Ark. Mat., **49** (2011), 129–148
DOI: 10.1007/s11512-010-0129-x

Homomorphisms of infinitely generated analytic sheaves

Vakhid Masagutov

Abstract. We prove that every homomorphism $\mathcal{O}^E_\zeta \to \mathcal{O}^F_\zeta$, with E and F Banach spaces and $\zeta \in \mathbb{C}^m$, is induced by a $\mathrm{Hom}(E,F)$-valued holomorphic germ, provided that $1 \le m < \infty$. A similar structure theorem is obtained for the homomorphisms of type $\mathcal{O}^E_\zeta \to \mathcal{S}_\zeta$, where $\mathcal{S}_\zeta$ is a stalk of a coherent sheaf of positive depth. We later extend these results to sheaf homomorphisms, obtaining a condition on coherent sheaves which guarantees the sheaf to be equipped with a unique analytic structure in the sense of Lempert–Patyi.

1. Introduction

The theory of coherent sheaves is one of the deeper and most developed subjects in complex analysis and geometry, see [GR]. Coherent sheaves are locally finitely generated. However, a number of problems even in finite-dimensional geometry lead to sheaves that are not finitely generated over the structure sheaf $\mathcal{O}$, such as the sheaf of holomorphic germs valued in a Banach space; and in infinite-dimensional problems infinitely generated sheaves are the rule rather than the exception. This paper is motivated by [LP], that introduced and studied the class of so called cohesive sheaves over Banach spaces; but here we shall almost exclusively deal with sheaves over $\mathbb{C}^m$. In a nutshell, we show that $\mathcal{O}$-homomorphisms among certain sheaves of $\mathcal{O}$-modules have strong continuity properties, and in fact arise by a simple construction.

Research partially supported by NSF grant DMS0700281 and the Mittag-Leffler Institute, Stockholm. I am grateful to both organizations, and I, particularly, would like to express my gratitude to the Mittag-Leffler Institute for their hospitality during my research leading to this paper. I am indebted to Professor Lempert for his guidance and for proposing questions that motivated this work. I am especially grateful for his suggestions and critical remarks that were invaluable at the research and writing phases. Lastly, I would like to thank the anonymous referee for devoting the time and effort to thoroughly review the manuscript and for suggesting numerous improvements.

We will consider two types of sheaves. The first type consists of coherent sheaves $\mathcal{S}$. The other consists of plain sheaves; these are the sheaves $\mathcal{O}^E$ of holomorphic germs valued in some fixed complex Banach space E. The base of the sheaves is $\mathbb{C}^m$ or an open set $\Omega\subset\mathbb{C}^m$. Thus $\mathcal{O}^E$ is a (sheaf of) $\mathcal{O}$-module(s). By an *$\mathcal{O}$-homomorphism* $\mathcal{O}^E\to\mathcal{O}^F$ or $\mathcal{O}^E\to\mathcal{S}$ we shall understand a sheaf map which commutes with multiplication by elements of the sheaf of local rings $\mathcal{O}$ and distributes across finite sums.

We denote the Banach space of continuous linear operators between Banach spaces E and F by $\operatorname{Hom}(E,F)$. Any holomorphic map $\Phi\colon \Omega\to\operatorname{Hom}(E,F)$ induces an $\mathcal{O}$-homomorphism $\phi\colon \mathcal{O}^E\to\mathcal{O}^F$. If $U\subset\Omega$ is open, $\zeta\in U$ and a holomorphic $e\colon U\to E$ represents a germ $e_\zeta\in\mathcal{O}^E_\zeta$, then $\phi(e)\in\mathcal{O}^F_\zeta$ is defined as the germ of the function $U\ni z\mapsto\Phi(z)e(z)\in E$. Following [LP], such homomorphisms will be called *plain*. In fact, if Φ is holomorphic only on some neighborhood of ζ it still defines a homomorphism $\mathcal{O}^E_\zeta\to\mathcal{O}^F_\zeta$ of the local modules over the local ring $\mathcal{O}_\zeta$. Again such homomorphisms will be called *plain*.

The first question we address is how restrictive it is for a homomorphism to be plain. It turns out it is not restrictive at all, provided $0<m<\infty$.

Theorem 1.1. *If $0<m<\infty$ and $\Omega\subset\mathbb{C}^m$ is open, then every $\mathcal{O}$-homomorphism $\mathcal{O}^E\to\mathcal{O}^F$ of plain sheaves is plain.*

This came as a surprise, because it fails in the simplest of all cases, when $m=0$. This was pointed out by Lempert. When $\Omega=\mathbb{C}^0=\{0\}$, $\mathcal{O}^E$, resp. $\mathcal{O}^F$, are identified with E and F, and the difference between $\mathcal{O}$-homomorphisms and plain homomorphisms boils down to the difference between linear and continuous linear operators $E\to F$. It would be interesting to decide whether Theorem 1.1 remains true if $\mathbb{C}^m$ is replaced by a Banach space.

Another surprising aspect of this result is that it fails for homomorphisms between modules of polynomial germs. If $\mathcal{P}$, resp. $\mathcal{P}^E$, denote the ring/module of $\mathbb{C}$- resp. E-valued polynomials, with $\dim E=\infty$, then any discontinuous $\mathbb{C}$-linear map $l\colon E\to\mathbb{C}$ induces a nonplain $\mathcal{P}$-homomorphism $\mathcal{P}^E\to\mathcal{P}$ via $p\mapsto l\circ p$.

A result similar to Theorem 1.1 has been obtained by Leiterer in [Lei, Proposition 1.3]. In it, Leiterer, however, assumed the homomorphism in question to be a priori continuous (AF-homomorphisms in the terminology of [Lei]), a condition that we omit.

Lempert observed that a variant of the original proof of Theorem 1.1 gives the corresponding theorem about local modules, and we shall derive Theorem 1.1 from it.

Theorem 1.2. *If* $0<m<\infty$ *and* $\zeta\in\mathbb{C}^m$, *then every* $\mathcal{O}_\zeta$*-homomorphism of plain modules* $\mathcal{O}_\zeta^E\to\mathcal{O}_\zeta^F$ *is plain.*

Next, we turn our attention to $\mathcal{O}$-homomorphisms from plain sheaves $\mathcal{O}^E$ to a coherent sheaf $\mathcal{S}$. On the level of stalks, such homomorphisms also have a simple description; however, this description applies only if the depth of each stalk $\mathcal{S}_\zeta$ is positive, a condition that corresponds to the positivity of m in Theorems 1.1 and 1.2. The notion of depth forms an important invariant of rings and modules in commutative and homological algebra; in our setting we can define it as follows. Let $\mathfrak{m}_\zeta\subset\mathcal{O}_\zeta$ denote the maximal ideal consisting of germs that vanish at ζ, and assume that $\mathfrak{m}_\zeta\neq0$. For a finitely generated $\mathcal{O}_\zeta$-module M, $\operatorname{depth} M=0$ if M has a nonzero submodule N such that $\mathfrak{m}_\zeta N=0$ (see Definition 4.1 and Proposition 4.3), and $\operatorname{depth} M>0$ otherwise. For example, the modules $\mathcal{O}_\mathbb{C}/z\mathcal{O}_\mathbb{C}$ and $\mathcal{O}_{\mathbb{C}^2}/(z_1^2, z_1z_2)\mathcal{O}_{\mathbb{C}^2}$ have zero-depth at the origin, while for comparison, the module $\mathcal{O}_{\mathbb{C}^2}/z_1\mathcal{O}_{\mathbb{C}^2}$ is of positive depth everywhere.

Theorem 1.3. *Let* $\zeta\in\mathbb{C}^m$, M *be a finite* $\mathcal{O}_\zeta$*-module, and* $p\colon\mathcal{O}_\zeta^n\to M$ *be an epimorphism. If* $\operatorname{depth} M>0$, *then, any* $\mathcal{O}_\zeta$*-homomorphism* $\phi\colon\mathcal{O}_\zeta^E\to M$ *factors through* p, *i.e.,* $\phi=p\psi$ *with an* $\mathcal{O}_\zeta$*-homomorphism* $\psi\colon\mathcal{O}_\zeta^E\to\mathcal{O}_\zeta^n$.

Here, and in what will follow, we adhere to the algebraic convention of simplifying "finitely generated" to "finite" when referring to modules. We also note that the above ψ is induced by a germ in $\mathcal{O}_\zeta^{\operatorname{Hom}(E,\mathbb{C}^n)}$, since the depth condition eliminates the possibility of a nonzero module for $m=0$. This condition is in fact necessary as shown in Theorem 5.1.

This result extends also to sheaves; however, its proof is not a simple application of Theorem 1.3. Instead, we initially obtain in Section 6 a weaker local extension, and only later, in Section 8, we obtain a global result after applying a cohomology vanishing theorem.

Theorem 1.4. *Let* $\mathcal{S}$ *be a coherent sheaf over an open pseudoconvex* $\Omega\subset\mathbb{C}^m$ *of positive depth at each stalk, and* E *be a Banach space. If* $p\colon\mathcal{O}^n\to\mathcal{S}$ *is an epimorphism, then any* $\mathcal{O}$*-homomorphism* $\mathcal{O}^E\to\mathcal{S}$ *factors through it.*

The following question motivated this line of research and is answered by the above theorems. In [LP] Lempert and Patyi introduced the notion of analytic structure on an infinitely generated $\mathcal{O}$-module and showed that this structure in general is not unique. However, the results of this paper, recast in their language, show that certain analytic structures coincide; in particular, on coherent sheaves, analytic structures are unique. This will be explained in detail in Section 7, together with the following corollary of Theorem 1.2.

Corollary 1.5. *If* $\zeta\in\mathbb{C}^m$, $0<m<\infty$, *and* E *is an infinite-dimensional Banach space, then the plain module* $\mathcal{O}^E_\zeta$ *is not free*; *it cannot even be embedded in a free module.*

2. Background

Here we quickly review a few notions of complex analysis. For more see [GR], [Muj], and [Ser]. Let X and E be Banach spaces (always over $\mathbb{C}$) and $\Omega\subset X$ be open.

Definition 2.1. A function $f\colon \Omega\to E$ is *holomorphic* if for all $x\in\Omega$ and $\xi\in X$

$$df(x,\xi)=\lim_{\lambda\to 0}\frac{f(x+\xi\lambda)-f(x)}{\lambda}$$

exists, and depends continuously on $(x,\xi)\in\Omega\times X$.

If $X=\mathbb{C}^m$ with coordinates $(z_1,...,z_m)$, then this is equivalent to requiring that in some neighborhood of each $a\in\Omega$ one can expand f in a uniformly convergent power series

$$f=\sum_J e_J(z-a)^J,\quad e_J\in E,$$

where multi-index notation is used. For general X one can only talk about homogeneous expansion. Recall that a function P between vector spaces V and W is an n-homogeneous polynomial if $P(v)=l(v,v,...,v)$ where $l\colon V^n\to W$ is an n-linear map. Given a ball $B\subset X$ centered at $a\in X$, any holomorphic $f\colon B\to E$ can be expanded in a series

$$f(x)=\sum_{n=0}^{\infty}P_n(x-a),\quad x\in B, \tag{1}$$

where the $P_n\colon X\to E$ are continuous n-homogeneous polynomials. The homogeneous components P_n are uniquely determined, and the series (1) converges locally uniformly on B.

We denote by f_x the germ at $x\in\Omega$ of a function $\Omega\to E$, and by $\mathcal{O}^E$ the sheaf over Ω of germs of E-valued holomorphic functions. The sheaf $\mathcal{O}^{\mathbb{C}}=\mathcal{O}$ is a sheaf of rings over Ω, and $\mathcal{O}^E$ is, in an obvious way, a sheaf of $\mathcal{O}$-modules. The sheaves $\mathcal{O}^E$ are called *plain sheaves*, and their stalks $\mathcal{O}^E_x$ *plain modules*. When $E=\mathbb{C}^n$, we write $\mathcal{O}^n$, resp. $\mathcal{O}^n_x$, for $\mathcal{O}^E$, resp. $\mathcal{O}^E_x$.

As said in the introduction, $\mathrm{Hom}(E,F)$ denotes the space of continuous linear operators between Banach spaces E and F, endowed with the operator norm. Any holomorphic function $\Phi\colon \Omega\to\mathrm{Hom}(E,F)$ induces an $\mathcal{O}$-homomorphism $\mathcal{O}^E\to\mathcal{O}^F$ and any $\Psi\in\mathcal{O}_x^{\mathrm{Hom}(E,F)}$ induces an $\mathcal{O}_x$-homomorphism $\mathcal{O}_x^E\to\mathcal{O}_x^F$. The homomorphisms obtained in this manner are called *plain homomorphisms.*

3. Homomorphisms of plain sheaves and modules

We shall deduce Theorem 1.2 from a weaker variant, which, however, is valid in an arbitrary Banach space.

Theorem 3.1. *Let X, E and F be Banach spaces, with* $\dim X>0$. *Let $\zeta\in X$ and $\phi\colon \mathcal{O}_\zeta^E\to\mathcal{O}_\zeta^F$ be an $\mathcal{O}_\zeta$-homomorphism. Then there is a plain homomorphism $\psi\colon \mathcal{O}_\zeta^E\to\mathcal{O}_\zeta^F$ that agrees with ϕ on constant germs.*

We need two auxiliary results to prove this result.

Proposition 3.2. *Let X and G be Banach spaces and $\pi_n\colon X\to G$ be continuous homogeneous polynomials of degree $n=0,1,2,\dots$. If for every $x\in X$ there is an $\varepsilon_x>0$ such that $\sup_n\|\pi_n(\varepsilon_x x)\|<\infty$, then there is an $\varepsilon>0$ such that*

$$\sup_n \sup_{\|x\|<\varepsilon} \|\pi_n(x)\|<\infty.$$

Here, and in the following, we indiscriminately use $\|\cdot\|$ for the norms on X,G, and whatever Banach spaces we encounter.

Proof. For numbers A and δ, consider the closed sets

$$X_{A,\delta}=\Big\{x\in X:\sup_n\|\pi_n(\delta x)\|\le A\Big\}.$$

By Baire's theorem $X_{A,\delta}$ contains a ball $\{x_0+y:\|y\|<r\}$ for some $A,\delta,r>0$. As a consequence of the polarization formula [Muj, Theorem 1.10],

$$\pi_n(\xi)=\sum_{\sigma_j=\pm1}\frac{\sigma_1\dots\sigma_n}{2^n n!}\pi_n(\delta x_0+\sigma_1\xi+\dots+\sigma_n\xi)\quad\text{for }\xi\in X,$$

see also [Muj, Exercise 2M]. Therefore if $\|\xi\|<\delta r/n$, then $\pi_n(\xi)\le A/n!$, and by homogeneity, for $\|x\|<\delta r/e$,

$$\|\pi_n(x)\|=n^n e^{-n}\|\pi_n(ex/n)\|\le An^n/e^n n!\le A.\quad\square$$

Proposition 3.3. *Let X, E, F be Banach spaces, $\Omega \subset X$ be open, and $g\colon \Omega \to \operatorname{Hom}(E, F)$ be a function. If for every $v \in E$ the function $gv\colon X \to F$ is holomorphic, then g itself is holomorphic.*

Proof. This is Exercise 8.E in [Muj]. First one shows using the principle of uniform boundedness that g is locally bounded. Standard one-variable Cauchy representation formulas then show that g is continuous and ultimately holomorphic. $\square$

Proof of Theorem 3.1. If $v \in E$ we write $\tilde{v} \in \mathcal{O}^E_\zeta$ for the constant germ whose value is v. Without loss of generality we can take $\zeta = 0$. Let the germ $\phi(\tilde{v}) \in \mathcal{O}^F_0$ have homogeneous series

$$\sum_{n=0}^{\infty} P_n(x, v). \tag{2}$$

Thus, P_n is $\mathbb{C}$-linear in v, and for fixed v, $P_n(\,\cdot\,, v)$ is a continuous n-homogeneous polynomial. For each $v \in E$, (2) converges if $\|x\|$ is sufficiently small.

Now let $\lambda \in \operatorname{Hom}(X, \mathbb{C})$, and suppose that, with $v_j \in E$, the series $\sum_{i=0}^{\infty} v_i \lambda^i$ represents a germ $e \in \mathcal{O}^E_0$. For example, this will be the case if the v_i are unit vectors. With an arbitrary $N \in \mathbb{N}$ and some $f \in \mathcal{O}^F_0$,

$$\begin{aligned} \phi(e) &= \sum_{i<N} \phi(\tilde{v}_i)\lambda^i + \lambda^N f = \sum_{i<N} \sum_{n=0}^{\infty} P_n(\,\cdot\,, v_i)\lambda^i + \lambda^N f \\ &= \sum_{j<N} \sum_{n=0}^{j} P_n(\,\cdot\,, v_{j-n})\lambda^{j-n} + \lambda^N g, \end{aligned}$$

where $g \in \mathcal{O}^F_0$. Hence the homogeneous components of $\phi(e)$ are

$$Q_j(x) = \sum_{n=0}^{j} P_n(x, v_{j-n})\lambda^{j-n}(x), \quad j = 0, 1, 2, \dots. \tag{3}$$

We use this to prove, by induction on n, that for any $x \in X$ the map $v \mapsto P_n(x, v)$ is not only linear but also continuous.

Suppose this is true for $n < k$. Take an $x \in X$, which can be supposed to be nonzero, and $\lambda \in \operatorname{Hom}(X, \mathbb{C})$ so that $\lambda(x) = 1$. If $v \mapsto P_k(x, v)$ were not continuous, we could inductively select unit vectors $v_i \in E$ so that

$$\|P_k(x, v_{j-k})\| > \sum_{n=0}^{k-1} \|P_n(x, \cdot\,)\| + j^j + \sum_{n=k+1}^{j} \|P_n(x, v_{j-n})\|$$

for $j=k,k+1,\dots$. Here $\|P_n(x,\cdot)\|$ stands for the operator norm of the homomorphism $P_n(x,\cdot)\in\mathrm{Hom}(E,F)$, $n<k$. However, (3) would then imply

$$\|Q_j(x)\|\geq\|P_k(x,v_{j-k})\|-\sum_{n=0}^{k-1}\|P_n(x,\cdot)\|-\sum_{n=k+1}^{j}\|P_n(x,v_{j-n})\|>j^j,$$

which would preclude $\sum_{j=0}^{\infty}Q_j$ from converging in any neighborhood of $0\in X$. The contradiction shows that $P_k(x,\cdot)\in\mathrm{Hom}(E,F)$, in fact for every k and $x\in X$. Let us write $\pi_k(x)$ for $P_k(x,\cdot)$.

Now, for fixed $v\in E$, $P_n(\cdot,v)=\pi_n v$ is a continuous n-homogeneous polynomial and, thus, holomorphic. We can apply Proposition 3.3 to conclude, in turn, that $\pi_n\colon X\to\mathrm{Hom}(E,F)$ is a holomorphic, n-homogeneous polynomial.

Next we estimate $\|\pi_n(x)\|$ for fixed $x\in X$. Suppose a sequence $\delta_n\geq 0$ goes to 0 superexponentially, in the sense that $\delta_n=o(\varepsilon^n)$ for all $\varepsilon>0$. Then for any homogeneous series $\sum_{n=0}^{\infty}p_n$ representing a germ $f\in\mathcal{O}_0^F$ we have $\sup_n\delta_n\|p_n(x)\|<\infty$.

In particular, $\sup_n\delta_n\|\pi_n(x)v\|<\infty$ for all $v\in E$, and by the principle of uniform boundedness, $\delta_n\|\pi_n(x)\|$ is bounded. This being so, there is an $\varepsilon=\varepsilon_x>0$ such that $\varepsilon^n\|\pi_n(x)\|$ is bounded. Indeed, otherwise we could find $n_1<n_2<\dots$ so that

$$\|\pi_{n_t}(x)\|>t^{n_t},\quad t=1,2,\dots.$$

But then the sequence

$$\delta_n=\begin{cases}t^{-n_t/2}, & \text{if } n=n_t,\\ 0, & \text{otherwise,}\end{cases}$$

would go to 0 superexponentially and yet $\delta_{n_t}\|\pi_{n_t}(x)\|\to\infty$; a contradiction.

Thus, for each x we have found $\varepsilon_x>0$ so that $\sup_n\|\pi_n(\varepsilon_x x)\|$ is bounded. By Proposition 3.2, the π_n are uniformly bounded on some ball $\{x:\|x\|<\varepsilon\}$. Therefore the series

$$\sum_{n=0}^{\infty}\pi_n(x)=\Phi(x)$$

converges uniformly on some neighborhood of $0\in X$, and represents a $\mathrm{Hom}(E,F)$-valued holomorphic function there. By the construction of $\pi_n(x)v=P_n(x,v)$, see (2), the plain homomorphism $\mathcal{O}_0^E\to\mathcal{O}_0^F$ induced by Φ agrees with ϕ on constant germs $\tilde{v}$, and the proof is complete. □

Proof of Theorem 1.2. In view of Theorem 3.1, all we have to show is that (for $X=\mathbb{C}^m, 0<m<\infty$) if an $\mathcal{O}_\zeta$-homomorphism $\phi\colon\mathcal{O}_\zeta^E\to\mathcal{O}_\zeta^F$ annihilates constant germs then it is in fact 0. This we formulate in a slightly greater generality.

Lemma 3.4. *Let $\zeta\in\mathbb{C}^m$, and M be an $\mathcal{O}_\zeta$-module. Denote by $\mathfrak{m}_\zeta$ the maximal ideal of $\mathcal{O}_\zeta$. If a homomorphism $\theta\colon \mathcal{O}_\zeta^F\to M$ annihilates all constant germs, then*

$$\operatorname{Im}\theta\subset\bigcap_{k=0}^{\infty}\mathfrak{m}_\zeta^k M. \tag{4}$$

Proof. Along with constants, θ will annihilate the $\mathcal{O}_\zeta$-module generated by constants, in particular, the polynomial germs. Since any $e\in\mathcal{O}_\zeta^E$ is congruent, modulo an arbitrary power of the maximal ideal, to a polynomial, and furthermore, $\theta(\mathfrak{m}_\zeta^k\,\mathcal{O}_\zeta^E)\subset\mathfrak{m}_\zeta^k M$, (4) follows. □

This then completes the proof of Theorem 1.2 since $\bigcap_{k=0}^{\infty}\mathfrak{m}_\zeta^k\,\mathcal{O}_\zeta^F=0$. □

Proof of Theorem 1.1. Let $\phi\colon \mathcal{O}^E\to\mathcal{O}^F$ be an $\mathcal{O}$-homomorphism. For $v\in E$ let $\hat{v}\colon \Omega\to\mathcal{O}^E$ be the section that associates with $\zeta\in\Omega$ the germ at ζ of the constant function $\Omega\ni z\mapsto v$. Then $\phi(\hat{v})$ is a section of $\mathcal{O}^F$ and so there is a holomorphic function $f(\cdot\,,v)\colon \Omega\to F$ whose germs $f(\cdot\,,v)_z$ at various $z\in\Omega$ agree with $\phi(\tilde{v})(z)$. By Theorem 1.2, for each $\zeta\in\Omega$ we can find a germ $\Phi^\zeta\in\mathcal{O}_\zeta^{\operatorname{Hom}(E,F)}$ such that

$$f(\cdot\,,v)_\zeta=\Phi^\zeta v.$$

Therefore for fixed ζ, $f(\zeta,\cdot)\in\operatorname{Hom}(E,F)$. Let $\Phi(\zeta)=f(\zeta,\cdot)$. Proposition 3.3 implies that $\Phi\colon \Omega\to\operatorname{Hom}(E,F)$ is holomorphic and by construction induces ϕ on constant germs.

This means that Φ^ζ above and the germ Φ_ζ of Φ induce homomorphisms $\mathcal{O}_\zeta^E\to\mathcal{O}_\zeta^F$ that agree on constant germs. Since we are talking about plain homomorphisms, the two induced homomorphisms in fact agree. Hence ϕ is induced by Φ. □

4. Auxiliary results on depth

For a general definition of depth we refer to [Eis, pp. 423, 429] or [Mat, p. 102]. Our interest in this notion is centered on its properties in the case of finite modules over Noetherian local rings. The following definition suffices for our needs.

Definition 4.1. Let $(R,\mathfrak{m})$ be a Noetherian local ring (always commutative and unital) and $M\neq 0$ be a finite R-module. We say the depth of M, $\operatorname{depth} M$, is positive if there is a nonzero divisor $r\in\mathfrak{m}$ on M; otherwise the depth is 0. If $M=0$, the convention is that the depth is $+\infty$.

Alternatively, the depth of M is zero if and only if $\operatorname{Hom}_R(R/\mathfrak{m},M)\neq 0$, see [Eis, Proposition 18.4].

Remark 4.2. When R is a field, the maximal ideal is $\mathfrak{m}=0$; hence, the depth of a finite-dimensional vector space M is positive (infinity) if and only if $M=0$.

When R is not a field, there is an alternative criterion for the positivity of depth.

Proposition 4.3. *For a Noetherian local ring* $(R,\mathfrak{m})$, *not a field, a finite* R*-module* M *has* $\operatorname{depth} M=0$ *if and only if there is a nonzero submodule* $L\subset M$ *such that* $\mathfrak{m}\,L=0$.

Proof. If $\operatorname{depth} M=0$, then $\operatorname{Hom}_R(R/\mathfrak{m},M)\neq 0$ and there is a nonzero R-homomorphism $\phi\colon R/\mathfrak{m}\to M$. Take $L=\operatorname{Im}\phi$ to get $\mathfrak{m}\,L=0$.

Conversely, suppose $L\subset M$ is a nonzero submodule such that $\mathfrak{m}\,L=0$. Then, $M\neq 0$ and every $r\in\mathfrak{m}\setminus\{0\}$ is a zero divisor. So, $\operatorname{depth} M=0$. ☐

For the proof of Theorem 1.3 we shall need a number of lemmas that are algebraic in nature. Recall the notion of localization at a prime. Suppose R is a ring, $\mathfrak{p}\subset R$ a prime ideal, and M an R-module. Consider the multiplicatively closed set $S=R\setminus\mathfrak{p}$, then the localization of M at $\mathfrak{p}$ is

$$M_{\mathfrak{p}}=(M\times S)/\sim,$$

where $(v,s)\sim(w,t)$ means that $q(vt-ws)=0$ for some $q\in S$. Elements of $M_{\mathfrak{p}}$ are written as fractions v/s. The usual rules for operating with fractions turn $R_{\mathfrak{p}}$ into a ring and $M_{\mathfrak{p}}$ into a module over it. Localization is a functor, in particular, a homomorphism $\alpha\colon M\to M'$ of R-modules induces a homomorphism $\alpha_{\mathfrak{p}}\colon M_{\mathfrak{p}}\to M'_{\mathfrak{p}}$.

Lemma 4.4. *Let* R *be a unique factorization domain and* $\mathfrak{p}\subset R$ *be a principal prime ideal. If* $N\subset R^m$ *is a finite module, then there is a finite free submodule* $F\subset N$ *such that* $F_{\mathfrak{p}}=N_{\mathfrak{p}}$.

Proof. Let p be a generator of $\mathfrak{p}$; any element of $R_{\mathfrak{p}}$ is either invertible or divisible by p. As R is a unique factorization domain, any nontrivial linear relation

$$\sum_{j=1}^{k} r_j u_j=0, \quad \text{with } u_1,...,u_k\in R_{\mathfrak{p}}^m \text{ and coefficients } r_1,...,r_k\in R_{\mathfrak{p}},$$

can be solved for some u_j. Hence, finitely generated submodules of $R_{\mathfrak{p}}^m$ are free. In particular, $N_{\mathfrak{p}}$ has a free generating set $\{v_1/s_1,...,v_k/s_k\}$. We can, therefore, take F to be the module generated by v_j's. ☐

Lemma 4.5. *Let $(R,\mathfrak{m})$ be a local ring which is a unique factorization domain, and Q be its field of fractions. Let $\rho\colon A\to B$ be a homomorphism of finite free R-modules. If $\operatorname{depth}\operatorname{coker}\rho>0$, then there are a finite free R-module C and a homomorphism $\theta\colon B\to C$ such that*

(i) $\ker\rho=\ker\theta\rho$;
(ii) $(\operatorname{coker}\theta\rho)\otimes Q=0$;
(iii) $\operatorname{depth}\operatorname{coker}\theta\rho>0$.

Proof. Let $\rho^*\colon \operatorname{Hom}_R(B,R)\to\operatorname{Hom}_R(A,R)$ be the pull-back operator. With the principal prime ideal $\mathfrak{p}$ to be chosen later, take a free R-module $F\subset N=\operatorname{Im}\rho^*$ such that $F_\mathfrak{p}=N_\mathfrak{p}$, as in Lemma 4.4. Define a homomorphism $\sigma\colon F\to\operatorname{Hom}_R(B,R)$ by specifying its values on a free generating set so that $\rho^*\sigma$ is the inclusion map $i\colon F\hookrightarrow\operatorname{Hom}_R(A,R)$. We will show that with a suitable choice of $\mathfrak{p}$ we can take

$$C=\operatorname{Hom}_R(F,R)\quad\text{and}\quad\theta=\sigma^*\colon B\longrightarrow C,$$

where B is canonically identified with $\operatorname{Hom}_R(\operatorname{Hom}_R(B,R),R)$. The following commutative diagrams summarize the homomorphisms in question:

$$\begin{tikzcd} A \arrow[rr, "\rho"] \arrow[dr, dashed, "i^*"'] & & B \arrow[dl, dashed, "\theta=\sigma^*"] \\ & C=\operatorname{Hom}(F,R), & \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Hom}(A,R) & & \operatorname{Hom}(B,R) \arrow[ll, "\rho^*"'] \\ & F. \arrow[ul, "i"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

(i) After localizing, $(\operatorname{Im}\rho^*)_\mathfrak{p}=F_\mathfrak{p}=(\operatorname{Im}i)_\mathfrak{p}=\operatorname{Im}\rho^*_\mathfrak{p}\sigma_\mathfrak{p}$, and, therefore, $\ker\rho_\mathfrak{p}=\ker\theta_\mathfrak{p}\rho_\mathfrak{p}$. Pulling back by the injective localizing map $A\to A_\mathfrak{p}$ (since A is free), we obtain $\ker\rho=\ker\theta\rho$.

(ii) Since $(\operatorname{Im}\rho^*_\mathfrak{p}\sigma_\mathfrak{p})\otimes Q=(\operatorname{Im}i)_\mathfrak{p}\otimes Q=F\otimes Q$ are vector spaces, we obtain that

$$\dim_Q(\operatorname{Im}\theta\rho)\otimes Q=\dim_Q(\operatorname{Im}\rho^*\sigma)\otimes Q=\dim_Q F\otimes Q=\dim_Q C\otimes Q.$$

Consequently, $(\operatorname{coker}\theta\rho)\otimes Q=0$.

(iii) If R is a field, the statement is trivial. So, assume that R is not a field and that $\operatorname{depth}\operatorname{coker}\rho>0$. Then, there is an $r\in\mathfrak{m}\setminus\{0\}$, a nonzero divisor on $\operatorname{coker}\rho$; let p be one of its prime factors. Note that

$$p\beta\in\operatorname{Im}\rho\Longrightarrow\beta\in\operatorname{Im}\rho\quad\text{for all }\beta\in B. \tag{5}$$

We will show that if, in the construction above, $\mathfrak{p}=(p)$, then p is a nonzero divisor on $\operatorname{coker}\theta\rho$, and thus, $\operatorname{depth}\operatorname{coker}\theta\rho>0$. Suppose not, i.e., suppose that there is $\gamma\in C$ and $\alpha\in A$ with $p\gamma=\theta\rho\alpha$. Identify the elements of A with the elements of its double dual using the canonical isomorphism. Then, after localizing, $p\gamma_{\mathfrak{p}}=i_{\mathfrak{p}}^*\alpha_{\mathfrak{p}}=\sigma_{\mathfrak{p}}^*\rho_{\mathfrak{p}}\alpha_{\mathfrak{p}}$. Since, by construction, $\operatorname{Im} i_{\mathfrak{p}}=F_{\mathfrak{p}}=N_{\mathfrak{p}}=\operatorname{Im}\rho_{\mathfrak{p}}^*$, the restriction $\alpha_{\mathfrak{p}}|_{F_{\mathfrak{p}}}$ of the homomorphism $\alpha_{\mathfrak{p}}\colon \operatorname{Hom}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}},R_{\mathfrak{p}})\to R_{\mathfrak{p}}$ is divisible by p. Hence, $\rho_{\mathfrak{p}}\alpha_{\mathfrak{p}}$ is also divisible by p. Now, for any $s\in R$, p divides s in R precisely when p divides $(s/1)$ in $R_{\mathfrak{p}}$. Thus, $\rho\alpha$ is also divisible by p, i.e., there is $\beta\in B$ with $\rho\alpha=p\beta$. In view of (5), $\beta\in\operatorname{Im}\rho$ and $\gamma=\theta\beta\in\operatorname{Im}\theta\rho$. This shows that p is a nonzero divisor on $\operatorname{coker}\theta\rho$, which completes the proof. $\square$

Remark 4.6. The following version of Lemma 4.5 will be needed for later use.

Let $\mathcal{M}$ be the sheaf of meromorphic germs on $\Omega\subset\mathbb{C}^m$, $\rho\colon\mathcal{A}\to\mathcal{B}$ be a homomorphism of finite free $\mathcal{O}$-modules, and $\zeta\in\Omega$. Then, there is a finite free $\mathcal{O}$-module $\mathcal{C}$, a neighborhood U of ζ, and a homomorphism $\theta\colon\mathcal{B}|_U\to\mathcal{C}|_U$, such that $\ker\rho_\zeta=\ker(\theta\rho)_\zeta$, *and* $(\operatorname{coker}\theta\rho)|_U\otimes\mathcal{M}|_U=0$.

While this version of the lemma is stated under slightly different conditions, its proof undergoes few changes. We only need to extend Lemma 4.4 to the local case, which is done below:

Let $\mathcal{N}\subset\mathcal{O}^n$ be a locally finitely generated $\mathcal{O}$-module over $\Omega\subset\mathbb{C}^m$. Then for any $\zeta\in\Omega$ there are an open U about ζ and a free submodule $\mathcal{F}\subset\mathcal{N}|_U$ such that $\mathcal{F}\otimes\mathcal{M}|_U=\mathcal{N}|_U\otimes\mathcal{M}|_U$.

Indeed, choose a basis $f_\zeta^1,\dots,f_\zeta^k\in\mathcal{O}_\zeta^n$ of the vector space $\mathcal{N}_\zeta\otimes\mathcal{M}_\zeta$. There is a neighborhood U of ζ so that every germ f_ζ^i extends to a section of $\mathcal{N}$ over U. By linear independence at ζ, there is a square submatrix M of $[f^1,\dots,f^k]\in\mathcal{O}^{n\times k}$, so that $(\det M)_\zeta\neq0$. Since $\det M$ is a $\mathbb{C}$-valued holomorphic function, $(\det M)_\eta$ is nonzero also for $\eta\neq\zeta$. So, $f_\eta^1,\dots,f_\eta^k$ are linearly independent. The same reasoning shows also that, if $g\in\mathcal{N}(U)$, the collection of sections $g,f^1,\dots,f^k$ is linearly dependent. Thus, $\mathcal{F}=(f^1,\dots,f^k)\mathcal{O}_U\subset\mathcal{N}|_U$ is a free module with $\mathcal{F}\otimes\mathcal{M}|_U=\mathcal{N}|_U\otimes\mathcal{M}|_U$.

In the next lemma we use the following notation. As before, $\mathcal{O}_0$ is the local ring at $0\in\mathbb{C}^m$, $m\geq1$. The subring of germs independent of the last coordinate z_m of $z\in\mathbb{C}^m$ is denoted $\mathcal{O}_0'$; the maximal ideals in $\mathcal{O}_0$, $\mathcal{O}_0'$ are $\mathfrak{m}$ and $\mathfrak{m}'$, respectively. Any $\mathcal{O}_0$-module M is automatically an $\mathcal{O}_0'$-module. We will denote the $\mathcal{O}_0'$-module structure on M by writing M' instead of M. We refer to [GR, Section 2.1.2] for the notion of a Weierstrass polynomial.

Lemma 4.7. *Suppose $h\in\mathcal{O}_0$ is the germ of a Weierstrass polynomial and M is a finite $\mathcal{O}_0$-module such that $hM=0$. Then* $\operatorname{depth}M=0$ *if and only if* $\operatorname{depth}M'=0$.

Proof. We note first that M' is a finite $\mathcal{O}_0'$-module. Indeed, M is an $\mathcal{O}_0/h\mathcal{O}_0$-module, and finite as such. Since $\mathcal{O}_0/h\mathcal{O}_0$ is also finitely generated as an $\mathcal{O}_0'$-module, our claim follows.

If $M=0$, then $\operatorname{depth} M=\operatorname{depth} M'=\infty$. So, we will assume that $M\neq 0$. Suppose first that $\operatorname{depth} M'=0$. We claim that there is a nonzero $u\in M$ such that $\mathfrak{m}'\,u=0$. Indeed, this is obvious when $m=1$, since $\mathcal{O}_0'$ is a field. On the other hand, when $m\geq 2$, we arrive at this conclusion by applying Proposition 4.3.

Write $h=z_m^d+\sum_{j=0}^{d-1} a_j z_m^j$, where $a_j\in\mathfrak{m}'$, $d>0$. As $u\neq 0$ but

$$z_m^d u = hu - \sum_{j=0}^{d-1} a_j u z_m^j = 0,$$

there is a largest $k=0,1,\dots,d-1$ such that $v=z_m^k u\neq 0$. Then $z_m v=0$, whence $\mathfrak{m}\,v=0$, and, since $\mathcal{O}_0$ is not a field when $m\geq 2$, we conclude by Proposition 4.3 that $\operatorname{depth} M=0$.

Conversely, suppose that $\operatorname{depth} M=0$. By Proposition 4.3, there is a nonzero submodule $L\subset M$ with $\mathfrak{m}\,L=0$. We claim that $\operatorname{depth} M'=0$. Indeed, since $\mathfrak{m}'\subset\mathfrak{m}$, this is a consequence of Proposition 4.3 when $m\geq 2$. On the other hand, this is obvious if $m=1$, for in this case, $\mathcal{O}_0'$ is a field and $M\neq 0$. $\square$

5. The proof of Theorem 1.3

Let us write (T_m) for the statement of Theorem 1.3, to indicate the number of variables involved. We prove it by induction on $m\geq 0$. When $m=0$, the depth assumption does not hold, unless $M=0$, and so, the claim is obvious.

Now assume that (T_{m-1}) holds for some $m\geq 1$, and prove (T_m). We are free to take $\zeta=0$. Let Q be the field of fractions of $\mathcal{O}_0$. We first verify (T_m) for torsion modules M, i.e., those for which $M\otimes Q=0$.

Since each generator $v\in M$ is annihilated by some nonzero $h_v\in\mathcal{O}_0$, there is a nonzero $h\in\mathcal{O}_0$ that annihilates all of M. By the Weierstrass preparation theorem, we can assume that h is (the germ of) a Weierstrass polynomial of degree $d\geq 1$ in z_m. We write $z=(z',z_m)$ for $z\in\mathbb{C}^m$. Let $\mathcal{O}_0'$ denote the ring of $\mathbb{C}$-valued holomorphic germs at 0 in $\mathbb{C}^{m-1}$ and, for any Banach space F, $\mathcal{O}_0'^F$ denote the $\mathcal{O}'$-module of F-valued holomorphic germs at 0 in $\mathbb{C}^{m-1}$. As before we embed $\mathcal{O}_0'\subset\mathcal{O}_0$ and $\mathcal{O}_0'^F\subset\mathcal{O}_0^F$. This makes any $\mathcal{O}_0$-module into an $\mathcal{O}_0'$-module, and any homomorphism $\phi\colon N_1\to N_2$ of $\mathcal{O}_0$-modules descends to an $\mathcal{O}_0'$-homomorphism

$$\phi'\colon N_1/hN_1 \longrightarrow N_2/hN_2. \tag{6}$$

A version of Weierstrass' division theorem remains true for holomorphic germs valued in a Banach space F (the proof in [GR] applies). Concretely, we can write any $f\in\mathcal{O}_0^F$ uniquely as

$$f=hf_0+\sum_{j=0}^{d-1} f'_j z_m^j, \quad f_0\in\mathcal{O}_0^F,\ f'_j\in\mathcal{O}'^F_0. \tag{7}$$

Clearly, the $\mathcal{O}'_0$-homomorphism

$$\mathcal{O}_0^F\ni f\longmapsto (f'_0,\dots,f'_{d-1})\in(\mathcal{O}'^F_0)^{\oplus d} \tag{8}$$

descends to an isomorphism

$$\mathcal{O}_0^F/h\mathcal{O}_0^F \xrightarrow{\approx} (\mathcal{O}'^F_0)^{\oplus d} \tag{9}$$

of $\mathcal{O}'_0$-modules. Composing this with the embedding

$$(\mathcal{O}'^F_0)^{\oplus d}\ni (f'_0,\dots,f'_{d-1})\longmapsto \sum_{j=0}^{d-1} f'_j z_m^j\in\mathcal{O}_0^F, \tag{10}$$

we obtain an $\mathcal{O}'_0$-homomorphism

$$\mathcal{O}_0^F/h\mathcal{O}_0^F\longrightarrow\mathcal{O}_0^F, \tag{11}$$

which is a right inverse of the canonical projection $\mathcal{O}_0^F\to\mathcal{O}_0^F/h\mathcal{O}_0^F$.

Now $p\colon\mathcal{O}_0^n\to M$ and $\phi\colon\mathcal{O}_0^E\to M$ of Theorem 1.3 induce $\mathcal{O}'_0$-homomorphisms

$$p'\colon\mathcal{O}_0^n/h\mathcal{O}_0^n\longrightarrow M \quad\text{and}\quad \phi'\colon\mathcal{O}_0^E/h\mathcal{O}_0^E\longrightarrow M,$$

as in (6), remembering that $hM=0$. Clearly, p' is surjective. Also, by Lemma 4.7, $\operatorname{depth} M'>0$. Because of the isomorphism (9), (T_{m-1}) implies that there is an $\mathcal{O}'_0$-homomorphism

$$\bar\chi\colon\mathcal{O}_0^E/h\mathcal{O}_0^E\longrightarrow\mathcal{O}_0^n/h\mathcal{O}_0^n$$

such that $\phi'=p'\bar\chi$. Since the projection $\mathcal{O}_0^n\to\mathcal{O}_0^n/h\mathcal{O}_0^n$ has a right inverse, cf. (11), $\bar\chi$ is induced by an $\mathcal{O}'_0$-homomorphism $\chi\colon\mathcal{O}_0^E\to\mathcal{O}_0^n$, which then satisfies $\phi=p\chi$. All that remains is to replace χ by an $\mathcal{O}_0$-homomorphism ψ, which we achieve as follows.

If a holomorphic germ, say f, at 0, valued in a Banach space $(F,\|\cdot\|_F)$, has a representative on a connected neighborhood V of 0, we write

$$[f]_V=\sup_{v\in V}\|f(v)\|_F\le\infty,$$

where f on the right-hand side of equality stands for the representative. Now consider the composition of $\chi|_{\mathcal{O}_0'^E}$ with (8), where $F{=}\mathbb{C}^n$,

$$\mathcal{O}_0'^E \xrightarrow{\chi} \mathcal{O}_0^n \longrightarrow \mathcal{O}_0'^{nd}. \tag{12}$$

By Theorem 1.1, this $\mathcal{O}_0'$-homomorphism is induced by a $\mathrm{Hom}(E,\mathbb{C}^{nd})$-valued holomorphic function, defined on some neighborhood U of $0{\in}\mathbb{C}^{m-1}$. It follows that if $V'{\Subset}U$ and $V''{\Subset}\mathbb{C}$ are connected neighborhoods of $0{\in}\mathbb{C}^{m-1}$, resp. $0{\in}\mathbb{C}$, then there is a constant C such that for each $e'{\in}\mathcal{O}_0'^E$ that has a representative defined on V'

$$[\chi(e')]_{V'\times V''} \le C[e']_{V'}. \tag{13}$$

Indeed, χ is obtained by composing (12) with (10) (again, $F{=}\mathbb{C}^n$), and this latter is trivial to estimate.

Now define $\psi\colon \mathcal{O}_0^E{\to}\mathcal{O}_0^n$ by $\psi(e){=}\sum_{j=0}^{\infty}\chi(e_j')z_m^j$, where $e{=}\sum_{k=0}^{\infty}e_j'z_m^j{\in}\mathcal{O}_0^E$. Cauchy estimates for e_j' and (13) together imply that the series above indeed represents a germ $\psi(e){\in}\mathcal{O}_0^n$. It is straightforward that ϕ is an $\mathcal{O}_0$-homomorphism. Because of this, $p\psi{=}\phi$ holds on $h\mathcal{O}_0^E$, both sides being zero. It also holds on polynomials $e{=}\sum_{j=0}^{k}e_j'z_m^j$, as

$$(p\psi)(e) = p\sum_{j=0}^{k}\chi(e_j')z_m^j = \sum_{j=0}^{k}\phi(e_j')z_m^j = \phi(e).$$

The division formula (7), this time with $F{=}E$, now implies that $p\psi{=}\phi$ on all $\mathcal{O}_0^E$.

Having taken care of torsion modules, consider a general module M as in the theorem. Since $\mathcal{O}_0$ is Noetherian, $\ker p$ is finitely generated; let

$$\rho\colon \mathcal{O}_0^r \longrightarrow \mathcal{O}_0^n$$

have image $\ker p$. So, $M{\approx}\operatorname{coker}\rho$. Construct a free $\mathcal{O}_0$-module C together with a homomorphism $\theta\colon \mathcal{O}_0^n{\to}C$ as in Lemma 4.5. Let $\pi\colon C{\to}\operatorname{coker}\theta\rho$ be the canonical projection. Here is a diagram to keep track of all the homomorphisms in question:

$$\begin{tikzcd}
\mathcal{O}_0^r \arrow[d, "\rho"'] & & \\
\mathcal{O}_0^n \arrow[r, "\theta"] \arrow[d, "p"'] & C \arrow[r, "\pi"] & \operatorname{coker}\theta\rho \\
M & & \mathcal{O}_0^E. \arrow[ll, "\phi"'] \arrow[u, "\tilde{\phi}"'] \arrow[ul, dashed, "\tilde{\psi}"']
\end{tikzcd}$$

We are yet to introduce $\tilde{\phi}$ and $\tilde{\psi}$. For $e\in\mathcal{O}_0^E$ choose $v\in\mathcal{O}_0^n$ so that $p(v)=\phi(e)$. Then $\pi\theta(v)$ is independent of which v we choose, since any two choices differ by an element of $\ker p=\operatorname{Im}\rho$, which $\pi\theta$ then maps to 0. We let $\tilde{\phi}(e)=\pi\theta(v)$. We want to lift $\tilde{\phi}$ to C; this certainly can be done if $\operatorname{coker}\theta\rho=0$. Otherwise Lemma 4.5 guarantees (as $\operatorname{depth} M=\operatorname{depth}\operatorname{coker}\rho>0$) that $\operatorname{depth}\operatorname{coker}\theta\rho>0$ and $(\operatorname{coker}\theta\rho)\otimes Q=0$. Hence we can apply the first part of this proof to obtain a homomorphism $\tilde{\psi}\colon \mathcal{O}_0^E\to C$ such that $\pi\tilde{\psi}=\tilde{\phi}$.

Finally, we lift $\tilde{\psi}$ to $\mathcal{O}_0^n$ as follows. For $e\in\mathcal{O}_0^E$ choose $v\in\mathcal{O}_0^n$ and $w\in\mathcal{O}_0^r$ so that $\phi(e)=p(v)$ and $\tilde{\psi}(e)=\theta(v)+\theta\rho(w)$. Again $v+\rho(w)\in\mathcal{O}_\zeta^n$ is independent of the choices. (It suffices to verify this for $e=0$. Then $v\in\ker p=\operatorname{Im}\rho$; let $v\in\rho(u)$, $u\in\mathcal{O}_0^r$. Hence $0=\theta(v)+\theta\rho(w)=\theta\rho(u+w)$. By Lemma 4.5 this implies that $0=\rho(u+w)=v+\rho w$ as claimed.)

Therefore we can define a homomorphism $\psi\colon \mathcal{O}_0^E\to\mathcal{O}_0^n$ by letting $\psi(e)=v+\rho w$. Since $p\psi(e)=p(v)=\phi(e)$, ψ is the homomorphism we were looking for, and the proof of Theorem 1.3 is complete.

We conclude this section by showing that the depth condition in Theorem 1.3 is also necessary.

Theorem 5.1. *Let $m,n\geq 1$, $\zeta\in\mathbb{C}^m$, $M\neq 0$ be a finite $\mathcal{O}_\zeta$-module, and $p\colon \mathcal{O}_\zeta^n\to M$ be an epimorphism. If* $\operatorname{depth} M=0$, *then for any infinite-dimensional Banach space E there is a homomorphism $\phi\colon \mathcal{O}_\zeta^E\to M$ that does not factor through p.*

Proof. We can view $\mathbb{C}\approx\mathcal{O}_\zeta/\mathfrak{m}_\zeta$ as an $\mathcal{O}_\zeta$-module. As $\operatorname{depth} M=0$, we have $\operatorname{Hom}_{\mathcal{O}_\zeta}(\mathbb{C},M)\neq 0$ by Definition 4.1. So, there is a nonzero $\mathcal{O}_\zeta$-homomorphism $\phi\colon \mathbb{C}\to M$. Consider a homomorphism $\varepsilon\colon \mathcal{O}_\zeta^E\to E$ of $\mathbb{C}$-vector spaces given by $\varepsilon(e)=e(0)$ and take $l\colon E\to\mathbb{C}$ to be a discontinuous $\mathbb{C}$-linear map. Then, $\phi l\varepsilon\colon \mathcal{O}_\zeta^E\to M$ is an $\mathcal{O}_\zeta$-homomorphism, which in view of Theorem 1.2, does not factor through $\mathcal{O}_\zeta^n$. $\square$

6. A local theorem

Theorem 6.1. *Let $\mathcal{S}$ be a coherent sheaf over an open set $\Omega\subset\mathbb{C}^m$ such that the depth of each nonzero stalk is positive. Suppose that $p\colon \mathcal{O}^n\to\mathcal{S}$ is an epimorphism and E is a Banach space. Then, for any $\zeta\in\Omega$, any $\mathcal{O}$-homomorphism $\mathcal{O}^E\to\mathcal{S}$ factors through p in some neighborhood of ζ.*

This theorem is a special case of the stronger Theorem 1.4. While the proof of the global result is postponed until Section 8, this local statement is a simple consequence of Theorem 1.3, once the following auxiliary result is shown.

Lemma 6.2. *Let $\mathcal{S}$ be a coherent sheaf over Ω. If $\zeta\in\Omega$, then there is a neighborhood $U\subset\Omega$ of ζ so that the evaluation map $\mathcal{S}(U)\to\mathcal{S}_\zeta$ is a monomorphism.*

Proof. We follow the outline given by the proof of Theorem 1.3. The lemma holds trivially for sheaves over $\Omega=\mathbb{C}^0=0$. For Ω lying in higher dimensions we proceed by induction. Initially, we verify the inductive step for torsion modules; then, the general case is proved by reduction to the torsion case.

We are free to take $\zeta=0$. As before, Q denotes the field of quotients of $\mathcal{O}_0$. Suppose that $\Omega\subset\mathbb{C}^m$, $m\geq 1$, and $\mathcal{S}_0\otimes Q=0$. Then $\mathcal{S}_0$ is annihilated by a nonzero $h\in\mathcal{O}_0$. After choosing a suitable neighborhood $U=U'\times U''\subset\mathbb{C}^{m-1}\times\mathbb{C}$ of 0, to be reduced further later, we can take h to be a Weierstrass polynomial of degree $d\geq 1$ in z_m and $h\mathcal{S}|_U=0$. We write $z=(z',z_m)$ and $\mathcal{O}'$ for the sheaf of germs in $\mathbb{C}^{m-1}$.

Let $|A|=\{z\in U'\times U''\colon h(z)=0\}$, $|B|=U'\times\{0\}$, and $\mathcal{O}_A=(\mathcal{O}/h\mathcal{O})|_{|A|}$. Define complex spaces $A=(|A|,\mathcal{O}_A)$ and $B=(|B|,\mathcal{O}')$. Since $h\mathcal{S}|_U=0$, $\operatorname{supp}\mathcal{S}|_U\subset|A|$ and $\mathcal{S}|_{|A|}$ has the structure of an $\mathcal{O}_A$-module. If

$$\mathcal{O}^r|_V \xrightarrow{\rho} \mathcal{O}^n|_V \xrightarrow{p} \mathcal{S}|_V \longrightarrow 0$$

is an exact sequence for some $V\subset U$, then the induced sequence

$$\mathcal{O}_A^r|_{|A|\cap V} \xrightarrow{\rho} \mathcal{O}_A^n|_{|A|\cap V} \xrightarrow{p} \mathcal{S}|_{|A|\cap V} \longrightarrow 0$$

is also exact. So, $\mathcal{S}|_{|A|}$ is also $\mathcal{O}_A$-coherent. The projection $U'\times U''\to U'$ induces a holomorphic Weierstrass map $\pi\colon A\to B$, see [GR, Section 2.3.4]. Since π is a finite map, [GR, Section 2.3.5], the direct image sheaf $\pi_*(\mathcal{S}|_{|A|})$ is a coherent sheaf over $U'\subset\mathbb{C}^{m-1}$.

Inductively we can assume that U' is such that the evaluation map

$$\pi_*(\mathcal{S}|_{|A|})(U') \longrightarrow \pi_*(\mathcal{S}|_{|A|})_0$$

is a monomorphism. On the other hand

$$\pi_*(\mathcal{S}|_{|A|})(U')=\mathcal{S}|_{|A|}(\pi^{-1}U')=\mathcal{S}(U) \quad \text{and} \quad \pi_*(\mathcal{S}|_{|A|})_0=\prod_{\xi\in\pi^{-1}(0)}\mathcal{S}_\xi=\mathcal{S}_0,$$

see [GR, Section 2.3.3]. So, $\mathcal{S}(U)\to\mathcal{S}_0$ is a monomorphism.

Now consider a general coherent sheaf $\mathcal{S}$. On some neighborhood U of ζ there exists an exact sequence

$$\mathcal{O}^r \xrightarrow{\rho} \mathcal{O}^n \xrightarrow{p} \mathcal{S}|_U \longrightarrow 0.$$

This neighborhood U can be taken so that there exists a homomorphism $\theta\colon \mathcal{O}^n|_U \to \mathcal{O}^s|_U$ as in Remark 4.6. Since $(\operatorname{coker}\theta\rho)_\zeta$ is a torsion $\mathcal{O}_\zeta$-module, we can apply the first part of the proof and assume that

$$(\operatorname{coker}\theta\rho)(U) \longrightarrow (\operatorname{coker}\theta\rho)_\zeta \quad \text{is a monomorphism.} \tag{14}$$

Furthermore, we can take U to be a pseudoconvex domain.

Suppose that $s\in\mathcal{S}(U)$ and $s_\zeta=0$. Let $v\in\mathcal{O}^n(U)$ be such that $p(v)=s$. Then $v_\zeta\in\operatorname{Im}\rho_\zeta$, say $v_\zeta=\rho_\zeta u_\zeta$ with $u_\zeta\in\mathcal{O}^r_\zeta$, and the class of $\theta_\zeta v_\zeta$ in $(\operatorname{coker}\theta\rho)_\zeta$ vanishes. By (14), θv represents the zero section in $\operatorname{coker}\theta\rho|_U$, i.e., there is $w\in\mathcal{O}^r(U)$ with $\theta v=\theta\rho w$. Hence, $(\theta\rho)_\zeta(u_\zeta-w_\zeta)=0$. From $\ker\rho_\zeta=\ker\theta_\zeta\rho_\zeta$ and $v_\zeta\in\operatorname{Im}\rho_\zeta$ we conclude that $\rho_\zeta(u_\zeta-w_\zeta)=0$ and $v_\zeta=\rho_\zeta w_\zeta$. On the other hand, v and ρw are holomorphic sections, over U, of the sheaf $\mathcal{O}^n$. Thus, $v=\rho w$ and $s=0$. $\square$

Proof of Theorem 6.1. Let $\phi\colon \mathcal{O}^E\to\mathcal{S}$ be an $\mathcal{O}$-homomorphism. If $\zeta\in\Omega$, then according to Theorem 1.2 there is a plain homomorphism $\psi^\zeta\colon \mathcal{O}^E_\zeta\to\mathcal{O}^n_\zeta$ so that

$$\phi|_{\mathcal{S}_\zeta}=p\psi^\zeta. \tag{15}$$

Since ψ^ζ is induced by a homomorphism-valued holomorphic map, ψ^ζ extends to a plain homomorphism $\psi_U\colon \mathcal{O}^E|_U\to\mathcal{O}^n|_U$ for a neighborhood $U\subset\Omega$ of ζ. By Lemma 6.2, we can assume that $\mathcal{S}(U)\to\mathcal{S}_\zeta$ is a monomorphism. In conjunction with (15), this implies that $\phi(v)-p\psi_U(v)=0$ for $v\in\mathcal{O}^E(U)$, in particular, if v is a constant section. Then, an application of Lemma 3.4 shows that $\operatorname{Im}(\phi_{\zeta'}-(p\psi_U)_{\zeta'})\subset\bigcap_{j=0}^\infty \mathfrak{m}^j_{\zeta'}\mathcal{S}_{\zeta'}=0$ for $\zeta'\in U$, i.e., that ϕ factors through p on U. $\square$

7. Applications

Our first application is Corollary 1.5, which depends on the following proposition.

Proposition 7.1. *Suppose that $(R,\mathfrak{m})$ is a local ring with residue field $k=R/\mathfrak{m}$, and M is a free R-module. If $c_\nu\in k$, for $\nu\in\mathbb{N}$, and $e_\nu\in M$ are such that their classes $\bar e_\nu$ in $M/\mathfrak{m}M$ are linearly independent over k, then there is an R-homomorphism $\phi\colon M\to R$ such that for every ν the class of $\phi(e_\nu)$ in k is c_ν.*

Proof. Let $\bar\phi\colon M/\mathfrak{m}M\to k$ be a k-linear map such that $\bar\phi(\bar e_\nu)=c_\nu$. Composing $\bar\phi$ with the projection $M\to M/\mathfrak{m}M$ we obtain an R-homomorphism $\psi\colon M\to k$ such that $\psi(e_\nu)=c_\nu$. If M is free then ψ can be lifted to a $\phi\colon M\to R$ as required. $\square$

Proof of Corollary 1.5. Let $e_\nu \in \mathcal{O}_\zeta^E$, $\nu=1,2,\ldots$, be germs such that $e_\nu(\zeta)\in E$ are $\mathbb{C}$-linearly independent unit vectors. Any $\mathcal{O}_\zeta$-homomorphism $\phi\colon \mathcal{O}_\zeta^E \to \mathcal{O}_\zeta$ is plain by Theorem 1.2, whence $\phi(e_\nu)(\zeta)\in\mathbb{C}$ is a bounded sequence. If $\mathcal{O}_\zeta^E$ were a submodule of a free module M then by Proposition 7.1 there would exist a homomorphism $\mathcal{O}_\zeta^E \to \mathcal{O}_\zeta$ such that $\phi(e_\nu)(\zeta)=\nu$, a contradiction. $\square$

For further applications we have to review some concepts introduced in [LP]. As there, in this review we place ourselves in an open subset Ω of a Banach space X; but our applications will only concern finite-dimensional X.

In the introduction we have already defined plain sheaves and homomorphisms. For sheaves $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{O}$-modules (always over Ω) we write $\mathbf{Hom}_{\mathcal{O}}(\mathcal{A},\mathcal{B})$ for the sheaf of $\mathcal{O}$-homomorphisms between them; if $\mathcal{A}$ and $\mathcal{B}$ are plain sheaves we write $\mathbf{Hom}_{\mathrm{plain}}(\mathcal{A},\mathcal{B})\subset\mathbf{Hom}_{\mathcal{O}}(\mathcal{A},\mathcal{B})$ for the sheaf of plain homomorphisms.

Definition 7.2. An analytic structure on a sheaf $\mathcal{S}$ is the choice, for each plain sheaf $\mathcal{E}$, of a submodule $\mathbf{Hom}(\mathcal{E},\mathcal{S})\subset\mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{S})$ subject to

(i) If $\mathcal{E}$ and $\mathcal{F}$ are plain sheaves, $x\in\Omega$, and $\varphi\in\mathbf{Hom}_{\mathrm{plain}}(\mathcal{E},\mathcal{F})_x$, then

$$\varphi^*\,\mathbf{Hom}(\mathcal{F},\mathcal{S})_x\subset\mathbf{Hom}(\mathcal{E},\mathcal{S})_x;$$

(ii) $\mathbf{Hom}(\mathcal{O},\mathcal{S})=\mathbf{Hom}_{\mathcal{O}}(\mathcal{O},\mathcal{S})$.

If $\mathcal{S}$ is endowed with an analytic structure, one also says that $\mathcal{S}$ is an *analytic sheaf*. This terminology is different from the traditional one, where "analytic sheaves" and "sheaves of $\mathcal{O}$-modules" mean one and the same thing.

If $U\subset\Omega$ is open, an $\mathcal{O}$-homomorphism $\psi\colon \mathcal{S}|_U\to\mathcal{S}'|_U$ of analytic sheaves is called *analytic* if $\psi_*\,\mathbf{Hom}(\mathcal{E}|_U,\mathcal{S}|_U)\subset\mathbf{Hom}(\mathcal{E}|_U,\mathcal{S}'|_U)$ for every plain sheaf $\mathcal{E}$.

Any plain sheaf $\mathcal{F}$ has a canonical analytic structure given by $\mathbf{Hom}(\mathcal{E},\mathcal{F})=\mathbf{Hom}_{\mathrm{plain}}(\mathcal{E},\mathcal{F})$. Further, on any $\mathcal{O}$-module $\mathcal{S}$ one can define a "maximal" analytic structure by $\mathbf{Hom}(\mathcal{E},\mathcal{S})=\mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{S})$; and also a "minimal" analytic structure, denoted by $\mathbf{Hom}_{\min}(\mathcal{E},\mathcal{S})$, consisting of germs α that can be written as a composition $\beta\gamma$ of

$$\gamma\in\mathbf{Hom}_{\mathrm{plain}}(\mathcal{E},\mathcal{O}^n)\quad\text{and}\quad\beta\in\mathbf{Hom}_{\mathcal{O}}(\mathcal{O}^n,\mathcal{S}),$$

where $n<\infty$. Definition 7.2 implies that

$$\mathbf{Hom}_{\min}(\mathcal{E},\mathcal{S})\subset\mathbf{Hom}(\mathcal{E},\mathcal{S})\subset\mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{S}).$$

In view of Theorems 1.1 and 6.1 we obtain the following uniqueness results.

Theorem 7.3. *For every plain sheaf $\mathcal{O}^F$ over an open set $\Omega\subset\mathbb{C}^m$, $0<m<\infty$, the canonical and the maximal analytic structures coincide.*

Proof. Let E be a Banach space, $U\subset\Omega$ be an open set, and $\phi\colon \mathcal{O}^E|_U\to\mathcal{O}^F|_U$ be an $\mathcal{O}$-homomorphism. By Theorem 1.1, ϕ is a plain homomorphism, and hence, an analytic homomorphism for the canonical analytic structure. Thus, $\mathbf{Hom}(\mathcal{O}^E,\mathcal{O}^F)=\mathbf{Hom}_{\mathcal{O}}(\mathcal{O}^E,\mathcal{O}^F)$. $\square$

Theorem 7.4. *Let $\mathcal{S}$ be a coherent sheaf of positive depth at each stalk. Then the minimal and the maximal analytic structures coincide, i.e., $\mathcal{S}$ has unique analytic structure.*

Proof. Denote by $\Omega\subset\mathbb{C}^m$ the base of the sheaf $\mathcal{S}$. Let E be a Banach space, $U\subset\Omega$ be an open set, and $\phi\colon \mathcal{O}^E|_U\to\mathcal{O}^F|_U$ be an $\mathcal{O}$-homomorphism. If $m=0$, the depth condition guarantees that $\mathcal{S}=0$ and the conclusion of the theorem follows. So, we may assume that $m\geq 1$.

Since $\mathcal{S}$ is a coherent sheaf, given $\zeta\in U$ there is an epimorphism $p\colon \mathcal{O}^n|_V\to\mathcal{S}|_V$, with $n<\infty$ and $V\subset U$, a suitable neighborhood of ζ. By Theorem 6.1, we can assume that $\phi|_V$ factors through $p|_V$, i.e., there is an $\mathcal{O}$-homomorphism $\psi\colon \mathcal{O}^E|_V\to\mathcal{O}^n|_V$ with $\phi|_V=p|_V\psi$. Then, by Theorem 1.1, ψ is a plain homomorphism, and so, $\phi_\zeta\in\mathbf{Hom}_{\min}(\mathcal{O}^E,\mathcal{O}^F)_\zeta$. Since ϕ and ζ were arbitrary, we have shown that $\mathbf{Hom}_{\min}(\mathcal{O}^E,\mathcal{O}^F)=\mathbf{Hom}_{\mathcal{O}}(\mathcal{O}^E,\mathcal{O}^F)$. $\square$

8. The proof of Theorem 1.4

This proof is based on [Lem, Theorem 4.3]: a coherent sheaf over $\Omega\subset\mathbb{C}^m$ endowed with its minimal analytic structure is cohesive. While [Lem] makes references to some of the results of the present paper, the proof of [Lem, Theorem 4.3] is independent of Theorem 1.4.

Let $\phi\colon \mathcal{O}^E\to\mathcal{S}$ be an $\mathcal{O}$-homomorphism. In view of Theorem 6.1, there is an open pseudoconvex cover $\mathfrak{V}$ of Ω such that on each $V\in\mathfrak{V}$ there is a homomorphism $\psi_V\colon \mathcal{O}^E|_V\to\mathcal{O}^n|_V$ with $\phi|_V=p\psi|_V$. If we let $\mathcal{K}=\ker p$, a coherent sheaf, then $\psi_{VW}=\psi_V-\psi_W$ maps $\mathcal{O}^E_{V\cap W}$ into $\mathcal{K}$, for $V,W\in\mathfrak{V}$. Thus, the $\mathcal{O}$-homomorphisms ψ_{VW} form a $\mathcal{K}$-valued 1-cocycle.

We can assume that $m\geq 1$, for otherwise the depth condition implies that $\mathcal{S}$ is the zero sheaf and there is nothing to prove. The module $\mathcal{K}\subset\mathcal{O}^n$ is torsion-free, i.e., $r_\zeta k_\zeta\neq 0$ for $\zeta\in\Omega$, $r_\zeta\in\mathcal{O}_\zeta$, and $k_\zeta\in\mathcal{K}_\zeta$, unless $r_\zeta=0$ or $k_\zeta=0$. Therefore, in view of Proposition 4.3, $\operatorname{depth}\mathcal{K}_\zeta>0$ for all $\zeta\in\operatorname{supp}\mathcal{K}$. We endow $\mathcal{K}$ with the minimal analytic structure, and note that, by Theorem 7.4, ψ_{VW} are analytic with respect to this structure. On the other hand, $\mathcal{K}$ is coherent, and hence, by [Lem, Theorem 4.3], is cohesive. Now $H^1(\Omega,\mathbf{Hom}(\mathcal{O}^E,\mathcal{K}))=0$, which is a special case of [LP, Theorem 9.1]. Consequently, $\psi_{VW}=\theta_V-\theta_W$ with some (analytic) homomorphisms

$\theta_V: \mathcal{O}^E|_V \to \mathcal{K}|_V$; defining $\psi: \mathcal{O}^E \to \mathcal{O}^n$ by

$$\psi|_V = \psi_V - \theta_V,$$

the resulting homomorphism satisfies $\phi = p\psi$.

References

[Eis] Eisenbud, D., *Commutative Algebra*, Graduate Texts in Mathematics, **150**, Springer, New York, 1995.

[GR] Grauert, H. and Remmert, R., *Coherent Analytic Sheaves*, Grundlehren der Mathematischen Wissenschaften **265**, Springer, Berlin, 1984.

[Lei] Leiterer, J., Banach coherent analytic Fréchet sheaves, *Math. Nachr.* **85** (1978), 91–109.

[Lem] Lempert, L., Coherent sheaves and cohesive sheaves, in *Complex Analysis*, Trends in Mathematics, pp. 227–244, Birkhäuser, Basel, 2010.

[LP] Lempert, L. and Patyi, I., Analytic sheaves in Banach spaces, *Ann. Sci. École Norm. Sup.* **40** (2007), 453–486.

[Mat] Matsumura, H., *Commutative Algebra*, 2nd ed., Mathematics Lecture Note Series **56**, Benjamin/Cummings, Reading, MA, 1980.

[Muj] Mujica, J., *Complex Analysis in Banach Spaces, Holomorphic Functions and Domains of Holomorphy in Finite and Infinite Dimensions*, North-Holland Mathematics Studies **120**, North-Holland, Amsterdam, 1986.

[Ser] Serre, J. P., Faisceaux algébriques cohérents, *Ann. of Math.* **61** (1955), 197–278.

Vakhid Masagutov
Department of Mathematics
Purdue University
West Lafayette, IN 47906
U.S.A.
vmasagut@hotmail.com

Received April 27, 2009
in revised form March 17, 2010
published online October 6, 2010