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Polynomial hulls and proper analytic disks

Armen Edigarian

Abstract. We show how to construct the Perron–Bremermann function by using proper analytic disks. We apply this result to the polynomial hull of a compact set K defined on the boundary of the unit ball.

1. Introduction

Let $K\subset\mathbb{C}^n$ be a compact set. The polynomial hull of K is defined as

$$\widehat{K}=\Big\{z\in\mathbb{C}^n:|p(z)|\le\sup_K|p| \text{ for any polynomial } p\Big\}. \tag{1}$$

Note that if $f\colon\mathbb{D}\to\mathbb{C}^n$ is a bounded holomorphic map from the unit disk $\mathbb{D}$ such that $\tilde f(\zeta)\in K$ for a.e. $\zeta\in\mathbb{T}$ then $f(0)\in\widehat{K}$. Here, $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$, and $\tilde f$ denotes the non-tangential values of f on $\mathbb{T}$ (see e.g. [12]). Polynomial hulls are not always constructed by analytic disks whose boundaries lie a.e. in the set. Indeed, the well-known examples of G. Stolzenberg [15] and J. Wermer [16] give a compact subset K of the unit sphere $\partial\mathbb{B}_n\subset\mathbb{C}^n$ such that $\widehat{K}\setminus K$ is non-trivial, however, for any $z_0\in\widehat{K}\setminus K$ there does not exist an analytic disk $f\colon\mathbb{D}\to\mathbb{C}^n$ such that $\tilde f\in K$ a.e. on $\mathbb{T}$ and $f(0)=z_0$. On the other hand, in some special cases it is true (see [14] and the references therein). Moreover, by E. Poletsky it holds approximately.

Theorem 1.1. (See [10], Theorem 7.1) *Let $D\subset\mathbb{C}^n$ be a Runge domain and let $K\subset D$ be a compact set. Then $z_0\in\widehat{K}$ if and only if for any $\varepsilon>0$ there exists an analytic disk $f\colon\mathbb{D}\to D$ such that $f(0)=z_0$ and*

$$\sigma(\{\zeta\in\mathbb{T}:\tilde f(\zeta)\in K_\varepsilon\})>1-\varepsilon, \tag{2}$$

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where σ *denotes the normalized Lebesgue measure on* $\mathbb{T}$ *and*

$$K_\varepsilon = \{z \in \mathbb{C}^n : \operatorname{dist}(z, K) < \varepsilon\}$$

is the ε*-neighborhood of* K*.*

Our aim is to study Theorem 1.1 in case $K \subset \partial D$, where D is a wide class of domains. We have the following result.

Theorem 1.2. *Let* $D \Subset \mathbb{C}^n$ *be a B-regular domain with* C^∞ *boundary and let* $K \subset \partial D$ *be a compact set. Assume that* $\widehat{K} \subset \overline{D}$*. Then* $z_0 \in \widehat{K} \cap D$ *if and only if for any* $\varepsilon > 0$ *there exists a proper analytic disk* $f : \mathbb{D} \to D$ *such that* $f(0) = z_0$ *and*

$$\sigma(\{\zeta \in \mathbb{T} : \tilde{f}(\zeta) \in K_\varepsilon \cap \partial D\}) > 1 - \varepsilon.$$

Let $D \Subset \mathbb{C}^n$ be a domain. Following N. Sibony (see [13]) we say that D is *B-regular* if for any continuous function φ on ∂D there is a continuous plurisubharmonic function $v \in \operatorname{PSH}(D) \cap C(\overline{D})$ such that $u = \varphi$ on ∂D. In particular, the balls in $\mathbb{C}^n$ are B-regular. For properties of B-regular domains see [1] and [13].

The proof of Theorem 1.2 is based on Poletsky's theory [10]. Let $D \Subset \mathbb{C}^n$ be a domain and let u be any function on D. For any $x \in \overline{D}$ we put

$$u^*(x) = \limsup_{D \ni y \to x} u(y).$$

In particular, u^* is an upper semicontinuous function on $\overline{D}$. For every function $\varphi \colon \overline{D} \to \mathbb{R}$ bounded from above, the function

$$P_\varphi = \sup\{v \in \operatorname{PSH}(D) : v^* \le \varphi\}$$

is called the *Perron–Bremermann envelope* of φ on D. Actually, in the literature (see e.g. [9]) appears the case when φ is defined only on ∂D (then we may extend φ to D by taking $\varphi = M$, where M is a sufficiently big real number) or the case when φ is defined only on D (then we may take $\varphi = M$ on ∂D, where M is again a sufficiently big real number). In this way, we get 'classical' definitions. For a subset $A \subset \partial D$ we put $\omega(\,\cdot\,, A, D) = P_{-\chi_A}$, where χ_A is the characteristic function of A. Note that for an upper semicontinuous function φ defined in D or on ∂D we have that P_φ is plurisubharmonic in D. One can easily show (see e.g. [10], the proof of Theorem 7.1) that for a Runge domain $D \subset \mathbb{C}^n$ and a compact set $K \subset D$ we have

$$\widehat{K} = \{z \in D : \omega(z, K, D) = -1\}. \tag{3}$$

The fundamental idea of Poletsky's theory is to give a description of some extremal plurisubharmonic functions by analytic disks. In particular, we have the equality

$$E_\varphi = P_\varphi \tag{4}$$

in case φ is an upper semicontinuous function defined inside a domain $D \subset \mathbb{C}^n$, where

$$E_\varphi(z) = \inf\left\{ \frac{1}{2\pi}\int_0^{2\pi} \varphi(f(e^{i\theta}))\,d\theta : f \in \mathcal{O}(\overline{\mathbb{D}}, D) \text{ and } f(0)=z \right\}. \tag{5}$$

Here, $\mathcal{O}(\overline{\mathbb{D}}, D)$ denotes the set of all holomorphic mappings $f\colon U_f \to D$, where U_f is an open neighborhood of $\overline{\mathbb{D}}$ (which may depend on f). In this way, we get a description of the polynomial hull by analytic disk. So, in order to prove Theorem 1.2 we have to prove boundary versions of the equality (3) (see Theorem 3.1 below) and the equality (4) stated as follows.

Theorem 1.3. *Let $D \Subset \mathbb{C}^n$ be a B-regular domain, $n \geq 2$. Then for any upper semicontinuous function $\varphi\colon \partial D \to [-\infty, \infty)$ we have $P_\varphi = \widetilde{E}_\varphi$ on D, where*

$$\widetilde{E}_\varphi(z) = \inf\left\{ \frac{1}{2\pi}\int_0^{2\pi} \varphi(\tilde{f}(e^{i\theta}))\,d\theta : f \in \mathcal{O}(\mathbb{D}, D) \textit{ is proper and } f(0)=z \right\}. \tag{6}$$

In case D is a strictly pseudoconvex domain with C^k boundary it is enough to take the infimum only over analytic disks which are $C^{k-\varepsilon}$ regular on $\overline{\mathbb{D}}$ for every $\varepsilon > 0$.

A weaker version of Theorem 1.3 (with proper analytic disks replaced by almost proper) was presented in Poletsky's paper [10]. In Remark 2.5 we will show that, in general, in the equality (6) one cannot replace 'inf' by 'min'.

2. Proof of Theorem 1.3

Recall the following result of F. Forstnerič and J. Globevnik (see [7] and [8]).

Theorem 2.1. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain, $n \geq 2$. Fix a metric d on D which induces the topology of D. Then for any $h \in \mathcal{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, D)$, for any $r \in (0,1)$, for any $\varepsilon > 0$, and for any finite set $A \subset \mathbb{D}$, there exists a proper analytic disk $f\colon \mathbb{D} \to D$ such that:*

(1) *$d(f(\zeta), h(\zeta)) < \varepsilon$ for any $|\zeta| < r$;*

(2) *$f(\zeta) = h(\zeta)$ for any $\zeta \in A$.*

Moreover, if D is a strictly pseudoconvex domain with C^k boundary, then there is an f as above that is of class $C^{k-\varepsilon}$ on $\overline{\mathbb{D}}$ for every $\varepsilon > 0$.

Proof of Theorem 1.3. Note that there exists a sequence of continuous functions φ_n defined on ∂D such that $\varphi_n \searrow \varphi$. For any two functions ψ_1 and ψ_2 defined on ∂D such that $\psi_1 \le \psi_2$, we have $P_{\psi_1} \le P_{\psi_2}$ and $\widetilde{E}_{\psi_1} \le \widetilde{E}_{\psi_2}$. So, $P_{\varphi_1} \ge P_{\varphi_2} \ge ... \ge P_\varphi$ and $\widetilde{E}_{\varphi_1} \ge \widetilde{E}_{\varphi_2} \ge ... \ge \widetilde{E}_\varphi$.

Let us show that $\lim_{n\to\infty} \widetilde{E}_{\varphi_n} = \widetilde{E}_\varphi$. Indeed, we know that $\lim_{n\to\infty} \widetilde{E}_{\varphi_n} \ge \widetilde{E}_\varphi$. Now fix $z_0 \in D$ and take any $a > E_\varphi(z_0)$. Then there exists a proper analytic disk $f : \mathbb{D} \to D$ such that $f(0) = z_0$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(\tilde{f}(e^{i\theta}))\, d\theta < a.$$

We have

$$\lim_{n\to\infty} \widetilde{E}_{\varphi_n}(z_0) \le \limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \varphi_n(\tilde{f}(e^{i\theta}))\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \limsup_{n\to\infty} \varphi_n(\tilde{f}(e^{i\theta}))\, d\theta < a.$$

Hence, $\lim_{n\to\infty} \widetilde{E}_{\varphi_n} \le \widetilde{E}_\varphi$.

Similarly, let us show that $\lim_{n\to\infty} P_{\varphi_n} = P_\varphi$. Put $u = \lim_{n\to\infty} P_{\varphi_n}$. Then $u \in \mathrm{PSH}(D)$ (or, $u \equiv -\infty$) and $P_\varphi \le u$. Moreover, $u \le P_{\varphi_n}$ for any $n \in \mathbb{N}$. Therefore, $u^* \le \varphi_n$ on ∂D for any $n \in \mathbb{N}$. Hence, letting $n \to \infty$ we get $u^* \le \varphi$ on ∂D, and from the definition of the Perron–Bremermann envelope $u \le P_\varphi$. Therefore, $u = P_\varphi$.

So, without loss of generality, we may assume that φ is continuous on ∂D. Since D is B-regular, there exists a plurisubharmonic function v on D, continuous on $\overline{D}$ such that $v = -\varphi$ on ∂D. By (5) for any $z \in D$ we have

$$P_{-v}(z) = \inf\left\{ -\frac{1}{2\pi}\int_0^{2\pi} v(f(e^{i\theta}))\, d\theta : f \in \mathcal{O}(\overline{\mathbb{D}}, D) \text{ and } f(0) = z \right\}. \tag{7}$$

Let us show that $P_\varphi = P_{-v}$. Indeed, for any plurisubharmonic function u on D such that $u^* \le \varphi = -v$ on ∂D, by the maximum principle for the plurisubharmonic function $u + v$ we have $u \le -v$ on D. Hence, $P_\varphi \le P_{-v}$. On the other hand, if u is a plurisubharmonic function on D such that $u \le -v$ on D, then $u^* \le -v = \varphi$ on ∂D. So, $P_{-v} \le P_\varphi$.

Fix $z_0 \in D$ and $\varepsilon > 0$. By (7) there exists a holomorphic mapping $f \in \mathcal{O}(\overline{\mathbb{D}}; D)$ such that $f(0) = z_0$ and

$$-\frac{1}{2\pi}\int_0^{2\pi} v(f(e^{i\theta}))\, d\theta < P_{-v}(z_0) + \varepsilon. \tag{8}$$

Then there exists an $r \in (0,1)$ such that

$$-\frac{1}{2\pi}\int_0^{2\pi} v(f(re^{i\theta}))\, d\theta < P_{-v}(z_0) + 2\varepsilon. \tag{9}$$

By Theorem 2.1 there exists a proper analytic disk $g\colon \mathbb{D}\to D$ such that $g(0)=f(0)=z_0$ and

$$-\frac{1}{2\pi}\int_0^{2\pi} v(g(re^{i\theta}))\,d\theta < P_{-v}(z_0)+3\varepsilon. \tag{10}$$

Since $v\circ g$ is subharmonic on $\mathbb{D}$, the averages $(1/2\pi)\int_0^{2\pi} v(g(r^{i\theta}))\,d\theta$ are increasing with respect to $r\in(0,1)$. We get

$$-\frac{1}{2\pi}\int_0^{2\pi} v(\tilde{g}(e^{i\theta}))\,d\theta \le -\frac{1}{2\pi}\int_0^{2\pi} v(g(re^{i\theta}))\,d\theta < P_{-v}(z_0)+3\varepsilon. \tag{11}$$

So, $\widetilde{E}_\varphi \le P_\varphi$. The inequality $P_\varphi\le\widetilde{E}_\varphi$ is elementary.

For the last statement of the theorem (regularity in the case of strictly pseudoconvex domains) use the last statement in Theorem 2.1. $\square$

As an immediate corollary of Theorem 1.3 we have the following example.

Example 2.2. Let $\varphi\colon \partial\mathbb{B}_n\to\mathbb{R}$ be an upper semicontinuous function. Then for any $k\in\mathbb{N}$ we have

$$P_\varphi(z)=\inf\Big\{\frac{1}{2\pi}\int_0^{2\pi}\varphi(f(e^{i\theta}))\,d\theta : f\in\mathcal{O}(\mathbb{D},\mathbb{B}_n)\cap C^k(\overline{\mathbb{D}},\overline{\mathbb{B}}_n), f(\mathbb{T})\subset\partial\mathbb{B}_n \text{ and } f(0)=z\Big\}. \tag{12}$$

In [2] it is shown that in the above equality one can take $k=\infty$.

Corollary 2.3. *Let $D\Subset\mathbb{C}^n$ be a B-regular domain and let $U\subset\partial D$ be a relatively open subset. Then*

$$\omega(z,U,D)=-\sup\{\sigma(\{e^{i\theta}: f(e^{i\theta})\in U\}) : f\in\mathcal{O}(\mathbb{D},D) \text{ is proper and } f(0)=z\}. \tag{13}$$

Proof. Put $\varphi=-\chi_U$ and use Theorem 1.3. $\square$

Remark 2.4. (a) Note that if $D\subset\mathbb{C}^n$ is a pseudoconvex domain, $n\ge2$, then from the Forstnerič–Globevnik result we get immediately that for any $p\in D$ we have

$$g_D(z;p)=\inf\Big\{\sum_{\lambda\in f^{-1}(p)}\log|\lambda| : f\in\mathcal{O}(\mathbb{D},D) \text{ is proper and } f(0)=z\Big\}, \tag{14}$$

where $g_D(\,\cdot\,,p)$ is the pluricomplex Green function in D with pole at p (see e.g. [5], [9] and [10]). We have a similar result for the multipole pluricomplex Green function [6].

(b) In a similar way, one can show easily that for the second disk functional in Poletsky's theory [10] one can also take the infimum just over proper analytic disks.

Remark 2.5. One may ask whether there exists a 'minimal' proper holomorphic mapping, i.e., whether there exists an $f\colon \mathbb{D}\to D$ which is proper and such that for any $z\in D$ we have $f(0)=z$ and

$$P_\varphi(z)=\frac{1}{2\pi}\int_0^{2\pi}\varphi(\tilde f(e^{i\theta}))\,d\theta. \tag{15}$$

Using the idea of Sibony (see e.g. [4]) we show that even in the unit ball for a non-empty set of $z\in\mathbb{B}$ there are no minimal proper holomorphic mappings. Namely, take a compact set $K\subset\partial\mathbb{B}$ such that its polynomial hull $\widehat{K}\cap\mathbb{B}$ does not contain analytic disks (use the examples of Stolzenberg or Wermer) and $\widehat{K}\setminus K\neq\varnothing$. Now take a continuous function φ on $\partial\mathbb{B}$ such that $\varphi=-1$ on K and $\varphi>-1$ on $\partial\mathbb{B}\setminus K$. Note that $\widehat{K}=\{z:P_\varphi(z)=-1\}$. Assume that $z_0\in\widehat{K}$. If there exists a proper holomorphic mapping $f\colon\mathbb{D}\to\mathbb{B}$ such that $f(0)=z_0$ and

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(\tilde f(e^{i\theta}))\,d\theta=P_\varphi(z_0)=-1, \tag{16}$$

then $\tilde f(\mathbb{T})\subset K$, and therefore $f(\mathbb{D})\subset\widehat{K}$. A contradiction.

Remark 2.6. It is worth noting that Theorem 1.3 can be used to show some regularity properties of the Perron–Bremermann function (see e.g. [9], Proposition 4.3.2 and Theorem 4.3.3). Namely take a C^2 function φ on the unit sphere $\partial\mathbb{B}_n$ and put $u=P_\varphi$. Then we can show that for any $\varepsilon\in(0,1)$ there exists a constant $C>0$ such that for all $z,h\in\mathbb{C}^n$ with $\|z\|\le 1-\varepsilon$ and $\|h\|\le\varepsilon/2$, we have

$$u(z+h)-2u(z)+u(z-h)\le C\|h\|^2. \tag{17}$$

Indeed, for a fixed z take a proper analytic disk $f\colon\mathbb{D}\to\mathbb{B}$ with $f(0)=z$. Now take automorphisms $\Phi_1,\Phi_2\colon\mathbb{B}\to\mathbb{B}$ of the unit ball such that $\Phi_1(z)=z+h$ and $\Phi_2(z)=z-h$. Then consider proper analytic disks $f_1=\Phi_1\circ f$ and $f_2=\Phi_2\circ f$ and use Theorem 1.3.

In a similar way one can show that u satisfies a Lipschitz condition in the unit ball.

Remark 2.7. As was pointed out by the referee, Theorem 1.3 can be proved for appropriately defined B-regular domains on a complex manifold. Indeed, in [3] the authors prove Theorem 2.1 for a class of complex manifolds. Also, J. P. Rosay proved Poletsky's theory on any complex manifold (see [11]).

3. Proof of Theorem 1.2

As we mentioned in Section 1, the following result is a basic tool in a characterization of polynomial hulls by analytic disks.

Theorem 3.1. *Let $D\Subset\mathbb{C}^n$ be a B-regular domain with C^∞-boundary and let $K\subset\partial D$ be a compact set. Assume that $\widehat{K}\subset\overline{D}$. Then*

$$\widehat{K}\cap D=\{z\in D:\omega(z,K,D)=-1\}. \tag{18}$$

Before we go into the proof recall the following results. The first one is due to F. Wikström (see [17]).

Theorem 3.2. *Let $D\Subset\mathbb{C}^n$ be a B-regular domain and let u be an upper bounded plurisubharmonic function on D. Then there exists a decreasing sequence $u_j\in\mathrm{PSH}(D)\cap C(\overline{D})$ such that $u_j\searrow u^*$ on $\overline{D}$.*

The second result is due to N. Sibony [13].

Theorem 3.3. *Let D be a pseudoconvex domain with C^∞ boundary. Then any $u\in\mathrm{PSH}(D)\cap C(\overline{D})$ is a uniform limit of $u_j\in\mathrm{PSH}(D_j)\cap C^\infty(D_j)$, where $D_j\supset\overline{D}$ are domains.*

Proof of Theorem 3.1. First note that for a point $z\in D$ the following conditions are equivalent:

(1) $z\in\widehat{K}$;

(2) $u(z)\le\sup_K u$ for any plurisubharmonic function u on D, which is continuous on $\overline{D}$;

(3) $u(z)\le\sup_K u$ for any negative plurisubharmonic function u on D.

Indeed, (2)$\Leftrightarrow$(3) holds by (3.2).

We have the equivalence of (1) and the following condition: $u(z)\le\sup_K u$ for u plurisubharmonic and continuous on $\mathbb{C}^n$; and hence for u plurisubharmonic and continuous on a Runge domain containing $\widehat{K}$. Thus, Theorem 3.3 and the fact that $\widehat{K}\subset\overline{D}$ show that (1) and (2) are equivalent.

Now assume that $z_0\in\widehat{K}\cap D$. Then for any negative plurisubharmonic function u on D such that $u^*\le-1$ on K we have $u(z_0)\le-1$. Hence, $\omega(z_0,K,D)=-1$.

Fix a point $z_0 \in D$ such that $\omega(z_0, K, D) = -1$. Take a negative plurisubharmonic function u on D and put $C = -\sup_K u^*$. We may assume that $C > 0$. Then $u_C = u/C$ is a negative plurisubharmonic function on D and $u_C^* \le -1$ on K. Hence, $u_C(z_0) \le \omega(z_0, K, D) = -1$, and therefore, $u(z_0) \le \sup_K u^*$. □

Proof of Theorem 1.2. It suffices to note that

$$\omega(z, K, D) = \lim_{\varepsilon \to 0} \omega(z, K_\varepsilon, D) \tag{19}$$

and use Theorem 3.1 and Corollary 2.3. □

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Armen Edigarian
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
PL-30-348 Kraków
Poland
armen.edigarian@im.uj.edu.pl

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