{-'} (i\xi)^\alpha A^{1-\varkappa-\mu} \hat{u} \|_{L_{p,\gamma}(R^n; E)}. \tag{2.7}$$

Moreover, for $u \in W^l_{p,\gamma}(R^n; E(A), E)$, we have

$$\begin{aligned} \|u\|_{W^l_{p,\gamma,t}(R^n; E(A), E)} &= \|u\|_{L_{p,\gamma}(R^n; E(A))} + \sum_{k=1}^{n} \| t_k D_k^{l_k} u \|_{L_{p,\gamma}(R^n; E)} \\ &= \| F^{-'} \hat{u} \|_{L_{p,\gamma}(R^n; E(A))} + \sum_{k=1}^{n} \| t_k F^{-'} [(i\xi_k)^{l_k} \hat{u}] \|_{L_{p,\gamma}(R^n; E)} \\ &\sim \| F^{-1} A \hat{u} \|_{L_{p,\gamma}(R^n; E)} + \sum_{k=1}^{n} \| t_k F^{-'} [(i\xi_k)^{l_k} \hat{u}] \|_{L_{p,\gamma}(R^n; E)}. \end{aligned} \tag{2.8}$$

Thus the inequality (2.6) for $\Omega = R^n$ will be proved if the estimate

$$\begin{aligned} &\prod_{k=1}^{n} t_k^{\alpha_k/l_k} \| F^{-'} (i\xi)^\alpha A^{1-\varkappa-\mu} \hat{u} \|_{L_{q,\gamma}(R^n, E)} \\ &\quad \leq C_\mu \Bigg[h^\mu \Bigg(\| F^{-'} A \hat{u} \|_{L_{p,\gamma}(R^n, E)} + \sum_{k=1}^{n} \| t_k F^{-'} [(i\xi_k)^{l_k} \hat{u}] \|_{L_{p,\gamma}(R^n, E)} \Bigg) \\ &\qquad + h^{-(1-\mu)} \| F^{-'} \hat{u} \|_{L_{p,\gamma}(R^n, E)} \Bigg) \Bigg] \end{aligned} \tag{2.9}$$

is provided for a suitable positive constant C_μ. Let

$$Q_{t,h}(\xi) = h^\mu \left(A + \sum_{k=1}^{n} t_k \, | \xi_k |^{l_k} \right) + h^{-(1-\mu)}. \tag{2.10}$$

By virtue of (2.8) it is easy to see that inequality (2.9) will follow immediately if we can prove that the operator-function $\Psi_{t,h} = \prod_{k=1}^{n} t_k^{\alpha_k/l_k}(i\xi)^{\alpha}A^{1-\varkappa-\mu}Q_{t,h}^{-1}(\xi)$ is a uniform collection of multipliers in $M_{p,\gamma}^{p,\gamma}(E)$ depend on parameters t and h. To see this, it is sufficing to show that the sets

$$\{\xi^{\beta}D^{\beta}\Psi_{t,h}(\xi) : \xi \in R^n/\{0\},\ \beta \in U_n\} \tag{2.11}$$

are R-bounded in E and the R-bounds do not depend on t and h. In fact, by using a similar technique as in [14, Lemma 3.1] we have

$$|\xi^{\beta}|\,\|D^{\beta}\Psi_{t,h}(\xi)\|_{B(E)} \leq C, \quad \xi \in R^n/\{0\},\ \beta \in U_n, \tag{2.12}$$

uniformly with respect to t and h. Due to R-positivity of operator A and by estimate (2.12) we obtain that the sets

$$\{AQ_{t,h}^{-1}(\xi) : \xi \in R^n/\{0\}\}, \qquad \left\{\left(1+\sum_{k=1}^{n} t_k|\xi_k|^{l_k}+h^{-1}\right)Q_{t,h}^{-1}(\xi) : \xi \in R^n/\{0\}\right\} \tag{2.13}$$

are R bounded uniformly with respect to t and h. Moreover, for $u_1, u_2, \dots, u_m \in E$, $m \in N$, and $\xi^j = (\xi_{1j}, \xi_{2j}, \dots, \xi_{nj}) \in R^n/\{0\}$, we have

$$\begin{aligned}
&\left\|\sum_{j=1}^{m} r_j(y)\Psi_{t,h}(\xi^j)u_j\right\|_{L_p} \\
&\quad= \left\|\sum_{j=1}^{m} r_j(y)\Phi(t)(\xi^j)^{\alpha}A^{1-\varkappa-\mu}Q_{t,h}^{-1}(\xi^j)u_j\right\|_{L_p} \\
&\quad= \left\|\sum_{j=1}^{m} r_j(y)\Phi(t)(\xi^j)^{\alpha}\left(1+\sum_{k=1}^{n} t_k|\xi_{kj}|^{l_k}+h^{-1}\right)^{-(\varkappa+\mu)}\right. \\
&\qquad\left.\times\left[\left(1+\sum_{k=1}^{n} t_k|\xi_{kj}|^{l_k}+h^{-1}\right)Q_{t,h}^{-1}(\xi^j)\right]^{(\varkappa+\mu)}\left[AQ_{t,h}^{-1}(\xi^j)\right]^{1-(\varkappa+\mu)}u_j\right\|_{L_p},
\end{aligned} \tag{2.14}$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. By virtue of Kahane's contraction principle [14, Lemma 3.5] we obtain from the above equality

$$\begin{aligned}
&\left\|\sum_{j=1}^{m} r_j(y)\Psi_{t,h}(\xi^j)u_j\right\|_{L_p(0,1;E)} \\
&\quad\leq 2M_0\left\|\sum_{j=1}^{m} r_j(y)\left[\left(1+\sum_{k=1}^{n} t_k|\xi_{kj}|^{l_k}+h^{-1}\right)Q_{t,h}^{-1}(\xi^j)\right]^{(\varkappa+\mu)}\times\left[AQ_{t,h}^{-1}(\xi^j)\right]^{1-(\varkappa+\mu)}u_j\right\|_{L_p(0,1;E)}.
\end{aligned} \tag{2.15}$$

Then by the above estimate, in view of (2.12), and by product properties of the collection of R-bounded operators (see, e.g., [14, Proposition 3.4]) we get that the set $\{\Psi_{t,h}(\xi) :$

$\xi \in R^n/\{0\}\}$ is R-bounded uniformly with respect to t and h. In a similar way, by using Kahane's contraction principle and by product and additional properties of the collection of R-bounded operators [14, Proposition 3.4], we obtain that the sets

$$\{\xi^\beta D^\beta \Psi_{t,h}(\xi) : \xi \in R^n/\{0\},\ \beta \in U_n\} \tag{2.16}$$

are R-bounded uniformly with respect to t and h. Then we obtain that operator-function $\Psi_{t,h}(\xi)$ is a uniform collection of multipliers in $M_{p,\gamma}^{q,\gamma}(E)$. Therefore, we obtain the estimate (2.12). Then by using an extension operator in $W_{p,\gamma}^l(\Omega; E(A), E)$, we obtain from (2.9) estimate (2.6). □

Theorem 2.4. *Suppose all conditions of Theorem 2.3 are satisfied; Ω is a bounded region on R^n satisfy the l-horn conditions and $A^{-1} \in \sigma_\infty(E)$. Let the weighted function γ satisfy Condition 1.4. Then for $0 < \mu \le 1 - \varkappa$, an embedding*

$$D^\alpha W_{p,\gamma}^l(\Omega; E(A), E) \subset L_{p,\gamma}(\Omega; E(A^{1-\varkappa-\mu})) \tag{2.17}$$

is compact.

Indeed putting in (2.6) $h = \|u\|_{L_{p,\gamma}(\Omega;E)} / \|u\|_{W_{p,\gamma}^l(\Omega;E(A),E)}$, the following multiplicative inequality is obtained:

$$||D^\alpha u||_{L_{p,\gamma}(\Omega;E(A^{1-\varkappa-\mu}))} \le C_\mu \|u\|_{L_{p,\gamma}(\Omega;E)}^{\mu} \|u\|_{W_{p,\gamma}^l(\Omega;E(A),E)}^{1-\mu}. \tag{2.18}$$

By virtue of [16, Theorem 2], the embedding

$$W_{p,\gamma}^l(\Omega; E(A), E) \subset L_{p,\gamma}(\Omega; E) \tag{2.19}$$

is compact. Then from the above estimate we obtain assertion of Theorem 2.4.

By a similar manner as Theorem 2.3, we have the following.

Theorem 2.5. *Suppose all conditions of Theorem 2.3 are satisfied. Then for $0 < \mu < 1 - \varkappa$, the embedding*

$$D^\alpha W_{p,\gamma}^l(\Omega; E(A), E) \subset L_{p,\gamma}(\Omega; (E(A), E)_{\varkappa,p}) \tag{2.20}$$

is continuous and there exists a positive constant C_μ such that

$$\prod_{k=1}^{n} t_k^{\alpha_k/l_k} ||D^\alpha u||_{L_{p,\gamma}(\Omega;(E(A),E)_{\varkappa,p})} \le C\|u\|_{W_{p,\gamma,t}^l(\Omega;E(A),E)} \tag{2.21}$$

for all $u \in W_{p,\gamma}^l(\Omega; E(A), E)$.

Proof. By reasoning as Theorem 2.3, it is sufficient to prove that an operator function $\Psi_t(\xi) = \prod_{k=1}^n t_k^{\alpha_k/l_k} \xi^\alpha [A + \sum_{k=1}^n t_k \xi_k^{l_k}]^{-1}$ is multiplier from $L_{p,\gamma}(R^n; E)$ to $L_{p,\gamma}(R^n; ((E(A), E)_{\varkappa,p}))$. It is shown by taking into account R-positivity of the operator A and by using the equivalent definition of the interpolation spaces [33, Section 1.14.5]. □

Theorem 2.6 [16]. *Let E be a Banach space, let A be a positive operator in E with bound M. Let m be a positive integer, $1 \le p < \infty$, and $\alpha \in (1/2p, m + 1/2p)$, $0 \le \nu < 2p\alpha - 1$. Then for*

$\lambda \in S(\varphi)$ the operator $-A_\lambda^{1/2}$ generates a semigroup $e^{-A_\lambda^{1/2}x}$, which is holomorphic for $x > 0$. Moreover, there exists a constant $C \in R^+$ (depending only on M, φ, m, α, ν, p) such that for every $u \in (E, E(A))_{\alpha/m-(1+\nu)/2mp,p}$ and $\lambda \in S(\varphi)$,

$$\int_0^\infty \left\|(A+\lambda I)^\alpha e^{-x(A+\lambda I)^{1/2}} u\right\|^p x^\nu dx \le C\left[\|u\|^p_{(E,E(A))_{\alpha/m-(1+\nu)/2mp,p}} + |\lambda|^{p\alpha-(1+\nu)/2}\|u\|_E^p\right]. \tag{2.22}$$

Proof. By using a similar technique as [8, Lemma 2.2], at first for a φ-positive operator A, where $\varphi \in (\pi/2, \pi)$, and for every $u \in E$ such that $\int_0^\infty \|x^{\alpha-(1+\nu)/p}(A(A+x)^{-1})^m u\|^p x^{\nu-1} dx < \infty$, using integral representation formula of holomorphic semigroup we obtain an estimate

$$\int_0^\infty \left\|A^\alpha e^{-xA} u\right\|^p x^\nu dx \le C\int_0^\infty \left\|x^{\alpha-(1+\nu)/p}\left(A(A+x)^{-1}\right)^m u\right\|^p \frac{dx}{x}. \tag{2.23}$$

Then by using the above estimate and [8, Lemmas 2.3–2.5] we obtain the assertion of Theorem 2.6. □

Theorem 2.7. *Let the following conditions be satisfied:*

(1) $0 \le \nu < 1 - 1/p$, *l and s are integer numbers, and* $0 \le s \le l-1$;

(2) $\theta_\gamma = (ps+1+\gamma)/pl$, $\theta = (ps+1)/pl$, $0 < t \le t_0 < \infty$, $x_0 \ne 0$, $0 < h \le h_0$.

Then, for $u \in W^l_{p,\gamma,t}(0,b;E_0,E)$ the following inequalities hold:

(a)

$$t^{\theta_\gamma}\left\|u^{(s)}(0)\right\|_{(E_0,E)_{\theta_\gamma,p}} \le C\left(\left\|tu^{(l)}\right\|_{L_{p,\gamma}(0,b;E)} + \|u\|_{L_{p,\gamma}(0,b;E_0)}\right); \tag{2.24}$$

(b)

$$t^{\theta}\left\|u^{(s)}(x_0)\right\|_{(E_0,E)_{\theta,p}} \le C\left(\left\|tu^{(l)}\right\|_{L_{p,\gamma}(0,b;E)} + \|u\|_{L_{p,\gamma}(0,b;E_0)}\right), \quad x_0 \ne 0; \tag{2.25}$$

(c)

$$t^{\theta_\gamma}\left\|u^{(s)}(0)\right\|_{E} \le C\left[h^{1-\theta_\gamma}\left\|tu^{(l)}\right\|_{L_{p,\gamma}(0,b;E)} + h^{-\theta_\gamma}\|u\|_{L_{p,\gamma}(0,b;E_0)}\right]; \tag{2.26}$$

(d)

$$t^{\theta}\left\|u^{(s)}(x_0)\right\|_{E} \le C\left(h^{1-\theta}\left\|tu^{(l)}\right\|_{L_{p,\gamma}(0,b;E)} + h^{-\theta}\|u\|_{L_{p,\gamma}(0,b;E_0)}\right), \quad x_0 \ne 0. \tag{2.27}$$

Proof. Really, by virtue of [32] for $u \in W^l_{p,\gamma}(0,b;E_0,E)$, the following inequality holds:

$$\left\|u^{(s)}(0)\right\|_{(E_0,E)_{\theta_\gamma,p}} \le C\left(\left\|u^{(l)}\right\|_{L_{p,\gamma}(0,b;E)} + \|u\|_{L_{p,\gamma}(0,b;E_0)}\right). \tag{2.28}$$

Moreover, in view of [4, Theorem 1.7.7/2] (only by replacing $|\lambda|^{-l}$ for $|\lambda| \ge \lambda_0 > 0$ with t) we obtain (a) and (b). Finally, (c) and (d) can be obtained from (a) and (b) by putting th in place of t.

Then by using the above transformation we get the estimate (c). In a similar way, we obtain the inequality (d).

Consider a differential-operator equation

$$Lu = u^{(m)}(x) + \sum_{k=1}^{m} a_k A^k u^{(m-k)}(x) = 0, \quad x \in (0,b). \tag{2.29}$$

Let $\omega_1, \omega_2, \dots, \omega_m$ be roots of the equation

$$\begin{gathered}\omega^m + a_1\omega^{m-1} + \cdots + a_m = 0,\\ \omega_m = \min\{\arg\omega_j,\ j = 1,\dots,d;\ \arg\omega_j + \pi j = d+1,\dots,m\},\\ \omega_M = \max\{\arg\omega_j,\ j = 1,\dots,d;\ \arg\omega_j + \pi j = d+1,\dots,m\}.\end{gathered} \tag{2.30}$$

A system of numbers $\omega_1, \omega_2, \dots, \omega_m$ is called d-separated if there exists a straight line P passing through 0 such that no value of the numbers ω_j lies on it, and $\omega_1, \omega_2, \dots, \omega_d$ are on one side of P while $\omega_{d+1}, \dots, \omega_m$ are on the other. □

By reasoning as [4, Lemma 5.3.2/1], we have the following.

Lemma 2.8. *Let the following conditions be satisfied:*
(1) $\gamma(x) = x^\nu$, $0 \le \nu < 1 - 1/p$, $p \in (1,\infty)$, $a_m \ne 0$, *and the roots* ω_j *are* d*-separated;*
(2) A *is a closed operator in a Banach space* E *with a dense domain* $D(A)$ *and*

$$\|(A-\lambda I)^{-1}\| \le C|\lambda|^{-1}, \quad -\frac{\pi}{2} - \omega_M \le \arg\lambda \le \frac{\pi}{2} - \omega_m, \quad |\lambda| \longrightarrow \infty. \tag{2.31}$$

Then for a function $u\ (x)$ *to be a solution of (2.29), which belongs to the space* $W^m_{p,\nu}$ $(0,b;E(A^m),E)$, *it is necessary and sufficient that*

$$u = \left[\sum_{k=1}^{d} e^{-x\omega_k A} g_k + \sum_{k=d+1}^{m} e^{-(b-x)\omega_k A} g_k\right], \tag{2.32}$$

where

$$g_k \in (E(A^m),E)_{(1+\nu)/mp,p}, \quad k = 1,\dots,d, \qquad g_k \in (E(A^m),E)_{1/mp,p}, \quad k = d+1,\dots,m. \tag{2.33}$$

3. A statement of the problem

In a Banach space E consider a degenerate nonlocal boundary value problem

$$Lu = -tu^{[2]}(x) + Au(x) + t^{1/2}B_1(x)u^{[1]}(x) + B_2(x)u(x) = f(x), \quad x \in (0,1), \tag{3.1}$$

$$\begin{gathered}L_1u = \alpha_0 t^{\theta_1} u^{[m_1]}(0) + \sum_{j=1}^{M_1} t^{\eta_{1j}} T_{1j} u(x_{1j}) = f_1,\\ L_2u = \beta_0 t^{\theta_2} u^{[m_2]}(1) + \sum_{j=1}^{M_2} t^{\eta_{2j}} T_{2j} u(x_{2j}) = f_2,\end{gathered} \tag{3.2}$$

where $x_{kj} \in [0,1]$, $\eta_{kj} = 1/2p(1-\nu)$, when $x_{kj} = 0$ and $\eta_{kj} = 1/2p$, when $x_{kj} \neq 0$, moreover,

$$\theta_1 = \frac{pm_1(1-\nu)+1}{2p(1-\nu)}, \qquad \theta_2 = \frac{pm_2+1}{2p}, \qquad u^{[i]} = \left(x^\nu \frac{d}{dx}\right)^i u(x), \\ \nu \geq 0,\ m_k \in \{0,1\},\ k = 1,2; \tag{3.3}$$

α_0, β_0 are complex numbers, t is a small parameter, and $f_k \in E_k = (E(A),E)_{\theta_k,p}$, $k = 1,2$, where A, $B_k(x)$, for $x \in [0,1]$, and T_{kj} are possible unbounded operators in E.

The function u that belongs to a space

$$\begin{aligned} W^{[2]}_{p,\nu}&(0,1;E(A),E) \\ &= \Big\{u;\ u \in L_p(0,1;E(A)),\ u^{[2]} \in L_p(0,1;E),\ \|u\|_{W^{[2]}_{p,\nu}(0.1;E(A),E)} \\ &\quad = \|Au\|_{L_p(0,1);E} + \left\|u^{[2]}\right\|_{L_p(0,1;E)} < \infty\Big\} \end{aligned} \tag{3.4}$$

and satisfies (3.1) a.e. on $(0,1)$ is said to be solution of (3.1).

Let

$$W^{[2]}_{p,\nu}(0,1;E(A),E,L_k) = \Big\{u; u \in W^{[2]}_{p,\nu}(0,1;E(A),E),\ L_k u = 0,\ k = 1,2\Big\}. \tag{3.5}$$

Remark 3.1. Under a substitution

$$y = (1-\nu)^{-1}x^{1-\nu}, \tag{3.6}$$

the spaces $L_p(0,1;E)$ and $W^{[2]}_{p,\nu}(0,1;E(A),E)$ are mapped isomorphically onto the weighted spaces $L_{p,\gamma}(0,b;E)$ and $W^2_{p,\gamma}(0,b;E(A),E)$, respectively, where

$$b = \frac{1}{1-\nu}, \qquad \gamma = (1-\nu)^{\nu/(1-\nu)}y^{\nu/(1-\nu)}. \tag{3.7}$$

Moreover, under the substitution (3.6), the problem (3.1)-(3.2) reduces to a nondegenerate BVP

$$\begin{gathered} Lu = -tu^{(2)}(y) + Au(y) + t^{1/2}\widetilde{B}_1(x)u^{(1)}(y) + \widetilde{B}_2(y)u(y) = f(y), \quad y \in (0,b), \\ L_1 u = \alpha_0 t^{\theta_1} u^{(m_1)}(0) + \sum_{j=1}^{M_1} t^{\eta_{1j}} T_{1j} u(y_{1j}) = f_1, \\ L_2 u = \beta_0 t^{\theta_2} u^{(m_2)}(b) + \sum_{j=1}^{M_2} t^{\eta_{2j}} T_{2j} u(y_{2j}) = f_2 \end{gathered} \tag{3.8}$$

in the weighted space $L_{p,\gamma}(0,b;E)$, where

$$\widetilde{B}_k = B_k\left((1-\nu)^{1/(1-\nu)}y^{1/(1-\nu)}\right), \quad y_{kj} = (1-\nu)^{-1}x_{kj}^{1-\nu}, \quad k = 1,2. \tag{3.9}$$

4. Homogeneous equations

Let us first consider a nonlocal boundary value problem

$$
\begin{gathered}
L_0(\lambda,t)u = -tu^{[2]}(x) + (A+\lambda)u(x) = 0, \\
L_1 u = \alpha_0 t^{\theta_1} u^{[m_1]}(0) = f_1, \qquad L_2 u = \beta_0 t^{\theta_2} u^{[m_2]}(1) = f_2,
\end{gathered}
\tag{4.1}
$$

where $m_k \in \{0,1\}$; α_k, β_k, δ_{kj} are complex numbers, A is, generally speaking, an unbounded operator in E.

Theorem 4.1. *Let A be a positive operator in a Banach space E for $\varphi \in (0,\pi]$, $0 \le \nu < 1 - 1/p$, $p \in (1,\infty)$, $0 < t \le t_0 < \infty$, $\alpha_0 \ne 0$, $\beta_0 \ne 0$. Then the problem (4.1) for $f_k \in E_k$, $|\arg\lambda| \le \pi - \varphi$, for sufficiently large $|\lambda|$ and t, has a unique solution u belongs to $W^{[2]}_{p,\nu}$ $(0,1;E(A),E)$, and the coercive uniform estimate*

$$
|\lambda| \|u\|_{L_p(0,1;E)} + \left\|tu^{[2]}\right\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \le M \sum_{k=1}^{2} \left(\left\|f_k\right\|_{E_k} + |\lambda|^{1-\theta_k} \left\|f_k\right\|_E \right) \tag{4.2}
$$

holds with respect to parameters t and λ.

Proof. Under the substitution (3.6), the problem (4.1) reduces to a nondegenerate problem

$$
L_0(\lambda,t)u = -tu^{(2)}(y) + (A+\lambda)u(y) = 0, \tag{4.3}
$$

$$
L_1 u = \alpha_0 t^{\eta_1} u^{(m_1)}(0) = f_1, \qquad L_2 u = \beta_0 t^{\eta_2} u^{(m_2)}(b) = f_2 \tag{4.4}
$$

in the weighted space $L_{p,\gamma}(0,b;E)$. Dividing both sides of (4.3) to $t > 0$, we obtain a boundary value problem

$$
L_0(\lambda,t)u = -u''(y) + t^{-1}(A+\lambda)u(y) = 0, \tag{4.5}
$$

$$
L_1 u = \alpha_0 t^{\theta_1} u^{(m_1)}(0) = f_1, \qquad L_2 u = \beta_0 t^{\theta_2} u^{(m_2)}(b) = f_2. \tag{4.6}
$$

Since A is the positive operator in E and $0 < t < t_0 < \infty$, then A/t is positive uniformly with respect to t, and for all $\lambda \in S_\varphi$, we have

$$
\left\| \left(\frac{A}{t} - \lambda I \right)^{-1} \right\| \le M \frac{t}{1 + t|\lambda|}. \tag{4.7}
$$

By virtue of condition (1) together with estimate (4.7) and by virtue of [4, Lemma 5.4.2/6], there is a holomorphic semigroup $e^{-x(t^{-1}A_\lambda)^{1/2}}$ for $x > 0$, which is strongly continuous for $x \ge 0$. Then by virtue of Lemma 2.8 an arbitrary solution of (4.5), for $|\arg\lambda| \le \pi - \varphi$, belonging to $W^2_{p,\gamma}(0,b;E(A),E)$ has the form

$$
u(y) = \left[e^{-yt^{-1/2}A_\lambda^{1/2}} g_1 + e^{-(b-y)t^{-1/2}A_\lambda^{1/2}} g_2 \right], \tag{4.8}
$$

where

$$A_\lambda = A+\lambda I, \qquad g_1 \in \left(E(A),E\right)_{1/2p(1-\nu),p}, \qquad g_2 \in \left(E(A),E\right)_{1/2p,p}. \tag{4.9}$$

Now taking into account the boundary conditions (4.6), we obtain algebraic linear equations with respect to g_1, g_2:

$$\begin{aligned} t^{1/2p(1-\nu)}\alpha_0 A_\lambda^{m_1/2}\left[(-1)^{m_1} g_1 + e^{-bt^{-1/2}A_\lambda^{1/2}} g_2\right] &= f_1, \\ t^{1/2p}\beta_0 A_\lambda^{m_2/2}\left[(-1)^{m_2} e^{-bt^{-1/2}A_\lambda^{1/2}} g_1 + g_2\right] &= f_2. \end{aligned} \tag{4.10}$$

The system (4.10) can be expressed as the following matrix-operator equation:

$$D(\lambda,t)\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{4.11}$$

where

$$D(\lambda,t) = \begin{bmatrix} (-1)^{m_1} t^{1/2p(1-\nu)}\alpha_0 A_\lambda^{m_1/2} & t^{1/2p(1-\nu)}\alpha_0 A_\lambda^{m_1/2} e^{-bt^{-1/2}A_\lambda^{1/2}} \\ (-1)^{m_2} t^{1/2p}\beta_0 A_\lambda^{m_2/2} e^{-bt^{-1/2}A_\lambda^{1/2}} & t^{1/2p}\beta_0 A_\lambda^{m_2/2} \end{bmatrix}. \tag{4.12}$$

Let $Q(\lambda,t)$ denote a determinant-operator of the matrix-operator $D(\lambda,t)$. It is clear that

$$Q(\lambda,t) = \alpha_0\beta_0 A_\lambda^{(m_1+m_2)/2} t^{(2-\nu)/2p(1-\nu)}\left[(-1)^{m_1} - (-1)^{m_2} e^{-2bt^{-1/2}A_\lambda^{1/2}}\right]. \tag{4.13}$$

Using the properties of positive operators and holomorphic semigroups (see [4, Lemma 5.4.2/6]) it is clear to see that for $|\arg\lambda| \le \pi-\varphi$, $|\lambda| \to \infty$ and $0 < t \le t_0$,

$$\left\|e^{-2t^{-1/2}A_\lambda^{1/2}}\right\| < 1. \tag{4.14}$$

The above estimate implies

$$\left\|\left[(-1)^{m_1} - (-1)^{m_2} e^{-2t^{-1/2}A_\lambda^{1/2}}\right]^{-1}\right\| \le C. \tag{4.15}$$

Due to the positivity of operator A in E and by (4.15) we obtain that operator $Q(\lambda,t)$ is invertible in $E^2 = E\times E$ and

$$Q^{-1}(\lambda,t) = \frac{1}{\alpha_0\beta_0} t^{(\nu-2)/2p(1-\nu)} A_\lambda^{-(m_1+m_2)/2} Q_0, \qquad Q_0 = \left[(-1)^{m_1} - (-1)^{m_2} e^{-2tb^{-1/2}A_\lambda^{1/2}}\right]^{-1}. \tag{4.16}$$

By virtue of estimate (4.15) it is clear that the operator $Q^{-1}(\lambda,t)$ is bounded uniformly with respect to the parameter λ, that is,

$$\left\|Q^{-1}(\lambda,t)\right\| \le C t^{(\nu-2)/2p(1-\nu)}. \tag{4.17}$$

Consequently, the system (4.10) has a unique solution for $|\arg\lambda| \le \pi - \varphi$, sufficiently large $|\lambda|$, and the solution can be expressed in the form

$$\begin{aligned} g_1 &= Q^{-1}\Big[t^{1/2p}\beta_0 A_\lambda^{m_2/2} f_1 - \alpha_0 t^{1/2p(1-\nu)} A_\lambda^{m_1/2} e^{-bt^{-1/2}A_\lambda^{1/2}} f_2\Big], \\ g_2 &= Q^{-1}\Big[(-1)^{m_1} t^{1/2p(1-\nu)}\alpha_0 A_\lambda^{m_1/2} f_2 - (-1)^{m_2} t^{1/2p}\beta_0 A_\lambda^{m_2/2} e^{-bt^{-1/2}A_\lambda^{1/2}} f_1\Big]. \end{aligned} \tag{4.18}$$

Substituting (4.16) and (4.18) into (4.8), we obtain a representation of the solution of the problem (4.5)-(4.6):

$$\begin{aligned} u(y) = {} & \frac{Q_0}{\alpha_0} t^{-1/2p(1-\nu)} A_\lambda^{-m_1/2}\big[e^{-yt^{-1/2}A_\lambda^{1/2}} - (-1)^{m_2} e^{-(2b-y)t^{-1/2}A_\lambda^{1/2}}\big] f_1 \\ & + \frac{Q_0}{\beta_0} t^{-1/2p} A_\lambda^{-m_2/2}\big[(-1)^{m_1} e^{-(b-y)t^{-1/2}A_\lambda^{1/2}} - e^{-(y+b)t^{-1/2}A_\lambda^{1/2}}\big] f_2. \end{aligned} \tag{4.19}$$

By virtue of the properties of the golomorphic semigroups [33, Section 1.13.1], in view of uniformly boundedness of Q_0, and by changing of variable, we obtain from (4.20) a uniformly estimate, with respect to t and λ,

$$\begin{aligned} & |\lambda|\,\|u\|_{L_{p,\gamma}} + \|tu''\|_{L_{p,\gamma}} + \|Au\|_{L_{p,\gamma}} \\ & \quad \le C\Big\{|\lambda|\Big[\big\|A_\lambda^{-m_1/2} e^{-zA_\lambda^{1/2}} f_1\big\|_{L_{p,\gamma}} + \big\|A_\lambda^{-m_2/2} e^{-(b-z)A_\lambda^{1/2}} f_2\big\|_{L_{p,\gamma}}\Big] \\ & \qquad + \big\|A_\lambda^{1-m_1/2} e^{-zA_\lambda^{1/2}} f_1\big\|_{L_{p,\gamma}} + \big\|A_\lambda^{1-m_2/2} e^{-(b-z)A_\lambda^{1/2}} f_2\big\|_{L_{p,\gamma}}\Big\}. \end{aligned} \tag{4.20}$$

By the properties of resolvent of positive operator A, we have

$$\begin{aligned} & |\lambda|\Big[\big\|A_\lambda^{-m_1/2} e^{-zA_\lambda^{1/2}} f_1\big\|_{L_{p,\gamma}} + \big\|A_\lambda^{-m_2/2} e^{-(b-z)A_\lambda^{1/2}} f_2\big\|_{L_{p,\gamma}}\Big] \\ & \quad \le |\lambda|\big\|A_\lambda^{-1}\big\|\Big[\big\|A_\lambda^{1-m_1/2} e^{-zA_\lambda^{1/2}} f_1\big\|_{L_{p,\gamma}} + \big\|A_\lambda^{1-m_2/2} e^{-(b-z)A_\lambda^{1/2}} f_2\big\|_{L_{p,\gamma}}\Big] \\ & \quad \le M\Big[\big\|A_\lambda^{1-m_1/2} e^{-zA_\lambda^{1/2}} f_1\big\|_{L_{p,\gamma}} + \big\|A_\lambda^{1-m_2/2} e^{-(b-z)A_\lambda^{1/2}} f_2\big\|_{L_{p,\gamma}}\Big]. \end{aligned} \tag{4.21}$$

By virtue of estimates (4.20), (4.21) and Theorem 2.6 we obtain

$$\begin{aligned} & |\lambda|\,\|u\|_{L_{p,\gamma}} + \|tu''\|_{L_{p,\gamma}} + \|Au\|_{L_{p,\gamma}} \\ & \quad \le M\Big[\big\|A_\lambda^{1-m_1/2} e^{-zA_\lambda^{1/2}} f_1\big\|_{L_{p,\gamma}} + \big\|A_\lambda^{1-m_2/2} e^{-(b-z)A_\lambda^{1/2}} f_2\big\|_{L_{p,\gamma}}\Big] \\ & \quad \le M\sum_{k=1}^{2}\Big[\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|\Big]. \end{aligned} \tag{4.22}$$

Then by virtue of estimate (4.22) and Remark 3.1 we obtain the estimate (4.2). □

5. Nonhomogeneous equations

Now consider a nonlocal boundary value problem for a nonhomogeneous equation with parameters t and λ in the space $L_p(0,1;E)$:

$$\begin{aligned} L_0(\lambda,t)u &= -tu^{[2]}(x)+(A+\lambda I)u(x) = f(x), \quad x\in(0,1),\\ L_1u &= \alpha_0 t^{\eta_1}u^{[m_1]}(0) = f_1, \qquad L_2u = \beta_0 t^{\eta_2}u^{[m_2]}(1) = f_2. \end{aligned} \tag{5.1}$$

Theorem 5.1. *Let the following conditions be satisfied:*

(1) *E is a Banach space that satisfies the multiplier condition with respect to p and weighted function $\gamma(y) = y^{\nu/(1-\nu)}$, $0\le \nu < 1-1/p$;*

(2) *A is an R-positive operator in E for $\varphi\in(0,\pi]$;*

(3) *$0<t\le t_0<\infty$ and $\alpha_0\neq 0$, $\beta_0\neq 0$. Then the operator $u\to D_0(\lambda,t)u = \{L_0(\lambda,t)u, L_{10}u, L_{20}u\}$ for $|\arg\lambda|\le \pi-\varphi$ and for sufficiently large $|\lambda|$ is an isomorphism from $W_{p,\gamma}^{[2]}(0,1;E(A),E)$ onto $L_p(0,1;E)+E_1+E_2$. Moreover, the coercive uniform estimate*

$$\begin{aligned} &|\lambda|\,\|u\|_{L_p(0,1;E)} + \left\|tu^{[2]}\right\|_{L_p(0,1:E)} + \|Au\|_{L_p(0,1:E)}\\ &\quad \le C\left[\|f\|_{L_p(0,1:E)} + \sum_{k=1}^{2}\left(\left\|f_k\right\|_{E_k} + |\lambda|^{1-\theta_k}\left\|f_k\right\|_E\right)\right] \end{aligned} \tag{5.2}$$

holds with respect to parameters λ and t.

Proof. By virtue of Remark 3.1, under the substitution (3.2), the problem (5.1) reduces to the nondependence problem

$$\begin{aligned} L_0(\lambda,t)u &= -tu^{(2)}(y)+(A+\lambda I)u(y) = f(y), \quad y\in(0,b),\\ L_1u &= \alpha_0 t^{\theta_1}u^{(m_1)}(0) = f_1, \qquad L_2u = \beta_0 t^{\theta_2}u^{(m_2)}(b) = f_2 \end{aligned} \tag{5.3}$$

in the weighted space $L_{p,\gamma}(0,b;E)$. It is clear that the problem (5.3) is equivalent to the problem

$$\begin{aligned} L_0(\lambda,t,D)u &= -u''(y)+\frac{1}{t}(A+\lambda I)u(y) = \frac{f(y)}{t}, \qquad x\in(0,b),\\ L_1u &= \alpha_0 t^{\theta_1}u^{(m_1)}(0) = f_1, \qquad L_2u = \beta_0 t^{\theta_2}u^{(m_2)}(b) = f_2. \end{aligned} \tag{5.4}$$

We have proved the uniqueness of the solution of the problem (5.3) in Theorem 4.1. Let us define

$$\overline{f}(y) = \begin{cases} f(y) & \text{if } y\in[0,b],\\ 0 & \text{if } y\notin[0,b]. \end{cases} \tag{5.5}$$

We now show that the solution of the problem (5.4) which belongs to the space $W_{p,\gamma}^2(0,b;E(A)E)$ can be represented as a sum $u(y) = u_1(y)+u_2(y)$, where u_1 is a restriction on $[0,b]$ of the solution u of the equation

$$L_0(\lambda,t)u=\overline{f}(y),\quad y\in R=(-\infty,\infty) \tag{5.6}$$

and u_2 is a solution of the problem

$$L_0(\lambda,t)u=0,\qquad L_{k0}u=f_k-L_{k0}u_1. \tag{5.7}$$

A solution of (5.6) is given by the formula

$$u(y)=F^{-1}L_0^{-1}(\lambda,t,\xi)F\overline{f}, \tag{5.8}$$

where $F\overline{f}$ is the Fourier transform of the function $\overline{f}$, and $L_0(\lambda,t,\xi)$ is a characteristic operator pencil of (5.6), that is,

$$L_0(\lambda,t,\xi)=(t\xi^2+\lambda)I+A. \tag{5.9}$$

It follows from the above expression that

$$\begin{aligned}
&|\lambda|\,\|u\|_{L_{p,\gamma}(R;E)}+\|u\|_{W^2_{p,\gamma,t}(R;E(A),E)}\\
&\quad=|\lambda|\,\|u\|_{L_{p,\gamma}(R;E)}+\|Au\|_{L_{p,\gamma}(R;E)}+\left\|tu^{(2)}\right\|_{L_{p,\gamma}(R;E)}\\
&\quad=\left\|F^{-1}\lambda L_0^{-1}(\lambda,t,\xi)F\overline{f}\right\|_{L_{p,\gamma}(R;E)}+\left\|F^{-1}AL_0^{-1}(\lambda,t,\xi)F\overline{f}\right\|_{L_{p,\gamma}(R;E)}\\
&\qquad+\left\|tF^{-1}\left[\xi^2L_0^{-1}(\lambda,t,\xi)F\overline{f}\right]\right\|_{L_{p,\gamma}(R;E)}.
\end{aligned} \tag{5.10}$$

By virtue of the R-positivity of operator A and due to that R-bounds are homogenous with respect to product by scalar and satisfy the triangle inequality (see, e.g., [14, Proposition 3.4]) for operator functions $H(\lambda,t,\xi)=\lambda L_0^{-1}(\lambda,t,\xi)$, $H_{k+1}(\lambda,t,\xi)=(t\xi)^{2k}A^{1-k}L_0^{-1}(\lambda,t,\xi)$, $k=0,1$, we have

$$\begin{aligned}
R\left(\left\{\xi\frac{d}{d\xi}H(\lambda,t,\xi)\right\}:\xi\in R\setminus\{0\}\right)&\le C,\\
R\left(\left\{\xi\frac{d}{d\xi}H_{k+1}(\lambda,t,\xi)\right\}:\xi\in R\setminus\{0\}\right)&\le C,\quad k=0,1.
\end{aligned} \tag{5.11}$$

Therefore, we obtain that the operator-valued functions $H\ (\lambda,t,\xi)$ and $H_{k+1}(\lambda,t,\xi)$ are uniformly R-bounded multipliers with respect to t,λ and R-bounds are independent of t and λ. Then in view of Definition 1.1, it follows that the operator-functions $H(\lambda,t,\xi)$, $H_{k+1}(\lambda,t,\xi)$ are uniformly Fourier multipliers in $L_{p,\gamma}(R;E)$. Then, by using the equality (5.10), we get

$$|\lambda|\,\|u\|_{L_{p,\gamma}}+\|Au\|_{L_{p,\gamma}}+\|tu''\|_{L_{p,\gamma}}\le C\|\overline{f}\|_{L_{p,\gamma}}. \tag{5.12}$$

That is (5.6) have a solution $u\in W^2_{p,\gamma}(R;E(A)E)$ and $u_1\in W^2_{p,\gamma}(0,b;E(A)E)$. By virtue of Theorem 2.7, we obtain

$$u_1^{(m_k)}(0)\in E_1,\qquad u_1^{(m_k)}(b)\in E_2. \tag{5.13}$$

Hence,

$$L_{0k}u_1 \in E_k, \quad k = 1,2. \tag{5.14}$$

Then by virtue of Theorem 4.1 the problem (5.7) has a unique solution $u_2(x)$ that belongs to the space $W^2_{p,\gamma}(0,b;E(A),E)$ for $|\arg\lambda| \leq \pi - \varphi$ and for sufficiently large $|\lambda|$. Moreover, for the solution of the problem (5.7), we have

$$\begin{aligned}
&|\lambda|\|u_2\|_{L_{p,\gamma}} + \|tu_2''\|_{L_{p,\gamma}} + \|Au_2\|_{L_{p,\gamma}} \\
&\quad\leq C\left[\sum_{k=1}^{2}\left(\|f_k - L_{0k}u_1\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k - L_{0k}u_1\|_E\right)\right] \\
&\quad\leq C\sum_{k=1}^{2}\left[\|f_k\|_{E_k} + \|L_{0k}u_1\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E + |\lambda|^{1-\theta_k}\|L_{0k}u_1\|_E\right. \\
&\quad= C\left[\left(\sum_{k=1}^{2}\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E\right) + \alpha_0 t^{\theta_1}\|u_1^{(m_1)}(0)\|_{E_1}\right. \\
&\qquad\left. + \alpha_0|\lambda|^{1-\theta_1}t^{\theta_1}\|u_1^{(m_1)}(0)\|_E + \beta_0 t^{\theta_2}\|u_1^{(m_2)}(b)\|_{E_2} + \beta_0|\lambda|^{1-\theta_2}t^{\theta_2}\|u_1^{(m_2)}(b)\|_E\right].
\end{aligned} \tag{5.15}$$

From (5.12), we obtain

$$|\lambda|\|u_1\|_{L_{p,\gamma}(0,b;E)} + \|tu_1^{(2)}\|_{L_{p,\gamma}(0,b:E)} + \|Au_1\|_{L_{p,\gamma}(0,b:E)} \leq C\|f\|_{L_{p,\gamma}(0,b;E)}. \tag{5.16}$$

Therefore, by Theorem 2.7 for $y = 0$, $k = 1$ and for $y = b$, $k = 2$, we have

$$t^{\theta_k}\|u_1^{(m_k)}(y)\|_{E_k} \leq C\|u_1\|_{W^2_{p,\gamma,t}(0,b;E(A),E)} \leq C\|f\|_{L_{p,\gamma}}. \tag{5.17}$$

By virtue of Theorem 2.7, for $h = |\lambda|^{-1}$ and $u \in W^2_{p,\gamma}(0,b;E(A),E)$, we get

$$|\lambda|^{1-\theta_k}t^{\theta_k}\|u^{(m_k)}(y)\|_E \leq C\left[\|u\|_{W^2_{p,\gamma,t}(0,b;E(A),E)} + |\lambda|\|u\|_{L_{p,\gamma}}\right]. \tag{5.18}$$

From (5.18), we obtain the estimate

$$|\lambda|^{1-\theta_k}t^{\theta_k}\|u_1^{(m_k)}(y)\|_E \leq C\left[\|tu_1^{(2)}\|_{L_{p,\gamma}} + \|Au_1\|_{L_{p,\gamma}} + |\lambda|\|u_1\|_{L_{p,\gamma}}\right] \leq C\|f\|_{L_{p,\gamma}} \tag{5.19}$$

uniformly with respect to t and λ. Hence from (5.15), (5.17), and (5.19), we have

$$\begin{aligned}
&|\lambda|\|u_2\|_{L_{p,\gamma}} + \|tu_2''\|_{L_{p,\gamma}} + \|Au_2\|_{L_{p,\gamma}} \\
&\quad\leq C\left[\|f\|_{L_{p,\gamma}} + \sum_{k=1}^{2}\left(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E\right)\right].
\end{aligned} \tag{5.20}$$

Then the estimates (5.16), (5.20), and Remark 3.1 imply (5.2). □

6. Coerciveness on the space variable and Fredholmness

Consider the problem (3.1)-(3.2).

THEOREM 6.1. *Let the following conditions be satisfied:*

(1) *E is a Banach space that satisfies the multiplier condition with respect to p and weighted function $\gamma(y) = y^{\nu/(1-\nu)}$, $0 \le \nu < 1 - 1/p$, $\theta_1 = m_1/2 + 1/2p(1-\nu)$, $\theta_2 = m_1/2 + 1/2p$, $p \in (1,\infty)$;*

(2) *A is an R-positive operator in E for $\varphi = \pi$ and $A^{-1} \in \sigma(E)$;*

(3) *$\alpha_0 \neq 0$, $\beta_0 \neq 0$, $0 < t \le t_0 < \infty$;*

(4) *for any $\varepsilon > 0$ and for almost all $x \in [0,1]$,*

$$\begin{aligned} \|B_1(x)u\| &\le \varepsilon \|A^{1/2}u\| + C(\varepsilon)\|u\|, \quad u \in E(A^{1/2}), \\ \|B_2(x)u\| &\le \varepsilon \|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A), \end{aligned} \tag{6.1}$$

for $u \in E(A^{1/2})$ the function $B_1(x)u$ and for $u \in D(A)$ the function $B_2(x)u$ are measurable on $[0,1]$ in E;

(5) *if $m_k = 0$, then $T_{kj} = 0$; if $m_k = 1$, then for $u \in (E(A),E)_{\sigma,p}$ and $\varepsilon > 0$,*

$$\|T_{kj}u\|_{E_k} \le \varepsilon \|u\|_{(E(A),E)_{\sigma,p}} + C(\varepsilon)\|u\|, \tag{6.2}$$

where $\sigma = 1/2p(1-\nu)$ if $x_{kj} = 0$, $\sigma = 1/2p$ if $x_{kj} \neq 0$.

Then

(a) *the coercive uniform estimate*

$$\begin{aligned} &\|tu^{[2]}\|_{L_p(0,1:E)} + \|Au\|_{L_p(0,1:E)} \\ &\quad \le C\left[\|Lu\|_{L_p(0,1;E)} + \sum_{k=1}^{2} \|L_k u\|_{(E(A),E)\theta_k,p} + \|u\|_{L_p(0,1;E)} \right] \end{aligned} \tag{6.3}$$

holds with respect to the parameter t for the solution u of the problem (3.1)-(3.2);

(b) *the operator $u \to D(t)u = \{Lu, L_1u, L_2u\}$ from $W_{p,\nu}^{[2]}(0,1;E(A),E)$ into*

$$L_p(0,1;E) + (E(A),E)_{\theta_1} + (E(A),E)_{\theta_2} \tag{6.4}$$

is bounded and Fredholm.

Proof. By Remark 3.1 it is sufficient to consider the problem (3.8) in $L_{p,\gamma}(0,b;E)$. The general case is reduced to the latter if the operator $A + \lambda_0 I$, for some sufficiently large $\lambda_0 > 0$, is considered instead of the operator A, and the operator $\widetilde{B}_2(x) - \lambda_0 I$ is considered instead of the operator $\widetilde{B}_2(x)$. Let $u \in W_{p,\gamma}^2(0,b;E(A),E)$ be a solution of the problem (3.8). Then $u(y)$ is a solution of the problem

$$\begin{aligned} &-\frac{d^2}{dy^2}u(y) + (A + \lambda I)u(y) = f(y) + \lambda u(y) - \widetilde{B}_1(y)\frac{d}{dy}u(y) - \widetilde{B}_2(y)u(y), \\ &L_{10}u = f_1 - \sum_{j=1}^{M_1} t^{1/2p(1-\nu)} T_{1j}u(y_{1j}), \qquad L_{20}u = f_2 - \sum_{j=1}^{M_2} t^{1/2p} T_{2j}u(y_{2j}), \end{aligned} \tag{6.5}$$

where L_{k0} are defined by (4.3). By Theorem 5.1 for some sufficiently large $\lambda_0 > 0$, we have the estimate

$$\begin{aligned}
&\|tu^{(2)}\|_{L_{p,\gamma}} + \|Au\|_{L_{p,\gamma}} \\
&\quad \le C\left[\|f + \lambda_0 u - t^{1/2}B_1 u^{(1)} - B_2 u\|_{L_{p,\gamma}} + \left\|f_1 - \sum_{j=1}^{M_1} t^{1/2p(1-\nu)} T_{1j} u(y_{1j})\right\|_{E_1}\right. \\
&\qquad \left. + \left\|f_2 - \sum_{j=1}^{M_2} t^{1/2p} T_{12} u(y_{2j})\right\|_{E_2}\right].
\end{aligned} \tag{6.6}$$

By virtue of condition (4) it follows that for $y \in (0,b)$,

$$\begin{aligned}
\|\widetilde{B}_1(y)u^{(1)}\| &\le \varepsilon\|A^{1/2}u^{(1)}(y)\| + C(\varepsilon)\|u^{(1)}(y)\|, \\
\|\widetilde{B}_2(y)u(y)\| &\le \varepsilon\|Au(y)\| + C(\varepsilon)\|u(y)\|,
\end{aligned} \tag{6.7}$$

hence

$$\begin{aligned}
\|\widetilde{B}_1 u^{(1)}\|_{L_{p,\gamma}} &\le \varepsilon\|A^{1/2}u^{(1)}\|_{L_{p,\gamma}} + C(\varepsilon)\|u^{(1)}\|_{L_{p,\gamma}}, \\
\|\widetilde{B}_2 u\|_{L_{p,\gamma}} &\le \varepsilon\|Au\|_{L_{p,\gamma}} + C(\varepsilon)\|u\|_{L_{p,\gamma}}, \quad \varepsilon > 0.
\end{aligned} \tag{6.8}$$

By virtue of Theorem 2.3, we have

$$\|t^{1/2}A^{1/2}u^{(1)}(y)\| \le c\|u\|_{W^2_{p,\gamma,t}(0,b;E(A),E)}. \tag{6.9}$$

Moreover, by virtue of Theorem 2.3 (by choosing A an identity operator in E) there exists $C > 0$ such that for $0 < t \le t_0$ and $0 < h \le h_0$,

$$\|t^{1/2}Du\|_{L_{p,\gamma}(0,b;E)} \le C\left(h^{1/2}\|tu^{(2)}\|_{L_{p,\gamma}} + h^{-1/2}\|u\|_{L_{p,\gamma}(0,b;E)}\right). \tag{6.10}$$

Therefore, we can conclude that

$$\begin{aligned}
\|t^{1/2}\widetilde{B}_1 u^{(1)}\|_{L_{p,\gamma}} &\le \varepsilon\|t^{1/2}A^{1/2}u^{(1)}\|_{L_{p,\gamma}} + C(\varepsilon)\|t^{1/2}u^{(1)}\|_{L_{p,\gamma}} \\
&\le C\left(\varepsilon + C(\varepsilon)h^{1/2}\right)\|u\|_{W^2_{p,\gamma,t}(0,b;E(A),E)} + CC(\varepsilon)h^{-1/2}\|u\|_{L_{p,\gamma}(0,b;E)}.
\end{aligned} \tag{6.11}$$

With a suitable choice of ε and h, $C(\varepsilon + C(\varepsilon)h^{1/2})$ can be made arbitrarily small, hence this proves that for every $\varepsilon > 0$ there exists $C(\varepsilon)$ independent of u and t such that

$$\|t^{1/2}\widetilde{B}_1 u^{(1)}\|_{L_{p,\gamma}} \le \varepsilon\|u\|_{W^2_{p,\gamma,t}(0,b;E(A),E)} + C(\varepsilon)\|u\|_{L_{p,\gamma}(0,b;E)}. \tag{6.12}$$

Moreover, it is clear that

$$\begin{aligned}
\|\widetilde{B}_2 u\|_{L_{p,\gamma}} &\le \varepsilon\|Au\|_{L_{p,\gamma}} + C(\varepsilon)\|u\|_{L_{p,\gamma}} \\
&\le \varepsilon\|u\|_{W^2_{p,\gamma,t}(0,b;E(A),E)} + C(\varepsilon)\|u\|_{L_{p,\gamma}(0,b;E)}.
\end{aligned} \tag{6.13}$$

In (6.6) it remains to estimate the terms

$$t^{1/2p(1-\nu)}||T_{kj}u(0)||_{E_k}, \qquad t^{1/2p}||T_{kj}u(y_{kj})||_{E_k} \tag{6.14}$$

with $y_{kj} \neq 0$; therefore, we have to prove that for every $\varepsilon > 0$ there exists $C(\varepsilon)$ such that

$$\begin{gathered} t^{1/2p(1-\nu)}||T_{kj}u(y_{kj})||_{E_k} \leq \varepsilon\|u\|_{W^2_{p,\gamma,t}(0,b;E(A),E)} + C(\varepsilon)\|u\|_{L_{p,\gamma}(0,b;E)}, \\ t^{1/2p}||T_{kj}u(y_{kj})||_{E_k} \leq \varepsilon\|u\|_{W^2_{p,\gamma,t}(0,b;E(A),E)} + C(\varepsilon)\|u\|_{L_{p,\gamma}(0,b;E)}. \end{gathered} \tag{6.15}$$

By hypothesis (5) for every $\delta > 0$ if $y_{kj} = 0$, we have

$$||u(y_{kj})||_{E_k} \leq \delta||u(y_{kj})||_{(E(A),E)_{1/2p(1-\nu),p}} + C(\delta)||u(y_{kj})||_{E}, \tag{6.16}$$

and if $y_{kj} \neq 0$, we have

$$||u(y_{kj})||_{E_k} \leq \delta||u(y_{kj})||_{(E(A),E)_{(1/2p),p}} + C(\delta)||u(y_{kj})||_{E}. \tag{6.17}$$

From Theorem 2.7, it follows that

$$t^{1/2p(1-\nu)}||u(0)||_{(E(A),E)_{\sigma,p}} \leq C[||tu^{(2)}||_{L_{p,\gamma}} + \|Au\|_{L_{p,\gamma}}], \tag{6.18}$$

and if $y_{kj} \neq 0$,

$$\begin{gathered} t^{1/2p}||u(y_{kj})||_{(E(A),E)_{\sigma,p}} \leq C\Big[||tu^{(2)}||_{L_{p,\gamma}} + \|Au\|_{L_{p,\gamma}}\Big], \\ t^{1/2p(1-\nu)}||u(0)||_{E} \leq C\Big[h^{1-1/2p(1-\nu)}||tu^{(2)}||_{L_{p,\gamma}} + h^{-1/2p(1-\nu)}\|u\|_{L_{p,\gamma}}\Big], \end{gathered} \tag{6.19}$$

and if $y_{kj} \neq 0$,

$$t^{1/2p}||u(y_{kj})||_{E} \leq C\Big[h^{1-1/2p}||tu^{(2)}||_{L_{p,\gamma}} + h^{-1/2p}\|u\|_{L_{p,\gamma}}\Big]. \tag{6.20}$$

Therefore, if $y_{kj} = 0$, we have

$$\begin{aligned} &t^{1/2p(1-\nu)}||T_{kj}u(y_{kj})||_{E_k} \\ &\quad\leq \delta t^{1/2p(1-\nu)}||u(y_{kj})||_{(E(A),E)_{1/2p(1-\nu),p}} + C(\delta)t^{1/2p(1-\nu)}||u(y_{kj})||_{E} \\ &\quad\leq C(\delta + C(\delta)h^{1-1/2p(1-\nu)})||tu^{(2)}||_{L_{p,\gamma}} + C\delta\|Au\|_{L_{p,\gamma}} + CC(\delta)h^{1-1/2p(1-\nu)}\|u\|_{L_{p,\gamma}}. \end{aligned} \tag{6.21}$$

By choosing a suitable δ and a suitable h, the quantities $(\delta + C(\delta)h^{1-1/2p(1-\nu)})$ and $C\delta$ can be made arbitrary small, hence the requested inequality (6.15) holds for case $y_{kj} = 0$. In the same way we can obtained the inequality for case $y_{kj} \neq 0$. Then in view of inequalities (6.12), (6.13), and (6.15) from (6.6), we get (6.3).

(b) The operator $D(t)$ can be rewritten in the form

$$D(t) = D_0(\lambda_0, t) + L_1, \tag{6.22}$$

where

$$D_0(\lambda_0, t)u = (L_0(\lambda, t)u, L_{10}, L_{20}) \tag{6.23}$$

are defined by (5.3) and

$$\begin{aligned} D_1(\lambda_0, t)u = \Bigg(& -\lambda_0 u(y) + t^{1/2} B_1(y) u^{(1)}(y) \\ & + B_2(y)u(y), \sum_{j=1}^{M_1} t^{1/2p(1-\nu)} T_{1j} u(y_{1j}), \sum_{j=1}^{M_2} t^{1/2p} T_{2j} u(y_{2j}) \Bigg). \end{aligned} \tag{6.24}$$

We can conclude from Theorem 5.1 that operator $D_0(\lambda_0, t)$ has an inverse from $L_p(0,1;E) + E_1 + E_2$ to

$$W_{p,\gamma}^2(0,b;E(A),E). \tag{6.25}$$

From estimates (6.12), (6.13), and (6.21) in view of Theorem 2.4 and [4, Lemma 1.2.7/2], it follows that the operator D_1 from $W_{p,\gamma}^2(0,b;E(A),E)$ into $L_{p,\gamma}(0,b;E) + E_1 + E_2$ is compact. Then in view of Theorem 5.1 and by the perturbation theory of linear operators [34, Section 14, Theorem 14.1] it follows that the operator $D(t)$ from $W_{p,\gamma}^2(0,b;E(A),E)$ into $L_{p,\gamma}(0,b;E) + E_1 + E_2$ is Fredholm operator. Then by Remark 3.1 we obtain assertion of Theorem 6.1. □

7. Nonlocal boundary value problems for degenerate elliptic equations with parameters

The Fredholm property of boundary value problems for elliptic equations with parameters in smooth domains was studied in [35, 36], also for nonsmooth domains was treated in [24, 37–39].

Let $G \subset R^m$, $m \geq 2$, be a bounded domain with an $(m-1)$-dimensional boundary ∂G which locally admits rectification. Let us consider a nonlocal boundary value problem on cylindrical domain $\Omega = [0,1] \times G$ for a degenerate elliptic differential equation with parameters

$$Lu = -tD_x^{[2]}u(x,y) - \sum_{k,j=1}^{m} a_{kj}(y)D_kD_ju(x,y) + t^{1/2}a(x,y)D_x^{[1]}u(x,y)$$

$$+\sum_{j=1}^{m} a_j(x,y)D_ju(x,y) + a_0(x,y)u(x,y) = f(x,y), \quad (x,y)\in\Omega,$$

$$L_1u = \alpha_0 t^{\theta_1}u^{[m_1]}(0,y) + \sum_{j=1}^{M_1} t^{\eta_j}T_{1j}u(x_{1j},y) = f_1(y), \tag{7.1}$$

$$L_2u = \beta_0 t^{\theta_2}u^{[m_2]}(1,y) + \sum_{j=1}^{M_2} t^{\eta_j}T_{2j}u(x_{2j},y) = f_2(y),$$

$$L_0u = \sum_{j=1}^{m} c_j(y')\frac{\partial}{\partial y_j}u(x,y') + c_0(y')u(x,y') = 0, \quad x\in(0,1),\ y'\in\partial G,$$

where $D^{[i]}u(x) = (x^{\nu}(d/dx))^i u(x)$, $\nu \geq 0$, $D_j = -i(\partial/\partial y_j)$, $m_k \in \{0,1\}$, α_k, β_k are complex numbers, $y = (y_1,\dots,y_m)$, T_{kj} are possible unbounded operators in $L_q(G)$, $x_{kj}\in[0,1]$; $\eta_j = 1/2p(1-\nu)$ when $x_{kj} = 0$ and $\eta_j = 1/2p$, when $x_{kj}\neq 0$; moreover,

$$\theta_1 = \frac{pm_1(1-\nu)+1}{2p(1-\nu)}, \qquad \theta_2 = \frac{pm_2+1}{2p}. \tag{7.2}$$

Let $r = \operatorname{ord} L_0$.

Theorem 7.1. *Let the following conditions be satisfied:*

(1) $a_{kj}\in C(\overline{G})$, $a_j, a_0 \in L_\infty(\overline{G})$, $c_0\in C(\overline{G})$, $a, c_j \in C^1(\overline{G})$, $\partial G\in C^\infty$;

(2) $c_0 \in C'(\overline{G})$ *for* $r = 1$ *and* $c_0\in C^2(\overline{G})$, $c_0(y')\neq 0$, $y'\in\partial G$, *for* $r = 0$;

(3) *for* $y\in G$, $\sigma\in R^m$, $\arg\lambda = \pi$, $|\sigma| + |\lambda| \neq 0$, $\lambda + \sum_{k,j=1}^{m} a_{kj}(y)\sigma_k\sigma_j \neq 0$;

(4) *for the tangent vector* σ' *and the normal vector* σ *to* ∂G *at the point* $y'\in\partial G$ *the following boundary value problem holds:*

$$\left[\lambda + \sum_{k,j=1}^{m} a_{kj}(y')\left(\sigma_k' - i\sigma_k\frac{d}{d\tau}\right)\left(\sigma_j' - i\sigma_j\frac{d}{d\tau}\right)\right]u(\tau) = 0, \quad \tau > 0,\ \lambda\leq 0,$$
$$\sum_{j=1}^{m} c_j(y')\left(\sigma_j' - i\sigma_j\frac{d}{d\tau}\right)u(\tau)\,|_{\tau=0} = d, \quad r = 1, \tag{7.3}$$

$$u(0) = d \quad \text{for } r = 0; \tag{7.4}$$

it is required that, for $r = 1$, *problem (7.3) (for* $r = 0$ *problem (7.3)-(7.4)) has one and only one solution, tending to zero including all its derivatives as* $y\to\infty$ *for any numbers* $d\in\mathbb{C}$;

(5) $0\leq\nu\leq 1-1/p$, $\alpha_0\neq 0$, $\beta_0\neq 0$, $0 < t\leq t_0 < \infty$;

(6) *if* $m_k = 0$, *then* $T_{kj} = 0$; *if* $m_k = 1$, *then for* $\varepsilon > 0$, $u \in B_{q,p}^{2-1/p(1-\nu)}(G; Lu = 0)$

$$||T_{1j}u||_{B_{q,p}^{1-1/p(1-\nu)}(G)} \leq \varepsilon \|u\|_{B_{q,p}^{2-1/p(1-\nu)}(G)} + c(\varepsilon)\|u\|_{L_q(G)} \tag{7.5}$$

and for $u \in B_{q,p}^{2-1/p}(G; Lu = 0)$,

$$||T_{2j}u||_{B_{q,p}^{1-1/p}(G)} \leq \varepsilon \|u\|_{B_{q,p}^{2-1/p}(G)} + c(\varepsilon)\|u\|_{L_q(G)}, \tag{7.6}$$

where $r < 1 - 1/p(1-\nu) - 1/p$, $q \in (1, \infty)$, $p \in (1, \infty)$.

Then

(a) *the coercive uniform estimate for the solution* $u \in W_{q,p,\nu}^{[2]}(\Omega)$ *of the problem (7.1)*

$$\begin{aligned} &||tD_x^{[2]}u||_{L_{q,p}(\Omega)} + \sum_{k=1}^{m} ||D_k^2 u||_{L_{q,p}(\Omega)} + \|u\|_{L_{q,p}(\Omega)} \\ &\quad \leq C\left[\|Lu\|_{L_{q,p}(\Omega)} + ||L_1 u||_{B_{q,p}^{2-m_1-1/p(1-\nu)}(G)} + ||L_2 u||_{B_{q,p}^{2-m_2-1/p}(G)} + \|u\|_{L_{q,p}(\Omega)} \right] \end{aligned} \tag{7.7}$$

holds with respect to the parameter t;

(b) *the operator* $u \to Q(t)u = \{Lu, L_1u, L_2u\}$ *from* $W_{q,p,\nu}^{[2]}(\Omega; L_0 u = 0)$ *into*

$$L_{q,p}(\Omega) \times B_{q,p}^{2-m_1-1/p(1-\nu)}(G, L_0 u = 0) \times B_{q,p}^{2-m_2-1/p}(G, L_0 u = 0) \tag{7.8}$$

is bounded uniformly with respect to the parameter t *and Fredholm.*

Proof. Let $E = L_q(G)$. Then by virtue of Theorem 2.2 the condition (1) of Theorem 6.1 is satisfied. Consider the following operator A which is defined by the equalities:

$$D(A) = W_q^2(G; L_0 u = 0), \qquad Au = -\sum_{k,j=1}^{m} a_{kj}(y) D_k D_j u(y). \tag{7.9}$$

For $x \in [0,1]$, also consider operators

$$B_1(x)u = a(x,y)u(y), \qquad B_2(x)u = \sum_{j=1}^{m} a_j(x,y) D_j u(y) + a_0(x,y)u(x,y). \tag{7.10}$$

Then the problem (7.1) can be rewritten in the form

$$\begin{gathered} -tD^{[2]}u(x) + Au(x) + t^{1/2}B_1(x)D^{[1]}u(x) + B_2(x)u(x) = f(x), \quad x \in (0,1), \\ L_1 u = \alpha_0 t^{\theta_1} u^{[m_1]}(0) + \sum_{j=1}^{M_1} t^{\eta_j} T_{1j} u(x_{1j}) = f_1, \\ L_2 u = \beta_0 t^{\theta_2} u^{[m_2]}(1) + \sum_{j=1}^{M_2} t^{\eta_j} T_{2j} u(x_{2j}) = f_2, \end{gathered} \tag{7.11}$$

where $u(x) = u(x, \cdot)$, $f(x) = f(x, \cdot)$ are functions with values in the Banach space $E = L_q(G)$, $f_k = f_k(\cdot)$.

Let us apply Theorem 6.1 to the problem (7.11). In view of Theorem 2.2 condition (1) of Theorem 6.1 holds. By virtue of [14, Theorem 8.2] the operator $A+\mu I$ for sufficiently large $\mu \geq 0$ is R-positive in L_q. Moreover, it is known that an embedding $W_q^2(G) \subset L_q(G)$ is compact (see, e.g., Triebel [33, Theorem 3.2.5]), then due to the positivity of $A+\mu I$ in $L_q(G)$ we obtain that $(A+\mu I)^{-1} \in \sigma_\infty(L_q(G))$. Therefore, the condition (2) of Theorem 6.1 is fulfilled. Condition 3 of Theorem 6.1 coincides with condition (5). By virtue of condition (1) of Theorem 7.1 the operators $B_1(x)$ in $L_q(G)$ and $B_2(x)$ from $W_q^1(G)$ to $L_q(G)$ are bounded. By virtue of condition (1), we have

$$\left\|B_1(x)u\right\|_{L_q} \leq \sup|a| \, \|u\|_{L_q}. \tag{7.12}$$

On the other hand, since the embedding $W_q^1(G) \subset L_q(G)$ is compact, then the operator $B_1(x)$ from $W_q^1(G)$ into $L_q(G)$ and, consequently, from $E(A^{1/2})$ into $L_q(G)$, is compact. Then by reasoning as [4, Lemma 1.2.1] we obtain that the operator $B_1(x)$ satisfies the condition (4) of Theorem 6.1. In a similar way we prove that the operator $B_2(x) - \mu I$ satisfies the condition (4) of Theorem 6.1 too. Using interpolation properties of Sobolev spaces (see, e.g., [21, Section 4]), it is clear to see that condition (5) of Theorem 6.1 is fulfilled too. By virtue of [21, Section 4.3.3], we have

$$(E(A),E)_{\theta_k,p} = \left(W_q^2(G,L_0),L_q(G)\right)_{\theta_k,p} = B_{q,p}^{2(1-\theta_k)}(G;L_0). \tag{7.13}$$

Hence, the condition (5) of Theorem 6.1 follows from the condition (6). □

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