Ark. Mat., **48** (2010), 323–333
DOI: 10.1007/s11512-009-0109-1

Maximal invariant subspaces for a class of operators

Kunyu Guo, Wei He and Shengzhao Hou

Abstract. In this note, we characterize maximal invariant subspaces for a class of operators. Let T be a Fredholm operator and $1-TT^*\in\mathcal{S}_p$ for some $p\geq 1$. It is shown that if M is an invariant subspace for T such that $\dim M\ominus TM<\infty$, then every maximal invariant subspace of M is of codimension 1 in M. As an immediate consequence, we obtain that if M is a shift invariant subspace of the Bergman space and $\dim M\ominus zM<\infty$, then every maximal invariant subspace of M is of codimension 1 in M. We also apply the result to translation operators and their invariant subspaces.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space, and let $T\in B(\mathcal{H})$, the set of bounded linear operators acting on $\mathcal{H}$. A subspace M of $\mathcal{H}$ is called *invariant* for T if it is closed and $TM\subseteq M$. The well-known invariant subspace problem is the following.

Question 1. Does every bounded linear operator on a separable Hilbert space have a nontrivial invariant subspace?

The invariant subspace problem has an equivalent form on the Bergman space. Recall that the Bergman space $L_a^2(\mathbb{D})$ over the unit disk $\mathbb{D}$ is the closed subspace of $L^2(\mathbb{D}, dA)$ consisting of analytic functions, where dA is the normalized area measure on $\mathbb{D}$. Let M_z be the Bergman shift, i.e., the operator of multiplication by the coordinate function z. The invariant subspace problem is equivalent to the following question (cf. [2] and [8]).

Question 2. For two M_z-invariant subspaces M and N of the Bergman space $L_a^2(\mathbb{D})$ with $N\subseteq M$ and $\dim M\ominus N\geq 2$, is there another M_z-invariant subspace L such that $N\subsetneq L\subsetneq M$?

Using methods from function theory and operator theory, Hedenmalm [6] got an affirmative answer to Question 2 in the case $M = L^2_a(\mathbb{D})$, by showing that every maximal invariant subspace of $L^2_a(\mathbb{D})$ is of codimension 1 and of the form

$$M_\alpha = \{f \in L^2_a(\mathbb{D}) : f(\alpha) = 0\}, \quad \alpha \in \mathbb{D}.$$

Using completely different methods from classical operator theory, Trent [10] obtained the same result and generalized that result to the $\mathbb{C}^n$-valued Bergman space. The result was also obtained in Atzmon's paper [1].

In this note, enlightened by [6] and [10], we will study the invariant subspace problem in a much more general setting. Let $T \in B(\mathcal{H})$ and let M be an invariant subspace for T. A T-invariant subspace N is called a *maximal invariant subspace* of M, if $N \subsetneq M$ and there is no T-invariant subspace L such that $N \subsetneq L \subsetneq M$.

One motivation for studying maximal invariant subspaces lies in the analogy between maximal invariant subspaces and maximal ideals of commutative Banach algebras in the Gelfand theory. Another motivation is that, if for any invariant subspace M for the Bergman shift M_z, the maximal invariant subspaces of M are of codimension 1 in M, then Question 2 will be answered affirmatively.

Our main result is as follows.

Theorem 1.1. *Suppose T is a Fredholm operator and $1 - TT^* \in \mathcal{S}_p$ for some $p \geq 1$. If M is an invariant subspace for T such that $\dim M \ominus TM < \infty$, then every maximal invariant subspace of M is of codimension 1 in M.*

Applying the above theorem to the Bergman shift M_z, we get an affirmative answer to Question 2 in the case $\dim M \ominus zM < \infty$. This covers the result of Hedenmalm [6]. In a sense, our result provides weak support for the invariant subspace problem. Obviously, the case $\dim M \ominus zM = \infty$ is the obstacle to the invariant subspace problem.

In Section 2, we will give the proof of the main theorem. It is worth mentioning that the proof will make essential use of a classical result from operator theory (see Lemma 2.2 below), which was brought into our sight by Trent's paper [10]. We also use some techniques in Trent's paper [10].

In Section 3, we will apply the main theorem to concrete operators, which are closely related to the invariant subspace problem.

2. Proof of the main theorem

In this section, we will give the proof of the main theorem. To this end, we first establish some lemmas.

For $p>0$, we use the notation $\mathcal{S}_p$ to denote the set of Schatten-p class operators. Recall that an operator $T\in\mathcal{S}_p$ if and only if the trace of $(T^*T)^{p/2}$ is finite.

Lemma 2.1. *If R is Fredholm and $1-R^*R\in\mathcal{S}_p$, then $1-RR^*\in\mathcal{S}_p$.*

Proof. Since R is Fredholm, there is an operator $Q\in B(\mathcal{H})$ and a finite-rank operator F such that $RQ=1+F$. Then

$$\begin{aligned} R(1-R^*R)Q &= (1-RR^*)RQ \\ &= (1-RR^*)(1+F) \\ &= 1-RR^*+(1-RR^*)F. \end{aligned}$$

As both $R(1-R^*R)Q$ and $(1-RR^*)F$ are in $\mathcal{S}_p$, we have $1-RR^*\in\mathcal{S}_p$. $\square$

Lemma 2.2. ([9], p. 107) *Suppose $T\in B(\mathcal{H})$ and $1-TT^*\in\mathcal{S}_p$ for some $p\geq 1$, then T has a nontrivial invariant subspace.*

In the following, we use the notation P_M to denote the orthogonal projection form $\mathcal{H}$ to the closed subspace M.

Lemma 2.3. *Suppose T is Fredholm and $1-TT^*\in\mathcal{S}_p$ for some $p\geq 1$. If M and N are two invariant subspaces for T such that $\dim M\ominus N\geq 2$ and $\dim M\ominus TM<\infty$, then there is an invariant subspace L for T such that $N\subsetneq L\subsetneq M$.*

Proof. Consider the operator $S=P_{M\ominus N}T|_{M\ominus N}$. To prove the lemma, we first prove the following claim.

Claim 1. There is an invariant subspace L for T such that $N\subsetneq L\subsetneq M$ if and only if there is an invariant subspace L_0 for S such that $0\subsetneq L_0\subsetneq M\ominus N$.

In fact, if L is an invariant subspace for T such that $N\subsetneq L\subsetneq M$, put $L_0=L\ominus N$. Then

$$SL_0=P_{M\ominus N}T|_{M\ominus N}(L\ominus N)\subseteq L\ominus N=L_0.$$

Conversely, if L_0 is an invariant subspace for S such that $0\subsetneq L_0\subsetneq M\ominus N$, put $L=L_0+N$. Since $P_{M\ominus N}T|_{M\ominus N}L_0\subseteq L_0$, we have $P_{N^\perp}TL_0\subseteq L_0$. Hence

$$TL_0=P_NTL_0+P_{N^\perp}TL_0\subseteq N+L_0=L,$$

and then

$$TL=TL_0+TN\subseteq L.$$

By Claim 1, it suffices to show that the operator S has a nontrivial invariant subspace. By Lemma 2.2, the proof will be accomplished if the following claim is true.

Claim 2. $1-SS^*\in\mathcal{S}_p$.

Let $R=T|_M$. To prove the claim, we first show that $1-RR^*\in\mathcal{S}_p$.

Since T is Fredholm and $1-TT^*\in\mathcal{S}_p$, we have $1-T^*T\in\mathcal{S}_p$ by Lemma 2.1. Thus

$$1-R^*R=1-P_MT^*TP_M=P_M(1-T^*T)P_M\in\mathcal{S}_p.$$

Since $\ker R\subseteq\ker T$, we have $\dim\ker R<\infty$ and

$$\dim\operatorname{coker} R=\dim M\ominus RM=\dim M\ominus TM<\infty.$$

So R is Fredholm. Using Lemma 2.1 again, we have $1-RR^*\in\mathcal{S}_p$.

Since $T^*N^\perp\subseteq N^\perp$, we have

$$\begin{aligned}SS^*&=P_{M\ominus N}TP_{M\ominus N}T^*P_{M\ominus N}\\&=P_{M\ominus N}T(P_M-P_N)T^*P_{M\ominus N}\\&=P_{M\ominus N}TP_MT^*P_{M\ominus N}-P_{M\ominus N}TP_NT^*P_{M\ominus N}\\&=P_{M\ominus N}TP_MT^*P_{M\ominus N}.\end{aligned}$$

Then

$$\begin{aligned}1-SS^*&=P_{M\ominus N}(1-TP_MT^*)P_{M\ominus N}\\&=P_{M\ominus N}(1-(TP_M)(TP_M)^*)P_{M\ominus N}\\&=P_{M\ominus N}(1-RR^*)P_{M\ominus N}.\end{aligned}$$

As we have proved that $1-RR^*\in\mathcal{S}_p$, we obtain that $1-SS^*\in\mathcal{S}_p$. So Claim 2 is true and the proof is complete. □

Now the main theorem is a direct consequence of the above lemma. For the convenience of the reader, we restate it here.

Theorem 2.4. *Suppose T is a Fredholm operator and $1-TT^*\in\mathcal{S}_p$ for some $p\geq1$. If M is an invariant subspace for T such that $\dim M\ominus TM<\infty$, then every maximal invariant subspace of M is of codimension 1 in M.*

Remark 2.5. In Theorem 2.4, if the conditions on the operator T is replaced by $1-T^*T\in\mathcal{S}_p$ for some $p\geq 1$, and T has a finite generating set, then the conclusion still holds. In fact, the assumption that $1-T^*T\in\mathcal{S}_p$ implies that the range of T^* is closed and has finite codimension. Thus T is left semi-Fredholm. Moreover, the fact that T has a finite generating set implies that the range of T has finite codimension. Therefore, T is Fredholm. This reasoning is essentially due to [3]. By Lemma 2.1, $1-TT^*\in\mathcal{S}_p$. Thus the conditions on T in Theorem 2.4 are satisfied and the conclusion follows.

3. Applications

In this section, we will apply the main theorem to concrete operators on separable Hilbert spaces, which are closely related to the invariant subspace problem.

Example 3.1. Consider the Bergman shift M_z. As mentioned in the introduction, the invariant subspace problem is equivalent to Question 2 about the invariant subspaces for the Bergman shift M_z. Now let us see what information we can get from the main theorem.

Obviously, M_z is Fredholm. Let $e_n=z^n/\|z^n\|$. Then $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $L_a^2(\mathbb{D})$. It is easy to verify that

$$1-M_zM_z^*=\sum_{n=0}^{\infty}\frac{1}{n+1}e_n\otimes e_n,$$

so $1-M_zM_z^*\in\mathcal{S}_p$ for any $p>1$. Hence, if M is an invariant subspace for M_z such that $\dim M\ominus zM<\infty$, then by Theorem 1.1, every maximal invariant subspace of M is of codimension 1 in M. This answers Question 2 affirmatively in the case $\dim M\ominus zM<\infty$.

Given an invariant subspace M of $L_a^2(\mathbb{D})$ such that $\dim M\ominus zM<\infty$, what do the maximal invariant subspaces of M look like? To discuss this, we first need a fact. Let $\varphi_\lambda(z)=(z-\lambda)/(1-\bar{\lambda}z)$ be the Möbius transform. If M is an invariant subspace such that $\dim M\ominus zM<\infty$, then

$$\dim M\ominus\varphi_\lambda M=\dim M\ominus zM \quad \text{for any } \lambda\in\mathbb{D}.$$

In fact, since $\dim M\ominus zM<\infty$, $\sigma_e(M_z|_M)=\partial\mathbb{D}$ (cf. [11]). Hence for any $\lambda\in\mathbb{D}$, $\lambda-M_z$ is Fredholm. Thus

$$\dim M\ominus\varphi_\lambda M=-\operatorname{Index}(\lambda-M_z)=-\operatorname{Index}M_z=\dim M\ominus zM.$$

Now consider the case $\dim M\ominus zM=1$. By the above fact, for any $\lambda\in\mathbb{D}$, $\varphi_\lambda M$ is an invariant subspace of codimension 1 in M, and thus maximal in M. In fact, there are no other forms of maximal invariant subspaces of M, which is the following theorem.

Theorem 3.2. *Let M be an invariant subspace of the Bergman space $L^2_a(\mathbb{D})$ such that $\dim M\ominus zM=1$. Then all the maximal invariant subspaces of M are of the form $\varphi_\lambda M$ for some $\lambda\in\mathbb{D}$.*

Proof. Let N be a maximal invariant subspace of M. By Theorem 1.1, N is of codimension 1 in M. This implies that there is a unit vector e in M such that $M=N\oplus\mathbb{C}e$. Define a linear functional ψ on the disk algebra $A(\mathbb{D})$ by

$$\psi(f)=\langle fe,e\rangle \quad \text{for } f\in A(\mathbb{D}).$$

We claim that ψ is multiplicative. To see this, note that for $g\in A(\mathbb{D})$, $ge\in M$. Thus ge can be decomposed as $ge=ge-\langle ge,e\rangle e+\langle ge,e\rangle e$, where $ge-\langle ge,e\rangle e\in N$ and $\langle ge,e\rangle e\perp N$. Since N is invariant for M_z, we have $f(ge-\langle ge,e\rangle e)\in N$ for $f\in A(\mathbb{D})$. Hence

$$\begin{aligned}\psi(fg)&=\langle fge,e\rangle\\&=\langle f(ge-\langle ge,e\rangle e+\langle ge,e\rangle e),e\rangle\\&=\langle f(ge-\langle ge,e\rangle e),e\rangle+\langle fe,e\rangle\langle ge,e\rangle\\&=\langle fe,e\rangle\langle ge,e\rangle\\&=\psi(f)\psi(g).\end{aligned}$$

As ψ is multiplicative, there is a $\lambda_0\in\overline{\mathbb{D}}$ such that

$$\psi(f)=\langle fe,e\rangle=f(\lambda_0).$$

We claim that $\lambda_0\in\mathbb{D}$. To complete the proof, suppose that $|\lambda_0|=1$. Put

$$B=\{f\in A(\mathbb{D}):f(\lambda_0)=0\}.$$

For any $f\in B$, we have

$$\langle fe,e\rangle=\int_{\mathbb{D}} f|e|^2\,dA=f(\lambda_0)=0.$$

Since B is weak-star dense in $H^\infty(\mathbb{D})$, we have

$$\int_{\mathbb{D}} f|e|^2\,dA=0 \quad \text{for any } f\in H^\infty(\mathbb{D}),$$

and hence $\int_{\mathbb{D}}|e|^2\,dA=0$, a contradiction. Therefore, we have $\lambda_0\in\mathbb{D}$ as desired.

As $\langle\varphi_{\lambda_0}e, e\rangle = \varphi_{\lambda_0}(\lambda_0) = 0$, we have $\varphi_{\lambda_0}e \perp e$, and thus $\varphi_{\lambda_0}e \in N$. Hence

$$\varphi_{\lambda_0}M = \varphi_{\lambda_0}(N \oplus \mathbb{C}e) \subseteq N \subseteq M.$$

Noting that as $\dim M \ominus \varphi_{\lambda_0}M = 1$ and N is maximal in M, we have $N = \varphi_{\lambda_0}M$, completing the proof. □

If $M = L_a^2(\mathbb{D})$, the above theorem is Hedenmalm's result [6].

Now consider the case $\dim M \ominus zM = n > 1$. By the fact preceding Theorem 3.2, for any $\lambda \in \mathbb{D}$, $\dim M \ominus \varphi_\lambda M = n$, and thus $\varphi_\lambda M$ is not maximal in M. Given $\lambda \in \mathbb{D}$, for any $f \in M$, there is a nonnegative integer n_f such that $f = \varphi_\lambda^{n_f} g$ for some analytic function g with $g(\lambda) \neq 0$. Let

$$n_0 = \inf\{n_f : f \in M\}. \tag{3.1}$$

Then M can be written as $M = \varphi_\lambda^{n_0} M'$. Let

$$M'' = \{f \in M' : f(\lambda) = 0\}.$$

Then $\varphi_\lambda^{n_0} M''$ is of codimension 1 in M, and thus maximal in M. But unlike the case $\dim M \ominus zM = 1$, these are not all the maximal invariant subspaces of M. Here is an example.

Example 3.3. For any positive integer n, there is a sampling sequence A for $L_a^2(\mathbb{D})$ and a decomposition of it as a finite disjoint union $A = \bigcup_{j=1}^n A_j$, with the property that each $A \setminus A_j$ is an interpolating sequence for $L_a^2(\mathbb{D})$ (cf. [5] or [7, Lemma 6.3]). This can be used to construct an invariant subspace I such that $\dim I \ominus zI = n$. Assume without loss of generality that $0 \notin A$, for if $0 \in A$, we can replace A with $\varphi_\lambda(A)$ for some $\lambda \notin A$.

For $1 \leq j \leq n$, let I_j be the set of functions in $L_a^2(\mathbb{D})$ vanishing on $A \setminus A_j$, then I_j is a zero-based invariant subspace. Let G_j be the extremal function of I_j, by [7, Theorem 3.31 and Corollary 6.16], we have $G_j(0) \neq 0$ and $I_j = [G_j]$, the invariant subspace generated by G_j. Let $I = I_1 \vee \ldots \vee I_n$, then I is an invariant subspace with the property $\dim I \ominus zI = n$ [7, Theorem 6.4].

Let $N = zI \vee \mathbb{C}G_2 \vee \ldots \vee \mathbb{C}G_n$. Since $I = zI \vee \mathbb{C}G_1 \vee \mathbb{C}G_2 \vee \ldots \vee \mathbb{C}G_n$ and $G_1, \ldots, G_n$ are linearly independent [7, Theorem 6.4], N is of codimension 1 in I and thus maximal in I. Note that the functions in I have no common zeros, so $n_0 = 0$, where n_0 is as defined in (3.1). We claim that N is a maximal invariant subspace of I that cannot be written as $\{f \in I : f(\lambda) = 0\}$ for any $\lambda \in \mathbb{D}$. In fact, since $G_2(0) \neq 0$, N cannot be written as $\{f \in I : f(0) = 0\}$. Moreover, N cannot be written as $\{f \in I : f(\lambda_0) = 0\}$ for any $0 \neq \lambda_0 \in \mathbb{D}$ either, because the functions in zI have no other common zeros except 0.

Now we turn to another operator whose invariant subspaces are also closely related to the invariant subspace problem, and see what the main theorem tells us about this operator.

Let $\mathcal{F}_1$ denote the Fock type space

$$\mathcal{F}_1=\left\{h\in \mathrm{Hol}(\mathbb{C}):\|h\|^2=\frac{1}{2\pi}\int_{\mathbb{C}}|h(z)|^2e^{-2|z|}\,dA(z)<\infty\right\},$$

where dA is the area measure over the complex plane $\mathbb{C}$. The translation operator T_b on $\mathcal{F}_1$ is defined by

$$T_bf(z)=f(z+b)\quad \text{for } f\in\mathcal{F}_1.$$

Let $D=d/dz$ be the differential operator. By [4], both T_b and D are bounded on $\mathcal{F}_1$. A closed subspace M of $\mathcal{F}_1$ is called *translation invariant* if it is invariant for all the translation operators. Since $T_b=e^{bD}$, one can verify that a closed subspace M is translation invariant if and only if it is invariant for D [4].

It is shown in [4] that the invariant subspace problem is equivalent to the following question on translation invariant subspaces of $\mathcal{F}_1$.

Question 3. Given two translation invariant subspaces M and N of $\mathcal{F}_1$ with $N\subseteq M$ and $\dim M\ominus N\geq 2$, is there another translation invariant subspace L such that $N\subsetneq L\subsetneq M$?

Intuitively, Question 3 is similar to Question 2 and we may get similar conclusions as in Example 3.1. But the following theorem contradicts the intuition.

Theorem 3.4. *The space $\mathcal{F}_1$ has no maximal translation invariant subspace.*

Proof. Suppose that N is a maximal translation invariant subspace of $\mathcal{F}_1$. This turns out to be equivalent to the fact that N is a maximal invariant subspace for the differential operator D. Obviously, the operator D is Fredholm. Let $e_n=z^n/\|z^n\|$. Then $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{F}_1$. A direct computation shows that

$$1-DD^*=\sum_{n=0}^{\infty}\frac{1}{2n+3}e_n\otimes e_n.$$

So $1-DD^*\in\mathcal{S}_p$ for any $p>1$. Note that $\dim\mathcal{F}_1\ominus D\mathcal{F}_1=0<\infty$, and hence if there exists a maximal invariant subspace, say N, then by Theorem 1.1, N is of codimension 1 in $\mathcal{F}_1$, that is, $\dim N^{\perp}=1$.

To reach a contradiction, let

$$F^2(\mathbb{D})=\left\{f=\sum_{n=0}^{\infty}a_nz^n\in \mathrm{Hol}(\mathbb{D}):\|f\|^2_{F^2(\mathbb{D})}=|a_0|^2+\sum_{n=1}^{\infty}\frac{|a_n|^2}{\sqrt{n}}<\infty\right\}.$$

Then M_z is a bounded multiplication operator defined by the coordinate function. By [4], there is an invertible operator $S\colon \mathcal{F}_1\to F^2(\mathbb{D})$ such that

$$SD^*S^{-1}=M_z.$$

Since N is invariant for D and $N\neq\mathcal{F}_1$, $SN^\perp$ is a nontrivial invariant subspace for M_z, and thus $\dim SN^\perp=\infty$. This implies that $\dim N^\perp=\infty$, a contradiction. Therefore, the space $\mathcal{F}_1$ has no maximal translation invariant subspace, which is the desired conclusion. □

In view of the above theorem, it is not feasible to study maximal translation invariant subspaces of $\mathcal{F}_1$. However, the following definition is natural in this setting. Let $T\in B(\mathcal{H})$ and let N be an invariant subspace for T. A T-invariant subspace M is called *minimal* over N if $N\subsetneq M$ and there is no T-invariant subspace L such that $N\subsetneq L\subsetneq M$. We have the following proposition which is a direct consequence of Theorem 1.1.

Proposition 3.5. *Suppose T is a Fredholm operator and $1-TT^*\in\mathcal{S}_p$ for some $p\geq 1$. Let M and N be two T-invariant subspaces. If M is minimal over N and* $\dim N^\perp\ominus T^*N^\perp<\infty$, *then* $\dim M\ominus N=1$.

Applying the above proposition to the differential operator D on $\mathcal{F}_1$, we get that if M and N are two translation invariant subspaces of $\mathcal{F}_1$, M is minimal over N and $\dim N^\perp\ominus D^*N^\perp<\infty$, then $\dim M\ominus N=1$. This answers Question 3 affirmatively in the case $\dim N^\perp\ominus D^*N^\perp<\infty$.

Given a translation invariant subspace N of $\mathcal{F}_1$ with the property that

$$\dim N^\perp\ominus D^*N^\perp<\infty,$$

what do the minimal translation invariant subspaces over N look like? Here are two examples.

Example 3.6. The minimal translation invariant subspaces of $\mathcal{F}_1$ are exactly those minimal translation invariant subspaces over $\{0\}$. By Proposition 3.5, they are all one-dimensional. An easy computation shows that they are of the form $\mathbb{C}e^{\lambda z}$ for $\lambda\in\mathbb{D}$.

Example 3.7. By [4], the space $\mathcal{F}_1$ possesses a class of translation invariant subspaces of the form

$$N=\overline{\operatorname{span}}\bigg\{\bigcup_{k=1}^{\infty} e^{\lambda_k z}\mathcal{P}_{n_k}\bigg\}, \tag{3.2}$$

where $\Lambda=\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of distinct points in the unit disk $\mathbb{D}$, $\{n_k\}_{k=1}^{\infty}$ is a sequence of nonnegative integers and $\mathcal{P}_{n_k}$ is the set of polynomials of degree less than or equal to n_k. Suppose $N\neq\mathcal{F}_1$. We will describe all the minimal translation invariant subspaces over N.

To this end, recall that in the proof of Theorem 3.4, there is an invertible operator $S\colon \mathcal{F}_1\to F^2(\mathbb{D})$ such that

$$SD^*S^{-1}=M_z.$$

One can also verify that

$$\langle f, z^l e^{\lambda z}\rangle=\frac{d^l(Sf)}{dz^l}(\bar{\lambda}) \quad \text{for } f\in\mathcal{F}_1. \tag{3.3}$$

By the above equality, we can deduce that if N is a translation invariant subspace of the form (3.2), then

$$SN^{\perp}=\{g\in F^2(\mathbb{D}): g(\bar{\lambda}_k)=\ldots=g^{(n_k)}(\bar{\lambda}_k)=0,\ k=1,2,\ldots\},$$

and $SN^{\perp}$ is invariant for M_z. Noting that $N\neq\mathcal{F}_1$, we obtain that $SN^{\perp}$ is nontrivial.

Since $SN^{\perp}$ is zero based, we have $\dim SN^{\perp}\ominus M_z SN^{\perp}=1$. Similar to the proof of Theorem 3.2, every maximal invariant subspace L of $SN^{\perp}$ is of the form $\varphi_\lambda SN^{\perp}$ for some $\lambda\in\mathbb{D}$. More precisely, if $\lambda\notin\Lambda$, then

$$L=\{g\in F^2(\mathbb{D}): g(\bar{\lambda})=g(\bar{\lambda}_k)=\ldots=g^{(n_k)}(\bar{\lambda}_k)=0,\ k=1,2,\ldots\}.$$

If $\lambda\in\Lambda$, say $\lambda=\lambda_{k_0}$ for some k_0, then

$$L=\{g\in F^2(\mathbb{D}): g^{(n_{k_0}+1)}(\bar{\lambda}_{k_0})=g(\bar{\lambda}_k)=\ldots=g^{(n_k)}(\bar{\lambda}_k)=0,\ k=1,2,\ldots\}.$$

Applying equality (3.3) again, we get that if $\lambda\notin\Lambda$, then the minimal translation invariant subspace M over N is

$$M=\overline{\operatorname{span}}\bigg\{\bigcup_{k=1}^{\infty} e^{\lambda_k z}\mathcal{P}_{n_k}, e^{\lambda z}\bigg\}.$$

If $\lambda=\lambda_{k_0}$ for some k_0, the minimal translation invariant subspace M over N is

$$M=\overline{\operatorname{span}}\bigg\{\bigcup_{k=1}^{\infty} e^{\lambda_k z}\mathcal{P}_{n_k}, e^{\lambda_{k_0} z}\mathcal{P}_{n_{k_0}+1}\bigg\}.$$

References

1. Atzmon, A., Maximal, minimal, and primary invariant subspaces, *J. Funct. Anal.* **185** (2001), 155–213.
2. Bercovici, H., Foiaş, C. and Pearcy, C., *Dual Algebras with Applications to Invariant Subspaces and Dilation Theory*, CBMS Regional Conference Series in Mathematics **56**, Amer. Math. Soc., Providence, RI, 1985.
3. Douglas, R. and Sarkar, J., Essentially reductive weighted shift Hilbert modules, *Preprint*, 2008. arXiv:0807.3922v1
4. Guo, K. and Izuchi, K., Composition operators on Fock type spaces, *Acta Sci. Math.* (*Szeged*) **74** (2008), 807–828.
5. Hedenmalm, H., An invariant subspace of the Bergman space having the codimension two property, *J. Reine Angew. Math.* **443** (1993), 1–9.
6. Hedenmalm, H., Maximal invariant subspaces in the Bergman space, *Ark. Mat.* **36** (1998), 97–101.
7. Hedenmalm, H., Korenblum, B. and Zhu, K., *Theory of Bergman Spaces*, Graduate Texts in Mathematics **199**, Springer, New York, 2000.
8. Hedenmalm, H., Richter, S. and Seip, K., Interpolating sequences and invariant subspaces of given index in the Bergman spaces, *J. Reine Angew. Math.* **477** (1996), 13–30.
9. Radjavi, H. and Rosenthal, P., *Invariant Subspaces*, Springer, New York, 1973.
10. Trent, T., Maximal invariant subspaces for $A^2_\alpha(D)$, *Proc. Amer. Math. Soc.* **132** (2004), 2429–2432.
11. Zhu, K., Restriction of the Bergman shift to an invariant subspace, *Q. J. Math.* **48** (1997), 519–532.

Kunyu Guo
School of Mathematical Sciences
Fudan University
Shanghai
P.R. China
kyguo@fudan.edu.cn

Shengzhao Hou
Department of Mathematics
Suzhou University
Suzhou, Jiangsu
P.R. China
shou@suda.edu.cn

Wei He
Department of Mathematics
Southeast University
Nanjing, Jiangsu
P.R. China
051018010@fudan.edu.cn

Received September 23, 2008
published online October 1, 2009