

Rigidity around Poisson submanifolds

by

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Introduction

Recall that a *Poisson structure* on a manifold M is a Lie bracket $\{\cdot, \cdot\}$ on the space $C^\infty(M)$ of smooth functions on M which acts as a derivation in each entry, that is

$$\{f, gh\} = \{f, g\}h + \{f, h\}g, \quad f, g, h \in C^\infty(M).$$

A Poisson structure can also be given by a bivector $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi, \pi] = 0$ for the Schouten bracket. The Lie bracket is related to π by the formula

$$\langle \pi, df \wedge dg \rangle = \{f, g\}, \quad f, g \in C^\infty(M).$$

The *Hamiltonian* vector field of a function $f \in C^\infty(M)$ is

$$X_f = \{f, \cdot\} \in \mathfrak{X}(M).$$

These vector fields span an involutive singular distribution on M , which integrates to a partition of M into regularly immersed submanifolds called *symplectic leaves*. These leaves are symplectic manifolds, the symplectic structure on the leaf S is given by

$$\omega_S := \pi|_S^{-1} \in \Omega^2(S).$$

The zero-dimensional symplectic leaves are the points $x \in M$ where π vanishes. At such a fixed point x , the cotangent space $\mathfrak{g}_x = T_x^*M$ carries a Lie algebra structure, called the *isotropy Lie algebra* at x , with bracket given by

$$[d_x f, d_x g] := d_x \{f, g\}, \quad f, g \in C^\infty(M).$$

Conversely, starting from a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ there is an associated Poisson structure $\pi_{\mathfrak{g}}$ on the vector space \mathfrak{g}^* , called the *linear Poisson structure*, defined by

$$\{f, g\}_\xi := \langle \xi, [d_\xi f, d_\xi g] \rangle, \quad f, g \in C^\infty(\mathfrak{g}^*).$$

So, at a fixed point x , the tangent space $T_x M = \mathfrak{g}_x^*$ carries a canonical Poisson structure $\pi_{\mathfrak{g}_x}$ which plays the role of the first-order approximation of (M, π) around x in the realm of Poisson geometry. We recall Conn's linearization theorem [2].

CONN'S THEOREM. *Let (M, π) be a Poisson manifold and $x \in M$ be a fixed point of π . If the isotropy Lie algebra \mathfrak{g}_x is semisimple of compact type, then a neighborhood of x in (M, π) is Poisson-diffeomorphic to a neighborhood of the origin in $(\mathfrak{g}_x^*, \pi_{\mathfrak{g}_x})$.*

Conn's proof is analytic, it uses the fast convergence method of Nash and Moser. A new proof of Conn's theorem, which uses Poisson-geometric techniques, is now available in [6]. This geometric proof was adapted to the case of general symplectic leaves [7], and the outcome will be explained in the sequel.

Recall that the cotangent bundle of a Poisson manifold (M, π) is canonically a Lie algebroid $(T^*M, [\cdot, \cdot]_\pi, \pi^\sharp)$ with anchor given by the map

$$\begin{aligned} \pi^\sharp: T^*M &\longrightarrow TM, \\ \alpha &\longmapsto \pi(\alpha, \cdot), \quad \alpha \in T^*M, \end{aligned}$$

and the Lie bracket given by the expression

$$[\alpha, \beta]_\pi = L_{\pi^\sharp(\alpha)}(\beta) - L_{\pi^\sharp(\beta)}(\alpha) - d\pi(\alpha, \beta), \quad \alpha, \beta \in \Gamma(T^*M).$$

Generalizing the isotropy algebra from the case of fixed points, one associates with a symplectic leaf (S, ω_S) a transitive Lie algebroid $A_S := T^*M|_S$ over S , which is the restriction of T^*M to S , and is called the *restricted Lie algebroid*.

Conversely, using the data of a transitive Lie algebroid $(A, [\cdot, \cdot], \varrho)$ over a symplectic manifold (S, ω_S) , Vorobjev constructed in [23] a Poisson manifold $(N(A), \pi_A)$ which serves as the *first-order local model* of a Poisson structure around a symplectic leaf. The space $N(A)$ is an open set in $\mathfrak{g}(A)^*$, where $\mathfrak{g}(A) := \ker(\varrho)$ is the isotropy bundle. The Poisson manifold $(N(A), \pi_A)$ has (S, ω_S) (viewed as the zero section) as a symplectic leaf, and A can be recovered as the transitive Lie algebroid corresponding to this leaf: $A \cong A_S$. The construction depends on the choice of a linear left inverse to the inclusion $\mathfrak{g}(A) \subset A$, but, up to isomorphisms around S , the outcome does not depend on this choice (see §1.2 for more details).

In this setting, we recall the following normal form result (Theorem 1 in [7]).

THEOREM. (The normal form theorem from [7]) *Let (M, π) be a Poisson manifold, with (S, ω_S) a compact symplectic leaf. If the restricted Lie algebroid $A_S := T^*M|_S$ is integrable and the 1-connected Lie groupoid integrating it is compact and its s -fibers have vanishing de Rham cohomology in degree 2, then a neighborhood of S in (M, π) is Poisson-diffeomorphic to a neighborhood of the zero section in the local model $(N(A_S), \pi_{A_S})$.*

In the case of fixed points this is equivalent to Conn's result.

The original goal of this research was to reprove this theorem with methods similar to those of Conn's original approach. The main incentive for this is that Conn's analytic techniques are apparently more powerful than the geometric ones from [7]; in particular, as suggested to the author by Crainic, an analytic proof should imply rigidity of the Poisson structure. This is indeed the case, and the precise rigidity property that we obtain is the following:

Definition. A Poisson structure π on M is called C^p - C^1 -rigid around the compact submanifold $N \subset M$, if there are small enough open neighborhoods U of N , such that, for all open sets O with $N \subset O \subset \bar{O} \subset U$, there exist

- an open neighborhood $\mathcal{V}_O \subset \mathfrak{X}^2(U)$ of $\pi|_U$ in the compact-open C^p -topology;
- a function $\tilde{\pi} \mapsto \psi_{\tilde{\pi}}$, which associates with a Poisson structure $\tilde{\pi} \in \mathcal{V}_O$ a map $\psi_{\tilde{\pi}}: \bar{O} \rightarrow M$ which extends to an embedding of a neighborhood of \bar{O} , such that $\psi_{\tilde{\pi}}$ is a Poisson diffeomorphism

$$\psi_{\tilde{\pi}}: (O, \pi|_O) \xrightarrow{\sim} (\psi_{\tilde{\pi}}(O), \tilde{\pi}|_{\psi_{\tilde{\pi}}(O)}),$$

and ψ is continuous at $\tilde{\pi} = \pi$ (with $\psi_{\pi} = \text{Id}_{\bar{O}}$), with respect to the C^p -topology on the space of Poisson structures and the C^1 -topology on $C^\infty(\bar{O}, M)$.

We prove the following improvement of [7], which also includes rigidity.

THEOREM 1. *Let (M, π) be a Poisson manifold and (S, ω_S) be a compact symplectic leaf. If the Lie algebroid $A_S := T^*M|_S$ is integrable by a compact Lie groupoid whose s -fibers have vanishing de Rham cohomology in degree 2, then*

- (a) *in a neighborhood of S , π is Poisson diffeomorphic to its local model around S ;*
- (b) *π is C^p - C^1 -rigid around S .*

Already in the case of fixed points, the first part of this theorem gives a slight generalization of Conn's result, which cannot be obtained by an immediate adaptation of the arguments in [6] and [7]. Namely, a Lie algebra is integrable by a compact group with vanishing second de Rham cohomology if and only if it is compact and its center is at most 1-dimensional (see Lemma 2.3). The case when the center is trivial is Conn's result, and the 1-dimensional case is a consequence of a result of Monnier and Zung on smooth Levi decomposition of Poisson manifolds [20].

However, the main advantage of the approach of this paper over [7] is that it allows for a rigidity theorem around an arbitrary Poisson submanifold. Recall that a submanifold N of (M, π) such that π is tangent to N is called a *Poisson submanifold*. The symplectic leaves are the simplest type of Poisson submanifolds. The main result of this paper is the following rigidity theorem for integrable Poisson manifolds.

THEOREM 2. *Let (M, π) be a Poisson manifold for which the Lie algebroid T^*M is integrable by a Hausdorff Lie groupoid whose s -fibers are compact and their de Rham cohomology vanishes in degree 2. For every compact Poisson submanifold N of M we have that*

- (a) π is C^p - C^1 -rigid around N ;
- (b) up to isomorphism, π is determined around N by its first-order jet at N .

We prove Theorem 1 by applying part (b) of this result to the local model.

In both theorems, p has the (most probably not optimal) value

$$p = 7(\lfloor \frac{1}{2} \dim M \rfloor + 5).$$

In part (b) of Theorem 2 we prove that every Poisson structure $\tilde{\pi}$, defined around N , that satisfies $j^1\pi|_N = j^1\tilde{\pi}|_N$ is isomorphic to π around N by a diffeomorphism which is the identity on N up to first order.

The structure encoded by the first-order jet of π at N can be organized as an extension of Lie algebroids (see [15, Remark 2.2])

$$0 \longrightarrow \nu_N^* \longrightarrow T^*M|_N \longrightarrow T^*N \longrightarrow 0, \quad (1)$$

where $\nu_N^* \subset T^*M|_N$ is the conormal bundle and T^*N is the cotangent Lie algebroid of the Poisson manifold $(N, \pi|_N)$. With this, Theorem 1 follows easily from Theorem 2: if $S := N$ is a compact symplectic leaf, then the Poisson structures (M, π) and $(N(A_S), \pi_{A_S})$ have the same first-order jet around S (they induce the same exact sequence (1)); moreover, the hypothesis of Theorem 1 implies that Theorem 2 can be applied to the local model $(N(A_S), \pi_{A_S})$ (see Lemma 1.3).

One might try to follow the same line of reasoning and use Theorem 2 to prove a normal form theorem around Poisson submanifolds. Unfortunately, around general Poisson submanifolds, a first order local model does not seem to exist. Actually, there are Lie algebroid extensions as in (1) which do not arise as the first jet of Poisson structures (see [15, Example 2.3]). Nevertheless, one can use Theorem 2 to prove normal form results around particular classes of Poisson submanifolds.

The paper is organized as follows. In section §1, after recalling some properties of Lie groupoids and Lie algebroids, we describe in detail the local model around a leaf and a symplectic groupoid integrating it. We end the section by proving that Theorem 2 implies Theorem 1. §2 is an extended introduction to the paper, we give a list of applications, examples and connections with related literature. In §3 we prove Theorem 2 by using the Nash–Moser method. The appendices contain three general results

on Lie groupoids: existence of invariant tubular neighborhoods, integrability of the adjoint representation on a proper ideal, and the tame vanishing lemma. This last result provides tame homotopy operators for Lie algebroid cohomology with coefficients and, when combined with the Nash–Moser techniques, it is a very useful tool for handling similar geometric problems (see the appendix in [16]).

About the proof. The proof of the rigidity theorem is inspired mainly by Conn’s paper [2]. Conn uses a technique due to Nash and Moser to construct a sequence of changes of coordinates in which π converges to the linear Poisson structure $\pi_{\mathfrak{g}_x}$. At every step the new coordinates are found by solving some equations which are regarded as belonging to the complex computing the Poisson cohomology of $\pi_{\mathfrak{g}_x}$. To account for the “loss of derivatives” phenomenon during this procedure he uses smoothing operators. Finally, he proves uniform convergence of these changes of coordinates and of their higher derivatives on some ball around x .

Conn’s proof has been formalized in [18] and [20] into an abstract Nash–Moser normal form theorem. It is likely that part (a) of our Theorem 2 could be proven using [18, Theorem 6.8]. Due to some technical issues (see Remark 2), we cannot apply this result to conclude neither part (b) of our Theorem 2 nor the normal form Theorem 1, therefore we follow a direct approach.

We also simplified Conn’s argument by giving coordinate-free statements and working with flows of vector fields. For the expert: we gave up on the polynomial-type inequalities using instead only inequalities which assert tameness of certain maps, i.e. we work in Hamilton’s category of tame Fréchet spaces. Our proof deviates the most from Conn’s when constructing the homotopy operators. Conn recognizes the Poisson cohomology of $\pi_{\mathfrak{g}_x}$ as the Chevalley–Eilenberg cohomology of \mathfrak{g}_x with coefficients in the Fréchet space of smooth functions. By passing to the Lie group action on the corresponding Sobolev spaces, he proves existence of tame (in the sense of Hamilton [12]) homotopy operators for this complex. We, on the other hand, regard this cohomology as Lie algebroid cohomology, and prove a general tame vanishing result for the cohomology of Lie algebroids integrable by groupoids with compact s -fibers. This is done by further identifying this complex with the invariant part of the de Rham complex of s -foliated forms on the Lie groupoid, and by using the fiberwise inverse of the Laplace–Beltrami operator in order to construct the homotopy operators.

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1. Proof of the normal form theorem (Theorem 2 \Rightarrow Theorem 1)

In this section, we first recall some basic properties of Lie algebroids and Lie groupoids, next we describe the local model around a symplectic leaf from three different perspectives, and we conclude by showing that Theorem 1 is a consequence of Theorem 2.

1.1. Lie groupoids and Lie algebroids

We recall here some standard results about Lie groupoids and Lie algebroids, for definitions and other basic properties we recommend [13] and [19]. To fix notation, the anchor of a Lie algebroid $A \rightarrow M$ will be denoted by ϱ , the source and target maps of a Lie groupoid $\mathcal{G} \rightrightarrows M$ by s and t , respectively, and the unit map by u .

A Lie groupoid $\mathcal{G} \rightrightarrows M$ has an associated Lie algebroid $A(\mathcal{G})$ over M ; as a vector bundle, $A(\mathcal{G})$ is the restriction to M (i.e. pull-back by u) of the subbundle $T^s\mathcal{G}$ of $T\mathcal{G}$ consisting of vectors tangent to the s -fibers. The anchor is given by the differential of t . The Lie bracket comes from the identification between sections of $A(\mathcal{G})$ and right-invariant vector fields on \mathcal{G} .

A Lie algebroid $(A, [\cdot, \cdot], \varrho)$ is *integrable* if it is isomorphic to the Lie algebroid $A(\mathcal{G})$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$. Not every Lie algebroid is integrable (see [3]). If a Lie algebroid is integrable, then, as for Lie algebras, there exists, up to isomorphism, a unique Lie groupoid with 1-connected s -fibers integrating it.

A Lie algebroid $A \rightarrow M$ is *transitive* if ϱ is surjective. A Lie groupoid is *transitive* if the map $(s, t): \mathcal{G} \rightarrow M \times M$ is a surjective submersion. If \mathcal{G} is transitive then also $A(\mathcal{G})$ is transitive. Conversely, if $A \rightarrow M$ is transitive and M is connected, then every Lie groupoid integrating it is transitive as well.

Out of a principal bundle $q: P \rightarrow S$ with structure group G one can construct a transitive Lie groupoid $\mathcal{G}(P)$, called the *gauge groupoid* of P , as

$$\mathcal{G}(P) := P \times_G P \rightrightarrows S,$$

with structure maps given by

$$s([p_1, p_2]) := q(p_2), \quad t([p_1, p_2]) := q(p_1) \quad \text{and} \quad [p_1, p_2][p_2, p_3] := [p_1, p_3].$$

The Lie algebroid of $\mathcal{G}(P)$ is TP/G , where the Lie bracket is obtained by identifying sections of TP/G with G -invariant vector fields on P . Conversely, every transitive Lie groupoid \mathcal{G} is the gauge groupoid of a principal bundle: the bundle is any s -fiber of \mathcal{G} and the structure group is the isotropy group. So, a transitive Lie algebroid A is integrable if and only if there exists a principal G -bundle P such that A is isomorphic to TP/G .

A *symplectic groupoid* $(\mathcal{G}, \omega) \rightrightarrows M$ is a Lie groupoid $\mathcal{G} \rightrightarrows M$ endowed with a symplectic structure $\omega \in \Omega^2(\mathcal{G})$ for which the graph of the multiplication is a Lagrangian submanifold:

$$\{(g_1, g_2, g_3) : g_1 g_2 = g_3\} \subset (\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}, \text{pr}_1^*(\omega) + \text{pr}_2^*(\omega) - \text{pr}_3^*(\omega)).$$

This condition has several consequences. It implies that the base carries a Poisson structure π such that the source map is Poisson and the target map is anti-Poisson; and moreover, that \mathcal{G} integrates the cotangent Lie algebroid T^*M of π . Conversely, if for a given Poisson manifold (M, π) the Lie algebroid T^*M is integrable, then the s -fiber 1-connected integration of T^*M is canonically a symplectic groupoid [14].

1.2. The local model

Consider a Poisson manifold (M, π) and let (S, ω_S) be an embedded symplectic leaf. The local model of π around S , constructed first by Vorobjev in [23], is a Poisson structure defined on some open neighborhood of S in M , which plays the role of a first-order approximation of π around S .

The local model depends (up to diffeomorphisms around S that fix S) only on the first jet of π at S , denoted by $j^1\pi|_S$. Consider the transitive Lie algebroid

$$A_S := T^*M|_S$$

associated with S . Note that the anchor of A_S is given by the inverse of the symplectic structure ω_S , and that the isotropy bundle of A_S is the conormal bundle $\nu_S^* \subset A_S$. In fact, $j^1\pi|_S$ encodes precisely the Lie algebroid structure on A_S (see [16, Proposition 4.1.13]).

PROPOSITION 1.1. *Let π_1 and π_2 be two Poisson structures defined around S , such that S is a symplectic leaf for both. Then π_1 and π_2 induce the same Lie algebroid structure on $A_S = T^*M|_S$ if and only if $j^1\pi_1|_S = j^1\pi_2|_S$.*

We give three different descriptions of the local model, each of them bringing different insight into the construction. All three constructions avoid the explicit use of Vorobjev triples, by using instead Dirac geometric techniques. For the proofs of the claims made here, we refer the reader to [16, §4.1 and §4.2].

Description 1

Our first approach to the local model is a Dirac geometric interpretation of the linearization procedure from [5]; and it is very useful for explicit computations of the local model. Consider a tubular neighborhood

$$\Psi: \nu_S \longrightarrow M$$

of S in M , where $\nu_S := TM|_S/TS$ is the normal bundle to S . Set $E := \Psi(\nu_S)$, denote by $\mu_t: E \rightarrow E$ the map corresponding to multiplication by $t \in \mathbb{R}$ on ν_S , and by $p: E \rightarrow S$ the corresponding projection map. Consider the path of Poisson structures

$$\pi_t := t\mu_t^*(\pi^{(t-1)p^*(\omega_S)}), \quad t \in (0, 1], \quad (2)$$

where, for a closed 2-form β , π^β denotes the gauge transform of π by β (i.e. the leaves of π^β are the leaves of π , but the symplectic structures on them differ by the restrictions of β). In fact, π_t is well defined on the entire E only as a Dirac structure (see [1] for the basics of Dirac geometry), which is given by

$$L_t := t\mu_t^*(L_\pi^{(t-1)p^*(\omega_S)}) \subset TE \oplus T^*E,$$

where L_π is the Dirac structure corresponding to π , and, for a Dirac structure L and $\lambda \in \mathbb{R} \setminus \{0\}$, we denote by λL the Dirac structure $\{\lambda X + \xi: X + \xi \in L\}$. Now L_t extends smoothly at $t=0$, and we let $L_0 := \lim_{t \rightarrow 0} L_t$. On the other hand, we have that L_t has (S, ω_S) as a (pre)symplectic leaf, for all $t \in \mathbb{R}$, and therefore there is an open neighborhood U of S such that L_t corresponds to a Poisson structure π_t on U for all $t \in [0, 1]$. The limit Poisson structure

$$\pi_0 := \lim_{t \rightarrow 0} \pi_t,$$

defined on U , is the local model of π around S . We also have that

$$j^1\pi_t|_S = j^1\pi|_S, \quad t \in \mathbb{R},$$

and in particular, by Proposition 1.1, the local model π_0 induces the same Lie algebroid structure on $A_S = T^*M|_S$.

Different choices of tubular neighborhoods of S give rise to local models that are isomorphic around S by diffeomorphisms that fix S .

Note also that the Dirac-geometric nature of this construction allows one to define in a similar fashion the local model of a Dirac structure around an embedded presymplectic leaf; the outcome is a Dirac structure which is globally defined on E .

Description 2

The second description comes closest to Vorobjev’s original construction [23]. The construction uses the data encoded by the first jet of a Poisson structure at a leaf: a symplectic manifold (S, ω_S) and a transitive Lie algebroid $(A, [\cdot, \cdot]_A, \varrho)$ over S . Similar to the linear Poisson structure on the dual of a Lie algebra, the dual vector bundle A^* carries a linear Poisson structure $\pi_{\text{lin}}(A)$, with Poisson bracket determined by

$$\{p^*(f), p^*(g)\} = 0, \quad \{\tilde{\alpha}, p^*(g)\} = p^*(L_{\varrho(\alpha)}g) \quad \text{and} \quad \{\tilde{\alpha}, \tilde{\beta}\} = [\widetilde{\alpha}, \widetilde{\beta}]_A,$$

for all $f, g \in C^\infty(S)$ and all $\alpha, \beta \in \Gamma(A)$, where $p: A^* \rightarrow S$ denotes the projection, and $\tilde{\alpha}, \tilde{\beta} \in C^\infty(A^*)$ denote the corresponding fiberwise linear functions on A^* . Consider the gauge transform of $\pi_{\text{lin}}(A)$ by $p^*(\omega_S)$:

$$\pi_{\text{lin}}^{p^*(\omega_S)}(A).$$

A priori, this gauge transform is defined only as a Dirac structure on A^* , but because of the particular structure of the linear Poisson structure, $\pi_{\text{lin}}^{p^*(\omega_S)}(A)$ is in fact a well-defined Poisson structure on A^* .

Let $\mathfrak{g}(A) := \ker(\varrho) \subset A$ be the isotropy bundle. Consider a linear spitting $\sigma: A \rightarrow \mathfrak{g}(A)$ of the short exact sequence

$$0 \longrightarrow \mathfrak{g}(A) \longrightarrow A \xrightarrow{\varrho} TS \longrightarrow 0. \tag{3}$$

Using the dual of σ , we regard $\mathfrak{g}(A)^*$ as a subbundle of A^* . An open neighborhood $N(A)$ of S in $\mathfrak{g}(A)^*$ is a Poisson transversal for $\pi_{\text{lin}}^{p^*(\omega_S)}(A)$ (also called a cosymplectic submanifold in the literature), i.e. for each symplectic leaf (L, ω_L) of $\pi_{\text{lin}}^{p^*(\omega_S)}(A)$, we have that $N(A)$ is transverse to L , and that $L \cap N(A)$ is a symplectic submanifold of L . This property allows one to pull back $\pi_{\text{lin}}^{p^*(\omega_S)}(A)$ to a Poisson structure π_A on $N(A)$: the leaves of π_A are $(L \cap N(A), \omega_L|_{L \cap N(A)})$, where, as before, (L, ω_L) is a leaf of $\pi_{\text{lin}}^{p^*(\omega_S)}(A)$. The Poisson manifold

$$(N(A), \pi_A)$$

represents the second description of the local model. Also, (S, ω_S) , identified with the zero section, is a symplectic leaf of π_A and the induced transitive Lie algebroid A_S is isomorphic to A via the maps

$$A_S = T^*\mathfrak{g}(A)^*|_S \cong T^*S \oplus \mathfrak{g}(A) \xrightarrow{(\omega_S^{-1, \sharp} + \sigma)} A.$$

Different choices of the splitting σ give rise to local models that are isomorphic around S by diffeomorphisms that fix S .

We now describe an isomorphism between the two Poisson manifolds resulting from the two descriptions of the local model. Let (S, ω_S) be an embedded symplectic leaf of the Poisson manifold (M, π) . Consider a tubular neighborhood of S , denoted by $\Psi: \nu_S \rightarrow M$, and let π_0 be the corresponding local model from the first description. Note that the Lie algebroid $A_S := T^*M|_S$ has isotropy bundle $\mathfrak{g}(A) = \nu_S^*$, and that the dual of the differential of Ψ along S gives a splitting of the anchor for A_S :

$$\sigma := (d\Psi|_S)^*: A_S \longrightarrow \nu_S^*.$$

Consider the local model π_{A_S} on a neighborhood of S in $\nu_S = \mathfrak{g}(A)^*$, constructed with the aid of σ . The map Ψ gives a Poisson diffeomorphism in a neighborhood of S between the two descriptions of the local model:

$$\Psi_*(\pi_{A_S}) = \pi_0.$$

We remark that, in general, the submanifold $\mathfrak{g}(A)^* \subset A^*$ is not Poisson transverse everywhere. Nevertheless, one can always pull back the Poisson structure $\pi_{\text{lin}}^{p^*(\omega_S)}(A)$ to a globally defined Dirac structure on $\mathfrak{g}(A)^*$, which is Poisson on $N(A)$. Actually, also this second construction works in the Dirac setting; and the outcome is a second description of the local model of a Dirac structure around a presymplectic leaf.

Description 3

The third description works only when the restricted Lie algebroid is integrable, and as remarked by Vorobjev in [23], the resulting Poisson manifold appeared already in the work of Montgomery [21]. The construction is standard in symplectic geometry as it represents the local form of a Hamiltonian space around the zero set of the moment map (see e.g. [11]).

The starting data is an integrable transitive Lie algebroid A over a symplectic manifold (S, ω_S) . Since A is transitive, it is isomorphic to TP/G for a principal G -bundle $P \rightarrow S$. So, the relevant first-order data becomes a principal G -bundle $p: P \rightarrow S$ over a symplectic manifold (S, ω_S) . Let $\theta \in \Omega^1(P, \mathfrak{g})$ be a principal connection on P , where \mathfrak{g} denotes the Lie algebra of G . Consider the following closed 2-form on $P \times \mathfrak{g}^*$, which is invariant under the diagonal action of G :

$$\Omega = p^*(\omega_S) - d\langle \mu | \theta \rangle, \quad \text{where } \mu(p, \xi) := \xi.$$

The open set Σ , where Ω is non-degenerate, is G -invariant and contains $P \times \{0\}$. The action of G is Hamiltonian with G -equivariant moment map $\mu: \Sigma \rightarrow \mathfrak{g}^*$. The local model is obtained as the quotient Poisson manifold

$$(N(P), \pi_P) := (\Sigma, \Omega)/G,$$

where $N(P) := \Sigma/G$ is an open neighborhood of the zero section in the associated coadjoint bundle $P[\mathfrak{g}^*] := (P \times \mathfrak{g}^*)/G$. The resulting Poisson structure π_P has (S, ω_S) (regarded as $(P \times \{0\})/G$) as a symplectic leaf, and its restricted Lie algebroid $T^*N(P)|_S$ is isomorphic to TP/G .

To relate this construction to the second, note that the isotropy bundle of $A = TP/G$ can be identified with the quotient $\mathfrak{g}(A) = (P \times \mathfrak{g})/G$, and so $\mathfrak{g}(A)^* = P[\mathfrak{g}^*]$. Also, note that there is a natural one-to-one correspondence between

- connection 1-forms $\theta \in \Omega^1(P, \mathfrak{g})$ on P , and
- linear splittings $\sigma: A \rightarrow \mathfrak{g}(A)$ of the sequence (3).

Now, under these isomorphisms and this correspondence, the Poisson manifold $(N(P), \pi_P)$, constructed with the aid of θ , and the Poisson manifold $(N(A), \pi_A)$, constructed using the corresponding σ , coincide.

The Poisson manifold $(N(P), \pi_P)$ is integrable, and we describe below a symplectic groupoid integrating it. Since this result fits into a more general framework, we state the following lemma, which is a direct consequence of results in [9].

LEMMA 1.2. *Let (Σ, Ω) be a symplectic manifold endowed with a proper, free Hamiltonian action of a Lie group G , and equivariant moment map $\mu: \Sigma \rightarrow \mathfrak{g}^*$. Then the Poisson manifold Σ/G is integrable, a symplectic Lie groupoid integrating it is*

$$(\Sigma \times_{\mu} \Sigma)/G \rightrightarrows \Sigma/G,$$

and the symplectic structure pulls back to $\Sigma \times_{\mu} \Sigma$ as $(s^*(\Omega) - t^*(\Omega))|_{\Sigma \times_{\mu} \Sigma}$.

Proof. Consider the symplectic groupoid $\Sigma \times \Sigma \rightrightarrows \Sigma$, with symplectic structure

$$s^*(\Omega) - t^*(\Omega).$$

Then G acts on $\Sigma \times \Sigma$ by symplectic groupoid automorphism with equivariant moment map $J := s^*\mu - t^*\mu$, which is also a groupoid 1-cocycle. By [9, Proposition 4.6], the Marsden–Weinstein reduction

$$(\Sigma \times \Sigma)//G = J^{-1}(0)/G$$

is a symplectic groupoid integrating the Poisson manifold Σ/G . In our case $J^{-1}(0) = \Sigma \times_{\mu} \Sigma$, and the symplectic form pulls back to $\Sigma \times_{\mu} \Sigma$ as $s^*(\Omega) - t^*(\Omega)|_{\Sigma \times_{\mu} \Sigma}$ \square

In our setting, the lemma shows that the groupoid integrating the local model $(N(P), \pi_P)$ is just the restriction to $N(P)$ of the action groupoid

$$\mathcal{G} := (P \times P \times \mathfrak{g}^*)/G \rightrightarrows P[\mathfrak{g}^*],$$

corresponding to the representation of $P \times_G P$ on $P[\mathfrak{g}^*]$. If P is compact, note that $N(P)$ contains arbitrarily small open sets of the form $P[V] := (P \times V)/G$, where V is a G -invariant neighborhood of 0 in \mathfrak{g}^* . These neighborhoods are \mathcal{G} -invariant, and the restriction of \mathcal{G} to $P[V]$ is $(P \times P \times V)/G$. In particular, all its s -fibers are diffeomorphic to P . This proves the following result.

PROPOSITION 1.3. *The local model $(N(P), \pi_P)$ associated with a principal bundle P over a symplectic manifold (S, ω_S) is integrable by a Hausdorff symplectic Lie groupoid. If P is compact, then there are arbitrarily small invariant open neighborhoods U of S , such that all s -fibers over points in U are diffeomorphic to P .*

1.3. Proof of Theorem 2 \Rightarrow Theorem 1

Consider a tubular neighborhood $\Psi: \nu_S \rightarrow M$ of S in M , and denote by π_0 the resulting local model constructed using the first description. So π_0 is a Poisson structure on some open neighborhood of S , which coincides with π up to first order. On the other hand, π_0 is isomorphic around S to the local model π_{A_S} corresponding to the transitive Lie algebroid $A_S := T^*M|_S$. By assumption, A_S is integrable, and so there is a principal G -bundle $P \rightarrow S$ such that $A_S \cong TP/G$. Moreover, we can choose P to be compact with vanishing second de Rham cohomology. By Proposition 1.3, for arbitrary small open neighborhoods U of S in $N(P)$, we have that $(U, \pi_P|_U)$ satisfies the assumption of Theorem 2. Since π_P is isomorphic to π_{A_S} around S , and also π_{A_S} is isomorphic to π_0 around S , we conclude that S has arbitrary small neighborhoods U in M for which $(U, \pi_0|_U)$ also satisfies the hypothesis of Theorem 2. By part (a), π_0 is C^p - C^1 -rigid around S , and by part (b), π_0 and π are Poisson diffeomorphic around S . Thus π is also C^p - C^1 -rigid around S .

2. Remarks, examples and applications

In this section we give a list of examples and applications for our two theorems and we also show some links with other results from the literature.

2.1. A global conflict

Theorem 2 does not exclude the case when the Poisson submanifold S is the total space M . In conclusion, a compact Poisson manifold (M, π) for which T^*M is integrable by a compact Lie groupoid whose s -fibers have trivial second de Rham cohomology is globally rigid. Nevertheless, this result is useless, since no such Poisson manifolds exist in dimension greater than 1. In the case when the groupoid has 1-connected s -fibers,

this conflict was pointed out in [4], and we explain below the general case. In symplectic geometry, this non-rigidity phenomenon is expressed by the fact that, on a compact symplectic manifold (M, ω) , the symplectic structure allows the smooth deformation $t\omega$, for $t > 0$, which is non-trivial because the symplectic volume changes.

PROPOSITION 2.1. *Consider a compact connected Poisson manifold (M, π) for which T^*M is integrable by a compact Lie groupoid \mathcal{G} whose s -fibers have trivial second de Rham cohomology. Then M is at most 1-dimensional.*

In the proof of the proposition we will use the volume-function vh given below.

LEMMA 2.2. *Consider the setting of Proposition 2.1. The set M^{reg} where π has maximal rank is open and dense. Every regular symplectic leaf $(S, \omega_S) \subset M^{\text{reg}}$ has a finite holonomy group, which we denote by $\text{Hol}(S)$, and a finite symplectic volume, which we denote by $\text{Vol}(S)$. The function*

$$\begin{aligned} \text{vh}: M &\longrightarrow \mathbb{R}, \\ x &\longmapsto \begin{cases} \text{Vol}(S_x) |\text{Hol}(S_x)|, & \text{if } x \in M^{\text{reg}}, \\ 0, & \text{if } x \notin M^{\text{reg}}, \end{cases} \end{aligned}$$

where S_x denotes the symplectic leaf through x , is continuous.

Proof of Proposition 2.1. By Lemma C.1 in the appendix, the second Poisson cohomology of (M, π) vanishes. In particular, the class $[\pi]$ is trivial, so there exists a vector field X such that $L_X(\pi) = \pi$. This implies that the flow of X gives a Poisson diffeomorphism

$$\varphi_X^t: (M, \pi) \xrightarrow{\sim} (M, e^{-t}\pi). \quad (4)$$

This and the Poisson geometric description of vh imply that $\text{vh} \circ \varphi_X^t = e^{tk} \text{vh}$, where $2k$ denotes the maximal rank of π . By Lemma 2.2, vh is bounded, and hence $\pi = 0$. If $\pi = 0$, then $\mathcal{G} \rightarrow M$ is a bundle of tori, so by the cohomological condition its fibers are at most 1-dimensional. Hence M is at most 1-dimensional as well. \square

Proof of Lemma 2.2. Clearly, M^{reg} is open. To show that M^{reg} is dense, by connectedness of M , it suffices to show that its closure $\overline{M^{\text{reg}}}$ is open. This follows from the following property of π , which we prove below: every leaf has a saturated neighborhood U , such that U^{reg} (i.e. the regular part of $(U, \pi|_U)$) is dense in U .

Let (S, ω_S) be a symplectic leaf of M . Since $\mathcal{G}|_S$ integrates A_S , by Theorem 1, the local model holds around S . So, for a compact, connected principal G -bundle P , we have that (M, π) is Poisson isomorphic around S to an open set around S in

$$(N(P), \pi_P) = (\Sigma, \Omega)/G,$$

where $\Sigma \subset P \times \mathfrak{g}^*$, $\Omega = p^*(\omega_S) - d\langle \mu | \theta \rangle$, θ is a principal connection on P , and $\mu: \Sigma \rightarrow \mathfrak{g}^*$, where $\mu(p, \xi) = \xi$, is an equivariant moment for the action of G . The symplectic leaves of π_P are of the form

$$(O_\xi, \omega_\xi), \quad O_\xi := P \times_G (G \cdot \xi), \quad \xi \in \mathfrak{g}^*,$$

and hence they are the base of the principal G_ξ -bundle

$$p_\xi: P \times \{\xi\} \longrightarrow O_\xi,$$

where G_ξ is the stabilizer of ξ , and the symplectic structure is determined by

$$p_\xi^*(\omega_\xi) = \Omega|_{P \times \{\xi\}}. \quad (5)$$

This last equation follows from the fact that the action is Hamiltonian, and therefore, the symplectic leaves are canonically isomorphic to the reduced spaces

$$\mu^{-1}(\xi) // G_\xi = (P \times \{\xi\}) // G_\xi.$$

We will show that ν_h extends to a continuous map on $P \times_G \mathfrak{g}^*$. Let T be a maximal torus in G and let \mathfrak{t} be its Lie algebra. By compactness of G , we can consider an invariant metric on \mathfrak{g} . This metric allows us to regard \mathfrak{t}^* as a subspace in \mathfrak{g}^* (i.e. the orthogonal to \mathfrak{t}), and it gives an isomorphism between the adjoint and the coadjoint representation which sends \mathfrak{t} to \mathfrak{t}^* . For the adjoint representation it is well known (see e.g. [8]) that every orbit hits \mathfrak{t} , and hence also every orbit of the coadjoint action hits \mathfrak{t}^* . An element $\xi \in \mathfrak{t}^*$ is *regular* if $\mathfrak{g}_\xi = \mathfrak{t}$, where \mathfrak{g}_ξ is the Lie algebra of G_ξ . Denote by $\mathfrak{t}^{*\text{reg}}$ the set of regular elements. Then $\mathfrak{t}^{*\text{reg}}$ is open and dense in \mathfrak{t}^* and it coincides with the set of elements ξ for which G_ξ/T is finite (see e.g. [8]). Thus, for $\xi \in \mathfrak{t}^*$, a leaf O_ξ has maximal dimension if and only if $\xi \in \mathfrak{t}^{*\text{reg}}$, and hence the regular part of π_P equals

$$N(P)^{\text{reg}} = (P \times_G G \cdot \mathfrak{t}^{*\text{reg}}) \cap N(P).$$

This implies also the claims made about M^{reg} at the beginning of the proof.

Now, we fix $\xi \in \mathfrak{t}^{*\text{reg}}$. By [8, Theorem 3.7.1] we have that $(G^\circ)_\xi = T$. Therefore also $(G_\xi)^\circ = T$. Since P is connected, the last terms in the long exact sequence in homotopy associated with p_ξ are

$$\dots \longrightarrow \pi_1(O_\xi) \xrightarrow{\Theta} \pi_0(G_\xi) \longrightarrow 1. \quad (6)$$

Thus we obtain a surjective group homomorphism $\Theta: \pi_1(O_\xi) \rightarrow G_\xi/T$. Explicitly, let $[q, \xi] \in O_\xi$ and $\gamma(t)$ be a closed loop at this point. Consider a lift $\tilde{\gamma}(t)$ of γ to P , with $\tilde{\gamma}(0) = q$. Since $p_\xi(\tilde{\gamma}(1), \xi) = [q, \xi]$, it follows that $\tilde{\gamma}(1) = qg$, for some $g \in G_\xi$. The map in (6) is given by $\Theta(\gamma) = [g] \in G_\xi/T$.

Next, we compute the holonomy group of O_ξ . Notice first that

$$T_\xi(G \cdot \xi) = \mathfrak{g}_\xi^\circ = \mathfrak{t}^\circ \subset \mathfrak{g}^* \cong T_\xi \mathfrak{g}^*,$$

and, since $\mathfrak{t}^* = (\mathfrak{t}^\circ)^\perp$, it follows that $\xi + \mathfrak{t}^*$ is transverse at ξ to the coadjoint orbit. Hence also the submanifold

$$\mathcal{T} := \{q\} \times (\xi + \mathfrak{t}^*) \subset P \times_G \mathfrak{g}^*$$

is transverse to O_ξ at $[q, \xi]$. Let γ be a loop in O_ξ based at $[q, \xi]$, and $\tilde{\gamma}$ be a lift to P . Observe that, for $\eta \in \mathfrak{t}^*$, the path

$$t \longmapsto [\tilde{\gamma}(t), \xi + \eta] \in P \times_G \mathfrak{g}^*$$

stays in the leaf $O_{\xi+\eta}$, and therefore the map $[q, \xi + \eta] \mapsto [\tilde{\gamma}(1), \xi + \eta]$ is the holonomy action of γ on \mathcal{T} . Writing $\tilde{\gamma}(1) = qg$, for $g \in G_\xi$, it follows that the holonomy of γ corresponds to the action of $g = \Theta(\gamma)$ on \mathfrak{t}^* . This and the surjectivity of Θ imply that

$$\text{Hol}(O_\xi) \cong G_\xi / Z_G(T), \quad (7)$$

where $Z_G(T)$ denotes the set of elements in G which commute with all elements in T . In particular, the holonomy groups are finite.

Since every coadjoint orbit hits \mathfrak{t}^* , it follows that the map $P \times \mathfrak{t}^* \rightarrow P \times_G \mathfrak{g}^*$ is onto. As this map is T -invariant, the induced map

$$\text{pr}: (P/T) \times \mathfrak{t}^* \longrightarrow P \times_G \mathfrak{g}^*$$

is smooth and onto. Clearly, pr is a proper map. Therefore, to show that vh is continuous, it suffices to show that $\text{vh} \circ \text{pr}$ extends continuously. Note that, for $\xi \in \mathfrak{t}^{\text{reg}}$, the map pr restricts to a $|G_\xi/T|$ -covering projection of the leaf

$$\bar{p}_\xi: (P/T) \times \{\xi\} \longrightarrow P/G_\xi \cong O_\xi.$$

Thus, using also (7), we have that

$$\begin{aligned} \text{Vol}((P/T) \times \{\xi\}, \bar{p}_\xi^*(\omega_\xi)) &= |G_\xi/T| \text{Vol}(O_\xi, \omega_\xi) = \frac{|G_\xi/T|}{|G_\xi/Z_G(T)|} \text{vh}(O_\xi) \\ &= |Z_G(T)/T| \text{vh}(O_\xi). \end{aligned}$$

Hence it suffices to show that the map

$$\mathfrak{t}^* \ni \xi \longmapsto \text{Vol}((P/T) \times \{\xi\}, \bar{p}_\xi^*(\omega_\xi)) \quad (8)$$

is continuous. By (5), we have that the pull-back of $\bar{p}_\xi^*(\omega_\xi)$ to $P \times \{\xi\}$ is given by

$$\Omega|_{P \times \{\xi\}} = p^*(\omega_S) - \langle \xi, d\theta \rangle,$$

in particular it depends smoothly on ξ . Hence also $\bar{p}_\xi^*(\omega_\xi)$ depends smoothly on ξ , and so the map (8) is continuous. To conclude the proof, we have to check that this map vanishes for $\xi \notin \mathfrak{t}^{\text{reg}}$. For such ξ , since $\dim(G_\xi/T) > 0$, we have that

$$2l = \dim(O_\xi) = \dim(P/G_\xi) < \dim(P/T) = 2k.$$

This finishes the proof, since

$$\bigwedge^k \bar{p}_\xi^*(\omega_\xi) = \bar{p}_\xi^*\left(\bigwedge^k \omega_\xi\right) = 0. \quad \square$$

2.2. C^p - C^1 -rigidity and isotopies

In the definition of C^p - C^1 -rigid, we may assume that the maps $\psi_{\bar{\pi}}$ are isotopic to the inclusion $\text{Id}_{\bar{O}}$ of \bar{O} in M , through a path of maps in $C^\infty(\bar{O}, M)$ that extend to embeddings on some neighborhood of \bar{O} . This follows from the C^p - C^1 -continuity of ψ and the fact that $\text{Id}_{\bar{O}}$ has a path-connected C^1 -neighborhood in $C^\infty(\bar{O}, M)$ consisting of such maps.

2.3. A comparison with the local normal form theorem from [7]

Part (a) of Theorem 1 is a slight improvement of the normal form result from [7]. Both theorems require the same conditions on a Lie groupoid, for us this groupoid could be any integration of A_S , but in [7] it has to be the s -fiber 1-connected integration. In §2.4, resp. §2.7, we will study two extreme examples which already reveal the wider applicability of Theorem 1: the case of fixed points and the case of regular Poisson structures whose underlying foliation is simple.

2.4. The case of fixed points

Consider a Poisson manifold (M, π) and let $x \in M$ be a fixed point of π . In a chart centered at x , we write

$$\pi = \sum_{i,j} \frac{1}{2} \pi_{i,j}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \text{with } \pi_{i,j}(0) = 0. \quad (9)$$

The local model of π around 0 is given by its first jet at 0,

$$\sum_{i,j,k} \frac{1}{2} \frac{\partial \pi_{i,j}}{\partial x_k}(0) x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

The coefficients

$$C_{i,j}^k := \frac{\partial \pi_{i,j}}{\partial x_k}(0)$$

are the structure constants of the isotropy Lie algebra \mathfrak{g}_x (see the introduction). To apply Theorem 1 in this setting, we need that \mathfrak{g}_x be integrable by a compact Lie group with vanishing second de Rham cohomology. Such Lie algebras have the following structure.

LEMMA 2.3. *A Lie algebra \mathfrak{g} is integrable by a compact Lie group with vanishing second de Rham cohomology if and only if it is of the form*

$$\mathfrak{g} = \mathfrak{k} \quad \text{or} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathbb{R},$$

where \mathfrak{k} is a semisimple Lie algebra of compact type.

Proof. It is well known that a compact Lie algebra \mathfrak{g} (i.e. a Lie algebra that is integrable by a compact Lie group) decomposes as a product $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{z}$, where $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple of compact type and \mathfrak{z} is the center of \mathfrak{g} . Hence, the Eilenberg–Chevalley complex of \mathfrak{g} is the tensor product of the respective complexes of \mathfrak{k} and \mathfrak{z} . Therefore, by the Künneth formula, $H^*(\mathfrak{g}) \cong H^*(\mathfrak{k}) \otimes H^*(\mathfrak{z})$. Since \mathfrak{k} is semisimple, by Whitehead’s lemma, $H^1(\mathfrak{k}) = 0$ and $H^2(\mathfrak{k}) = 0$, and since \mathfrak{z} is abelian, $H^*(\mathfrak{z}) = \bigwedge^* \mathfrak{z}^*$. Thus, we obtain that

$$H^2(\mathfrak{g}) \cong \bigwedge^2 \mathfrak{z}^*. \quad (10)$$

Consider now any compact connected integration G of \mathfrak{g} . The cohomology of G can be computed using left-invariant differential forms, and therefore $H^*(G) \cong H^*(\mathfrak{g})$. By (10), we obtain that $H^2(G) = 0$ is equivalent to $\dim(\mathfrak{z}) \leq 1$. \square

So, for fixed points, Theorem 1 gives the following consequence.

COROLLARY 2.4. *Let (M, π) be a Poisson manifold with a fixed point x for which the isotropy Lie algebra \mathfrak{g}_x is compact and its center is at most 1-dimensional. Then π is rigid around x , and an open set around x is Poisson diffeomorphic to a neighborhood of 0 in the linear Poisson manifold $(\mathfrak{g}_x^*, \pi_{\mathfrak{g}_x})$.*

The linearization result in the semisimple case is Conn’s theorem [2] and the case when the isotropy has a 1-dimensional center is a consequence of the smooth Levi decomposition theorem of Monnier and Zung [20].

This fits into Weinstein’s notion of a non-degenerate Lie algebra [24]. Recall that a Lie algebra \mathfrak{g} is *non-degenerate* if every Poisson structure which has isotropy Lie algebra \mathfrak{g} at a fixed point x , is Poisson-diffeomorphic around x to the linear Poisson structure $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ around 0.

A Lie algebra \mathfrak{g} , for which $\pi_{\mathfrak{g}}$ is rigid around 0, is necessarily non-degenerate. To see this, consider a Poisson bivector π given in local coordinates by (9), and whose linearization at 0 is $\pi_{\mathfrak{g}}$. The path of Poisson bivectors π_t from the first description of the local model (2) satisfies $\pi_1=\pi$ and $\pi_0=\pi_{\mathfrak{g}}$, and for $t>0$ is given by

$$\pi_t := t\mu_t^*(\pi) = \sum_{i,j} \frac{1}{2t} \pi_{i,j}(tx) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where μ_t denotes multiplication by $t>0$. If $\pi_{\mathfrak{g}}$ is rigid around 0, then, for some $r>0$ and some $t>0$, there is a Poisson isomorphism

$$\psi: (B_r, \pi_t) \xrightarrow{\sim} (\psi(B_r), \pi_{\mathfrak{g}}).$$

Now $\xi:=\psi(0)$ is a fixed point of $\pi_{\mathfrak{g}}$, which is the same as an element in $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$. It is easy to see that translation by ξ is a Poisson isomorphism of $\pi_{\mathfrak{g}}$, and thus, replacing ψ by $\psi-\xi$, we may assume that $\psi(0)=0$. Linearity of $\pi_{\mathfrak{g}}$ implies that $\mu_t^*(\pi_{\mathfrak{g}})=\pi_{\mathfrak{g}}/t$, and therefore

$$\pi = \frac{1}{t} \mu_{1/t}^*(\pi_t) = \frac{1}{t} \mu_{1/t}^*(\psi^*(\pi_{\mathfrak{g}})) = \mu_{1/t}^* \circ \psi^* \circ \mu_t^*(\pi_{\mathfrak{g}}).$$

Hence, π is linearizable by the map

$$\mu_t \circ \psi \circ \mu_{1/t}: (B_{tr}, \pi) \longrightarrow (t\psi(B_r), \pi_{\mathfrak{g}}),$$

which maps 0 to 0. This shows that \mathfrak{g} is non-degenerate.

2.5. The Poisson sphere in \mathfrak{g}^*

Let \mathfrak{g} be a semisimple Lie algebra of compact type and let G be the compact, 1-connected Lie group integrating it. The linear Poisson structure $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ is integrable by the symplectic groupoid $(T^*G, \omega_{\text{can}}) \rightrightarrows \mathfrak{g}^*$, with source and target maps given by left and right trivialization. All s -fibers of T^*G are diffeomorphic to G and, since $H^2(G)=0$, we can apply Theorem 2 to any compact Poisson submanifold of \mathfrak{g}^* . An example of such a submanifold is the sphere $\mathbb{S}(\mathfrak{g}^*) \subset \mathfrak{g}^*$ with respect to some invariant metric. We obtain the following result, whose formal version appeared in [15] and served as an inspiration.

PROPOSITION 2.5. *Let \mathfrak{g} be a semisimple Lie algebra of compact type and denote by $\mathbb{S}(\mathfrak{g}^*) \subset \mathfrak{g}^*$ the unit sphere centered at the origin with respect to some invariant inner product. Then $\pi_{\mathfrak{g}}$ is C^p - C^1 -rigid around $\mathbb{S}(\mathfrak{g}^*)$ and, up to isomorphism, it is determined around $\mathbb{S}(\mathfrak{g}^*)$ by its first-order jet.*

Using this rigidity result, one can describe an open set around $\pi_{\mathfrak{S}} := \pi_{\mathfrak{g}}|_{\mathfrak{S}(\mathfrak{g}^*)}$ in the moduli space of all Poisson structures on the sphere $\mathfrak{S}(\mathfrak{g}^*)$. More precisely, any Poisson structure on $\mathfrak{S}(\mathfrak{g}^*)$ that is C^p -close to $\pi_{\mathfrak{S}}$ is Poisson diffeomorphic to one of the type $f\pi_{\mathfrak{S}}$, where f is a positive Casimir function. If the metric is $\text{Aut}(\mathfrak{g})$ -invariant, then two structures of this type $f_1\pi_{\mathfrak{S}}$ and $f_2\pi_{\mathfrak{S}}$ are isomorphic if and only if $f_1 = f_2 \circ \chi^*$ for some outer automorphism χ of the Lie algebra \mathfrak{g} . The details are given in [17].

2.6. Relation with stability of symplectic leaves

Recall from [5] that a symplectic leaf (S, ω_S) of a Poisson manifold (M, π) is C^p -strongly stable if for every open set U containing S there exists an open neighborhood $\mathcal{V} \subset \mathfrak{X}^2(U)$ of $\pi|_U$ with respect to the compact-open C^p -topology, such that every Poisson structure in \mathcal{V} has a leaf symplectomorphic to (S, ω_S) . Recall also the following result.

THEOREM. ([5, Theorem 2.2]) *If S is compact and the Lie algebroid $A_S := T^*M|_S$ satisfies $H^2(A_S) = 0$, then S is a strongly stable leaf.*

If π is C^p - C^1 -rigid around S , then S is a strongly stable leaf. Also, the hypotheses of our Theorem 1 imply those of [5, Theorem 2.2]. To see this, let $P \rightarrow S$ be a principal G -bundle for which $A_S \cong TP/G$. Then

$$H^\bullet(A_S) \cong H^\bullet(\Omega(P)^G).$$

If G is compact then, by averaging primitives, one easily shows that the inclusion $\Omega^\bullet(P)^G \subset \Omega^\bullet(P)$ induces an injection $H^\bullet(\Omega(P)^G) \rightarrow H^\bullet(P)$. So $H^2(P) = 0$ implies that $H^2(A_S) = 0$.

On the other hand, $H^2(A_S) = 0$ does not imply rigidity, counterexamples can be found even for fixed points. Weinstein [25] proves that a non-compact semisimple Lie algebra \mathfrak{g} of real rank at least 2 is degenerate, so $\pi_{\mathfrak{g}}$ is not rigid (see §2.4). However, 0 is a stable point for $\pi_{\mathfrak{g}}$, because by Whitehead's lemma $H^2(\mathfrak{g}) = 0$.

According to [5, Theorem 2.3], the condition $H^2(A_S) = 0$ is also necessary for strong stability of the symplectic leaf (S, ω_S) for Poisson structures of "first order", i.e. for Poisson structures which are isomorphic to their local model around S . So, for this type of Poisson structures, $H^2(A_S) = 0$ is also necessary for rigidity.

For regular Poisson structures whose underlying foliation is simple, we will prove below that the hypotheses of Theorem 1 and of [5, Theorem 2.2] are equivalent.

2.7. Simple symplectic foliations

We will now discuss rigidity and linearization of regular Poisson structures π on $S \times \mathbb{R}^n$ with symplectic leaves

$$(S \times \{y\}, \omega_y := \pi|_{S \times \{y\}}^{-1}), \quad y \in \mathbb{R}^n,$$

where $\{\omega_y\}_{y \in \mathbb{R}^n}$ is a smooth family of symplectic forms on S . Let (S, ω_S) be the symplectic leaf for $y=0$. To construct the local model around S , we use the first description. The path of Poisson structures π_t from (2), for $t \neq 0$, corresponds to the family of 2-forms on S ,

$$\omega_y^t := \omega_S + \frac{\omega_{ty} - \omega_S}{t}.$$

Therefore, the local model around S corresponds to the family of 2-forms

$$j_S^1(\omega)_y := \omega_S + \delta_S \omega_y,$$

where $\delta_S \omega_y$ is the ‘‘vertical derivative’’ of ω at S , that is

$$\delta_S \omega_y := \left. \frac{d}{d\varepsilon} \omega_{\varepsilon y} \right|_{\varepsilon=0} = y_1 \omega_1 + \dots + y_n \omega_n.$$

The local model is defined on an open set $U \subset S \times \mathbb{R}^n$ containing S , such that $j_S^1(\omega)_y$ is non-degenerate along $U \cap (S \times \{y\})$. Using the splitting $T^*(S \times \mathbb{R}^n)|_S = T^*S \times \mathbb{R}^n$ and the isomorphism of $\omega_S^\sharp: TS \xrightarrow{\sim} T^*S$, we identify $A_S \cong TS \times \mathbb{R}^n$. Under this identification, the Lie bracket becomes

$$\begin{aligned} & [(X, f_1, \dots, f_n), (Y, g_1, \dots, g_n)] \\ &= ([X, Y], X(g_1) - Y(f_1) + \omega_1(X, Y), \dots, X(g_n) - Y(f_n) + \omega_n(X, Y)). \end{aligned} \quad (11)$$

The conditions in Theorem 1 become more computable in this case.

LEMMA 2.6. *If S is compact, then the following are equivalent:*

- (a) A_S is integrable by a compact principal bundle P , with $H^2(P)=0$;
- (b) $H^2(A_S)=0$;
- (c) the cohomological variation $[\delta_S \omega]: \mathbb{R}^n \rightarrow H^2(S)$, $y \mapsto [\delta_S \omega_y]$, satisfies the following properties:

- (c₁) it is surjective;
- (c₂) its kernel is at most 1-dimensional;
- (c₃) the map $H^1(S) \otimes \mathbb{R}^n \rightarrow H^3(S)$, $\eta \otimes y \mapsto \eta \wedge [\delta_S \omega_y]$, is injective.

Proof. The complex computing $H^\bullet(A_S)$ can be identified with

$$\Omega^k(A_S) := \bigoplus_{p+q=k} \Omega^p(S) \otimes \bigwedge^q \mathbb{R}^n,$$

endowed with the differential

$$d_{A_S}(\alpha \otimes w) = (d\alpha) \otimes w + (-1)^{p+1} \alpha \wedge \delta_S \omega(w),$$

for $\alpha \in \Omega^p(S)$ and $w \in \bigwedge^q \mathbb{R}^n$, where the map

$$\delta_S \omega: \bigwedge^q \mathbb{R}^n \longrightarrow \Omega^2(S) \otimes \bigwedge^{q-1} \mathbb{R}^n$$

is induced by the vertical derivative of ω :

$$\delta_S \omega(y_1 \wedge \dots \wedge y_q) = \sum_{i=1}^q (-1)^{i-1} \delta_S \omega_{y_i} \otimes y_1 \wedge \dots \wedge y_{i-1} \wedge y_{i+1} \wedge \dots \wedge y_q.$$

Consider the filtration $F^p \Omega^*(A_S) := \Omega^p(S) \wedge \Omega^{*-p}(A_S)$ of this complex, and the corresponding spectral sequence (for general constructions of spectral sequences for computing Lie algebroid cohomology see, e.g., [13]). We have that

$$E_2^{p,q} = H^p(S) \otimes \bigwedge^q \mathbb{R}^n \Rightarrow H^{p+q}(A_S),$$

and the differentials on the second page E_2 are given by

$$\begin{aligned} [\delta_S \omega]: H^p(S) \otimes \bigwedge^q \mathbb{R}^n &\longrightarrow H^{p+2}(S) \otimes \bigwedge^{q-1} \mathbb{R}^n, \\ [\alpha] \otimes w &\longmapsto (-1)^{p+1} [\alpha \wedge \delta_S \omega(w)]. \end{aligned}$$

In total degree 2, the cohomology of E_2 is given by

$$\begin{aligned} E_3^{2,0} &:= \operatorname{coker}([\delta_S \omega]: \mathbb{R}^n \rightarrow H^2(S)), \\ E_3^{1,1} &:= \ker([\delta_S \omega]: H^1(S) \otimes \mathbb{R}^n \rightarrow H^3(S)), \\ E_3^{0,2} &:= \ker([\delta_S \omega]: \bigwedge^2 \mathbb{R}^n \rightarrow H^2(S) \otimes \mathbb{R}^n). \end{aligned}$$

We claim that the last group is also given by

$$E_3^{0,2} = \bigwedge^2 \ker([\delta_S \omega]: \mathbb{R}^n \rightarrow H^2(S)). \quad (12)$$

This is based on a simple result from linear algebra: namely, if $A: V \rightarrow W$ is a linear map between finite-dimensional vector spaces, then the kernel of the map

$$\begin{aligned} \bigwedge^2 V &\longrightarrow W \otimes V, \\ v_1 \wedge v_2 &\longmapsto A(v_1) \otimes v_2 - A(v_2) \otimes v_1, \end{aligned}$$

is given by $\bigwedge^2 \ker(A)$.

Next, we claim that the cohomology can be read from the third page:

$$H^2(A_S) = E_3^{2,0} \oplus E_3^{1,1} \oplus E_3^{0,2}. \tag{13}$$

Since $E_3^{2,0} = E_\infty^{2,0}$ and $E_3^{1,1} = E_\infty^{1,1}$, this is equivalent to the edge morphism $e_F: E_\infty^{0,2} \rightarrow E_3^{0,2}$ being an isomorphism, or to surjectivity of the map

$$\begin{aligned} H^2(A_S) &\longrightarrow E_3^{0,2}, \\ [\alpha] &\longmapsto [\alpha^{0,2}], \end{aligned} \tag{14}$$

where $\alpha^{0,2}$ denotes the component in $\bigwedge^2 \mathbb{R}^n$ of the closed form $\alpha \in \Omega^2(A_S)$. By (12), it suffices to show that every element of the form $v \wedge w$, with $[\delta_S \omega_v] = [\delta_S \omega_w] = 0$ is in the range of this map. Writing $\delta_S \omega_v = d\eta$ and $\delta_S \omega_w = d\theta$, for $\eta, \theta \in \Omega^1(S)$, one easily checks that

$$\xi := (\eta \wedge \theta, \eta \otimes w - \theta \otimes v, v \wedge w) \in \Omega^2(A_S)$$

is closed. Thus, the map in (14) maps $[\xi]$ to $v \wedge w$, which proves that it is surjective. Hence (13) holds.

The three conditions in (c) are equivalent to the vanishing of the three components of E_3^2 . So, by (13), (b) and (c) are equivalent.

The fact that (a) implies (b) was explained in §2.6.

We prove now that (b) and (c) imply (a). Part (c₁) implies that, by taking a different basis of \mathbb{R}^n , we may assume that $[\omega_1], \dots, [\omega_n] \in H^2(S, \mathbb{Z})$. Let $P \rightarrow S$ be a principal T^n bundle with connection form $(\theta_1, \dots, \theta_n)$ and curvature form $(-\omega_1, \dots, -\omega_n)$. We claim that the Lie algebroid TP/T^n is isomorphic to A_S . A section of TP/T^n is the same as a T^n -invariant vector field on P , and as such, it can be decomposed uniquely as

$$\tilde{X} + \sum_{i=1}^n f_i \partial_{\theta_i},$$

where \tilde{X} is the horizontal lift of a vector field X on S , f_1, \dots, f_n are smooth functions on S , and ∂_{θ_i} is the unique vertical vector field on P which satisfies

$$\theta_j(\partial_{\theta_i}) = \delta_{i,j}.$$

Using (11) for the bracket on A_S and that $d\theta_i = -p^*(\omega_i)$, it is straightforward to check that the map

$$\begin{aligned} TP/T^n &\xrightarrow{\sim} A_S, \\ \tilde{X} + \sum f_i \partial_{\theta_i} &\longmapsto (X, f_1, \dots, f_n), \end{aligned}$$

is a Lie algebroid isomorphism. Since T^n is compact and connected, using averaging, one shows that the complexes

$$(\Omega^\bullet(P)^{T^n}, d) \quad \text{and} \quad (\Omega^\bullet(P), d)$$

are quasi-isomorphic; in particular $H^2(P) \cong H^2(A_S)$. By (b), $H^2(P) = 0$, and so P satisfies the conditions from (a). This finishes the proof. \square

So, in the case of simple foliations, Theorem 1 becomes the following result.

COROLLARY 2.7. *Let $\{\omega_y \in \Omega^2(S)\}_{y \in \mathbb{R}^n}$ be a smooth family of symplectic structures on a compact manifold S . If the cohomological variation at 0,*

$$[\delta_S \omega]: \mathbb{R}^n \longrightarrow H^2(S),$$

satisfies the conditions from Lemma 2.6, then the Poisson manifold with leaves

$$(S \times \mathbb{R}^n, \{\omega_y^{-1}\}_{y \in \mathbb{R}^n})$$

is isomorphic to its local model at $S \times \{0\}$, and is C^p - C^1 -rigid around this leaf.

For simple symplectic foliations Lemma 2.6 shows that the condition in Theorem 1 is equivalent to the vanishing of $H^2(A_S)$. This is precisely the assumption of the stability result from [5, Theorem 2.2]. In [5], it is also proven that under this assumption there exists a smoothly parameterized family of symplectic leaves near S that are symplectomorphic to (S, ω_S) . To describe the parameter space, consider the cohomology $H^\bullet(A_S; S)$ of the quotient complex (here we use the notation from the proof of Lemma 2.6)

$$\Omega^\bullet(A_S; S) := \Omega^\bullet(A_S) / \Omega^\bullet(S),$$

and consider the canonical map induced by the quotient map

$$\Phi: H^\bullet(A_S) \longrightarrow H^\bullet(A_S; S).$$

Theorem 2.2 in [5] states that every Poisson structure near π has a family of symplectic leaves symplectomorphic to (S, ω_S) , which is smoothly parameterized by the image of the map

$$\Phi: H^1(A_S) \longrightarrow H^1(A_S; S).$$

Applying the same techniques as in the proof of Lemma 2.6, this map can be computed as follows.

LEMMA 2.8. *With the notation from the proof of Lemma 2.6, we have that*

$$H^1(A_S) \cong H^1(S) \oplus \ker([\delta_S \omega]: \mathbb{R}^n \rightarrow H^2(S)) \quad \text{and} \quad H^1(A_S; S) \cong \mathbb{R}^n.$$

Under these isomorphisms, the map $\Phi: H^1(A_S) \rightarrow H^1(A_S; S)$ becomes $([\eta], v) \mapsto v$.

So, under the assumptions from Corollary 2.7, the space of leaves symplectomorphic to (S, ω_S) is parameterized by

$$\ker([\delta_S \omega]: \mathbb{R}^n \rightarrow H^2(S)). \quad (15)$$

Of course, using the local model, this can be checked directly. By Lemma 2.6, this space is at most 1-dimensional. An example where the space (15) is indeed 1-dimensional, can be constructed as follows: consider the 2-sphere $S := \mathbb{S}^2$, endowed with a symplectic structure ω_S . Then the Poisson structure on $S \times \mathbb{R}^2$ with symplectic foliation given by

$$(S \times \{(y_1, y_2)\}, e^{y_1} \omega_S), \quad (y_1, y_2) \in \mathbb{R}^2, \quad (16)$$

satisfies the conditions of Lemma 2.6. Note that every leaf $S \times \{(y_1, y_2)\}$ is part of a 1-parameter family $S \times \{(y_1, y_2 + t)\}$, $t \in \mathbb{R}$, of symplectomorphic leaves.

We remark that the Poisson structure in this example is isomorphic to the regular part of the linear Poisson structure corresponding to the Lie algebra $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}$. In fact, for a semisimple Lie algebra of compact type \mathfrak{k} , the linear Poisson structure $\pi_{\mathfrak{g}}$ of the product $\mathfrak{g} = \mathfrak{k} \oplus \mathbb{R}$ is rigid (cf. Corollary 2.4), and the Poisson structure has a 1-parameter family of isomorphisms that do not preserve leaves: the translation by elements in \mathfrak{k}° . Thus, any leaf has a line of symplectomorphic leaves nearby.

In the case of simple symplectic foliations, we also have an improvement compared to the result of [7]; the hypothesis in there can be restated as follows (cf. [16, Corollary 4.1.22]):

- S is compact with finite fundamental group,
- the map $p^* \circ [\delta_S \omega]: \mathbb{R}^n \rightarrow H^2(\tilde{S})$ is an isomorphism,

where $p: \tilde{S} \rightarrow S$ is the universal cover of S . So, for example when S is simply connected, the difference between the assumptions is that, in our case, the map $[\delta_S \omega]$ might still have a 1-dimensional kernel, whereas in [7] it has to be injective. In particular, the example (16) above falls out of the framework of [7].

3. Proof of Theorem 2

We start by preparing the setting needed for applying the Nash–Moser method: we fix norms on the Fréchet spaces involved, we construct smoothing operators adapted

to the problem and we recall the interpolation inequalities. Next, we prove a series of inequalities which assert tameness of some natural operations such as: the Lie derivative, the flow of a vector field, and the pull-back; and then we prove some inequalities for the composition of local diffeomorphisms. We end the section with the proof of Theorem 2, which is mostly inspired by Conn’s proof [2].

Remark 1. A usual convention when dealing with the Nash–Moser techniques (see e.g. [12]), which we also adopt, is to denote all constants by the same symbol. In the series of preliminary results below we work with “big enough” constants C and C_n , and with “small enough” constants $\theta > 0$; these depend on the trivialization data for the vector bundle E and on the smoothing operators. In the proof of Proposition 3.12, C_n depends also on the Poisson structure π .

3.1. The ingredients of the tame category

We borrow the terminology from [12]. A Fréchet space F endowed with an increasing family of seminorms $\{\|\cdot\|_n\}_{n \geq 0}$ generating its topology is called a *graded Fréchet space*. A linear map $T: F_1 \rightarrow F_2$ between two graded Fréchet spaces is called *tame* of degree d and base b , if it satisfies inequalities of the form

$$\|Tf\|_n \leq C_n \|f\|_{n+d} \quad \text{for all } n \geq b \text{ and } f \in F_1.$$

Let $E \rightarrow N$ be a vector bundle over a compact manifold N and fix a metric on E . For $r > 0$, consider the closed tube in E of radius r ,

$$E_r := \{v \in E : |v| \leq r\}.$$

The space of multivector fields on E_r , denoted by $\mathfrak{X}^\bullet(E_r)$, when endowed with C^n -norms becomes a graded Fréchet space. We recall here the construction of such norms. Fix a finite open cover of N by domains of charts $\{\chi_i: O_i \xrightarrow{\sim} \mathbb{R}^d\}_{i=1}^I$ and vector bundle isomorphisms

$$\tilde{\chi}_i: E|_{O_i} \xrightarrow{\sim} \mathbb{R}^d \times \mathbb{R}^D$$

covering χ_i . We will assume that $\tilde{\chi}_i(E_r|_{O_i}) = \mathbb{R}^d \times \bar{B}_r$ and that the family

$$\{O_i^\delta := \chi_i^{-1}(B_\delta)\}_{i=1}^I$$

covers N for all $\delta \geq 1$. Moreover, we assume that the cover satisfies

$$\text{if } O_i^{3/2} \cap O_j^{3/2} \neq \emptyset \quad \text{then } O_j^1 \subset O_i^4. \quad (17)$$

This holds if $\chi_i^{-1}|_{B_4}: B_4 \rightarrow O_i$ is the exponential corresponding to some metric on N , with injectivity radius larger than 4.

For $W \in \mathfrak{X}^\bullet(E_r)$, denote its local expression in the chart $\tilde{\chi}_i$ by

$$W_i(z) := \sum_{1 \leq i_1 < \dots < i_p \leq d+D} W_i^{i_1, \dots, i_p}(z) \frac{\partial}{\partial z_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{i_p}},$$

and let the C^n -norm of W be given by

$$\|W\|_{n,r} := \sup_{i, i_1, \dots, i_p} \left\{ \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} W_i^{i_1, \dots, i_p}(z) \right| : z \in B_1 \times B_r \text{ and } 0 \leq |\alpha| \leq n \right\}.$$

For $s < r$, the restriction maps are norm decreasing

$$\mathfrak{X}^\bullet(E_r) \ni W \mapsto W|_s := W|_{E_s} \in \mathfrak{X}^\bullet(E_s), \quad \|W|_s\|_{n,s} \leq \|W\|_{n,r}.$$

We will work also with the closed subspaces of multivector fields on E_r whose first jet vanishes along N , which we denote by

$$\mathfrak{X}^k(E_r)^{(1)} := \{W \in \mathfrak{X}^k(E_r) : j^1 W|_N = 0\}.$$

The main technical tool used in the Nash–Moser method are the smoothing operators. We will call a family $\{S_t: F \rightarrow F\}_{t>1}$ of linear operators on the graded Fréchet space F *smoothing operators* of degree $d \geq 0$, if there exist constants $C_{n,m} > 0$ such that, for all $n, m \geq 0$ and $f \in F$, the following inequalities hold:

$$\|S_t(f)\|_{n+m} \leq t^{m+d} C_{n,m} \|f\|_n \quad \text{and} \quad \|S_t(f) - f\|_n \leq t^{-m} C_{n,m} \|f\|_{n+m+d}. \quad (18)$$

The construction of such operators is standard, but since we are dealing with a Fréchet space for each $r \in (0, 1]$, we give the explicit dependence of the constants $C_{n,m}$ from (18) on the parameter r .

LEMMA 3.1. *The family of graded Fréchet spaces $\{(\mathfrak{X}^k(E_r), \|\cdot\|_{n,r})\}_{r \in (0,1]}$ has a family of smoothing operators of degree $d=0$,*

$$\{S_t^r: \mathfrak{X}^k(E_r) \rightarrow \mathfrak{X}^k(E_r)\}_{t>1, 0 < r \leq 1},$$

which satisfy (18) with constants of the form $C_{n,m}(r) = C_{n,m} r^{-(n+m+k)}$.

Similarly, the family $\{(\mathfrak{X}^k(E_r)^{(1)}, \|\cdot\|_{n,r})\}_{r \in (0,1]}$ has smoothing operators

$$\{S_t^{r,1}: \mathfrak{X}^k(E_r)^{(1)} \rightarrow \mathfrak{X}^k(E_r)^{(1)}\}_{t>1, 0 < r \leq 1},$$

of degree $d=1$ and constants $C_{n,m}(r) = C_{n,m} r^{-(n+m+k+1)}$.

Proof. The existence of smoothing operators of degree zero on the Fréchet space of sections of a vector bundle over a compact manifold (possibly with boundary) is standard (see [12]). We fix such a family $\{S_t: \mathfrak{X}^k(E_1) \rightarrow \mathfrak{X}^k(E_1)\}_{t>1}$. Denote by

$$\begin{aligned}\mu_\varrho: E_R &\longrightarrow E_{\varrho R}, \\ v &\longmapsto \varrho v,\end{aligned}$$

the rescaling operators. For $r \in (0, 1]$, define S_t^r by conjugating S_t with μ_r^* ,

$$S_t^r := \mu_{r^{-1}}^* \circ S_t \circ \mu_r^*: \mathfrak{X}^k(E_r) \longrightarrow \mathfrak{X}^k(E_r).$$

Using the straightforward inequality

$$\|\mu_\varrho^*(W)\|_{n,R} \leq \max\{\varrho^{-k}, \varrho^n\} \|W\|_{n,\varrho R} \quad \text{for all } W \in \mathfrak{X}^k(E_{\varrho R}),$$

we obtain that S_t^r satisfies (18) with $C_{n,m}(r) = C_{n,m} r^{-(n+m+k)}$.

To construct the operators $S_t^{r,1}$, we first define a tame projection

$$P: \mathfrak{X}^k(E_r) \longrightarrow \mathfrak{X}^k(E_r)^{(1)}.$$

Choose a smooth partition of unit $\{\lambda_i\}_{i=1}^I$ on N subordinated to the cover $\{O_i^1\}_{i=1}^I$, and let $\{\tilde{\lambda}_i\}_{i=1}^I$ be the pull-back to E . For $W \in \mathfrak{X}^k(E_r)$, denote its local representatives by $W_i := \tilde{\chi}_{i,*}(W|_{E_r|_{O_i}}) \in \mathfrak{X}^k(\mathbb{R}^d \times \bar{B}_r)$. Define P as

$$P(W) := \sum_{i=1}^I \tilde{\lambda}_i \tilde{\chi}_{i,*}^{-1}(W_i - T_y^1(W_i)),$$

where $T_y^1(W_i)$ is the degree-1 Taylor polynomial of W_i in the fiber direction

$$T_y^1(W_i)(x, y) := W_i(x, 0) + \sum_{j=1}^D y_j \frac{\partial W_i}{\partial y_j}(x, 0).$$

If $W \in \mathfrak{X}^k(E_r)^{(1)}$, then $T_y^1(W_i) = 0$; so P is a projection. It is easy to check that P is tame of degree 1, that is, there are constants $C_n > 0$ such that

$$\|P(W)\|_{n,r} \leq C_n \|W\|_{n+1,r}.$$

Define the smoothing operators on $\mathfrak{X}^k(E_r)^{(1)}$ as

$$S_t^{r,1} := P \circ S_t^r: \mathfrak{X}^k(E_r)^{(1)} \longrightarrow \mathfrak{X}^k(E_r)^{(1)}.$$

Using tameness of P , the inequalities for $S_t^{r,1}$ are straightforward. \square

The norms $\|\cdot\|_{n,r}$ satisfy the classical interpolation inequalities with constants which are polynomials in r^{-1} .

LEMMA 3.2. *The norms $\|\cdot\|_{n,r}$ satisfy*

$$\|W\|_{l,r} \leq C_n r^{k-l} \|W\|_{k,r}^{(n-l)/(n-k)} \|W\|_{n,r}^{(l-k)/(n-k)} \quad \text{for } r \in (0, 1],$$

for all $0 \leq k \leq l \leq n$, not all equal, and all $W \in \mathfrak{X}^\bullet(E_r)$.

Proof. By the interpolation inequalities from [2], it follows that these inequalities hold for the C^n -norms on the spaces $C^\infty(\bar{B}_1 \times \bar{B}_r)$. Applying these to the components of the restrictions to the charts $(E_r|_{O_i}, \tilde{\chi}_i)$ of a multivector field in $\mathfrak{X}^\bullet(E_r)$, we obtain the interpolation inequalities on $\mathfrak{X}^\bullet(E_r)$. \square

3.2. Tameness of some natural operators

In this subsection we prove some tameness properties of the Lie bracket, the pull-back and the flow of vector fields.

The tame Fréchet Lie algebra of multivector fields

We prove that

$$(\mathfrak{X}^\bullet(E_r), [\cdot, \cdot], \{\|\cdot\|_{n,r}\}_{n \geq 0})$$

is a tame Fréchet graded Lie algebra.

LEMMA 3.3. *The Schouten bracket on $\mathfrak{X}^\bullet(E_r)$ satisfies*

$$\|[W, V]\|_{n,r} \leq C_n r^{-(n+1)} (\|W\|_{0,r} \|V\|_{n+1,r} + \|W\|_{n+1,r} \|V\|_{0,r}) \quad \text{for all } r \in (0, 1].$$

Proof. By a local computation, the bracket satisfies inequalities of the form

$$\|[W, V]\|_{n,r} \leq C_n \sum_{i+j=n+1} \|W\|_{i,r} \|V\|_{j,r}.$$

Using the interpolation inequalities, a term in this sum can be bounded by

$$\|W\|_{i,r} \|V\|_{j,r} \leq C_n r^{-(n+1)} (\|W\|_{0,r} \|V\|_{n+1,r})^{j/(n+1)} (\|V\|_{0,r} \|W\|_{n+1,r})^{i/(n+1)}.$$

The following inequality, which will be used again later, implies the conclusion

$$x^\lambda y^{1-\lambda} \leq x+y \quad \text{for all } x, y \geq 0 \text{ and all } \lambda \in [0, 1]. \tag{19}$$

\square

The space of local diffeomorphisms

We now consider the space of smooth maps $E_r \rightarrow E$ which are C^1 -close to the inclusion $I_r: E_r \hookrightarrow E$. We call a map $\varphi: E_r \rightarrow E$ a *local diffeomorphism*, if it can be extended on some open set to a diffeomorphism onto its image. Since E_r is compact, this is equivalent to injectivity of $d\varphi: TE_r \rightarrow TE$. To be able to measure C^n -norms of such maps, we work with the following open neighborhood of I_r in $C^\infty(E_r; E)$:

$$\mathcal{U}_r := \{\varphi: E_r \rightarrow E : \varphi(E_r|_{\bar{O}_i^1}) \subset E|_{O_i}, 1 \leq i \leq I\}.$$

Denote the local representatives of a map $\varphi \in \mathcal{U}_r$ by

$$\varphi_i: \bar{B}_1 \times \bar{B}_r \longrightarrow \mathbb{R}^d \times \mathbb{R}^D.$$

Define C^n -distances between maps $\varphi, \psi \in \mathcal{U}_r$ as

$$d(\varphi, \psi)_{n,r} := \sup_{1 \leq i \leq I} \left\{ \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} (\varphi_i - \psi_i)(z) \right| : z \in B_1 \times B_r \text{ and } 0 \leq |\alpha| \leq n \right\}.$$

To control compositions of maps, we will also need the following C^n -distances

$$d(\varphi, \psi)_{n,r,\delta} := \sup_{1 \leq i \leq I} \left\{ \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} (\varphi_i - \psi_i)(z) \right| : z \in B_\delta \times B_r \text{ and } 0 \leq |\alpha| \leq n \right\},$$

which are well-defined only on the open set

$$\mathcal{U}_r^\delta := \{\chi \in \mathcal{U}_r : \chi(E_r|_{\bar{O}_i^\delta}) \subset E|_{O_i}\}.$$

Similarly, we define also on $\mathfrak{X}^\bullet(E_r)$ norms $\|\cdot\|_{n,r,\delta}$ (these measure the C^n -norms in all our local charts on $B_\delta \times B_r$).

These norms and distances are equivalent.

LEMMA 3.4. *There exist $C_n > 0$ such that, for all $r \in (0, 1]$ and all $\delta \in [1, 4]$,*

$$d(\varphi, \psi)_{n,r} \leq d(\varphi, \psi)_{n,r,\delta} \leq C_n d(\varphi, \psi)_{n,r} \quad \text{for all } \varphi, \psi \in \mathcal{U}_r^\delta$$

and

$$\|W\|_{n,r} \leq \|W\|_{n,r,\delta} \leq C_n \|W\|_{n,r} \quad \text{for all } W \in \mathfrak{X}^\bullet(E_r).$$

We also use the simplified notation

$$d(\psi)_{n,r} := d(\psi, I_r)_{n,r} \quad \text{and} \quad d(\psi)_{n,r,\delta} := d(\psi, I_r)_{n,r,\delta}.$$

The lemma below is used to check that compositions are defined.

LEMMA 3.5. *There exists a constant $\theta > 0$ such that, for all $r \in (0, 1]$, all $\varepsilon \in (0, 1]$, all $\delta \in [1, 4]$ and all $\varphi \in \mathcal{U}_r$ satisfying $d(\varphi)_{0,r} < \varepsilon\theta$,*

$$\varphi(E_r|_{\bar{O}_i^\delta}) \subset E_{r+\varepsilon}|_{O_i^{\delta+\varepsilon}}.$$

We now prove that I_r has a C^1 -neighborhood of local diffeomorphisms.

LEMMA 3.6. *There exists a constant $\theta > 0$ such that, for all $r \in (0, 1]$, if $\psi \in \mathcal{U}_r$ satisfies $d(\psi)_{1,r} < \theta$, then ψ is a local diffeomorphism.*

Proof. By Lemma 3.5, if we shrink θ , we may assume that

$$\psi(E_r|_{\bar{O}_i^1}) \subset E|_{O_i^{3/2}} \quad \text{and} \quad \psi(E_r|_{\bar{O}_i^4}) \subset E|_{O_i}. \quad (20)$$

In a local chart, we write ψ as

$$\psi_i := \text{Id} + g_i: \bar{B}_4 \times \bar{B}_r \longrightarrow \mathbb{R}^d \times \mathbb{R}^D.$$

By Lemma 3.4, if we shrink θ , we may also assume that

$$\left| \frac{\partial g_i}{\partial z_j}(z) \right| < \frac{1}{2(d+D)} \quad \text{for all } z \in \bar{B}_4 \times \bar{B}_r. \quad (21)$$

This ensures that $\text{Id} + (dg_i)_z$ is close enough to Id so that it is invertible for all $z \in \bar{B}_1 \times \bar{B}_r$. Thus, $(d\psi)_p$ is invertible for all $p \in E_r$.

We now check the injectivity of ψ . Let $p^i \in E_r|_{O_i^1}$ and $p^j \in E_r|_{O_j^1}$ be such that

$$\psi(p^i) = q = \psi(p^j).$$

Then, by (20), $q \in E|_{O_i^{3/2}} \cap E|_{O_j^{3/2}}$, so, by the property (17), we know that $O_j^1 \subset O_i^4$, and hence $p^i, p^j \in E_r|_{O_i^4}$. Setting $w^i := \tilde{\chi}_i(p^i)$ and $w^j := \tilde{\chi}_i(p^j)$ we have that $w^i, w^j \in \bar{B}_4 \times \bar{B}_r$. Since $w^i + g_i(w^i) = w^j + g_i(w^j)$, using (21), we obtain that

$$|w^i - w^j| = |g_i(w^i) - g_i(w^j)| = \left| \int_0^1 \sum_{k=1}^{D+d} \frac{\partial g_i}{\partial z_k}(tw^i + (1-t)w^j)(w_k^i - w_k^j) dt \right| \leq \frac{1}{2} |w^i - w^j|.$$

Thus $w^i = w^j$, and so $p^i = p^j$. This finishes the proof. \square

The composition satisfies the following tame inequalities.

LEMMA 3.7. *There are $C_n > 0$ such that, for all $1 \leq \delta \leq \sigma \leq 4$ and all $0 < s \leq r \leq 1$, we have that, if $\varphi \in \mathcal{U}_s$ and $\psi \in \mathcal{U}_r$ satisfy*

$$\varphi(E_s|_{\bar{O}_i^\delta}) \subset E_r|_{O_i^\sigma} \quad \text{and} \quad \psi(E_r|_{\bar{O}_i^\sigma}) \subset E|_{O_i} \quad \text{for all } 1 \leq i \leq I,$$

and $d(\varphi)_{1,s} < 1$, then the following inequalities hold:

$$\begin{aligned} d(\psi \circ \varphi)_{n,s,\delta} &\leq d(\psi)_{n,r,\sigma} + d(\varphi)_{n,s,\delta} + C_n s^{-n} (d(\psi)_{n,r,\sigma} d(\varphi)_{1,s,\delta} + d(\varphi)_{n,s,\delta} d(\psi)_{1,r,\sigma}), \\ d(\psi \circ \varphi, \psi)_{n,s,\delta} &\leq d(\varphi)_{n,s,\delta} + C_n s^{-n} (d(\psi)_{n+1,r,\sigma} d(\varphi)_{1,s,\delta} + d(\varphi)_{n,s,\delta} d(\psi)_{1,r,\sigma}). \end{aligned}$$

Proof. Denote the local expressions of φ and ψ as

$$\begin{aligned}\varphi_i &:= \text{Id} + g_i: \bar{B}_\delta \times \bar{B}_s \longrightarrow B_\sigma \times B_r, \\ \psi_i &:= \text{Id} + f_i: \bar{B}_\sigma \times \bar{B}_r \longrightarrow \mathbb{R}^d \times \mathbb{R}^D.\end{aligned}$$

Then, for all $z \in \bar{B}_\delta \times \bar{B}_s$, we can write

$$\psi_i(\varphi_i(z)) - z = f_i(z + g_i(z)) + g_i(z).$$

By computing $\partial^{|\alpha|}/\partial z^\alpha$ of the right-hand side, for a multi-index α with $|\alpha|=n$, we obtain an expression of the form

$$\frac{\partial^{|\alpha|} g_i}{\partial z^\alpha}(z) + \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(\varphi_i(z)) + \sum_{\beta, \gamma_1, \dots, \gamma_p} \frac{\partial^{|\beta|} f_i}{\partial z^\beta}(\varphi_i(z)) \frac{\partial^{|\gamma_1|} g_i^{j_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_i^{j_p}}{\partial z^{\gamma_p}}(z),$$

where the multi-indices in the sum satisfy

$$1 \leq p \leq n, \quad 1 \leq |\beta|, |\gamma_j| \leq n \quad \text{and} \quad |\beta| + \sum_{j=1}^p (|\gamma_j| - 1) = n. \quad (22)$$

The first two terms can be bounded by $d(\psi)_{n,r,\sigma} + d(\varphi)_{n,s,\delta}$. For the last term we use the interpolation inequalities to obtain that

$$\begin{aligned}\|f_i\|_{|\beta|,r,\sigma} &\leq C_n s^{1-|\beta|} \|f_i\|_{1,r,\sigma}^{(n-|\beta|)/(n-1)} \|f_i\|_{n,r,\sigma}^{(|\beta|-1)/(n-1)}, \\ \|g_i\|_{|\gamma_i|,s,\delta} &\leq C_n s^{1-|\gamma_i|} \|g_i\|_{1,s,\delta}^{(n-|\gamma_i|)/(n-1)} \|g_i\|_{n,s,\delta}^{(|\gamma_i|-1)/(n-1)}.\end{aligned}$$

Multiplying all these, and using (22), the sum is bounded by

$$C_n s^{1-n} \|g_i\|_{1,s,\delta}^{p-1} (\|f_i\|_{1,r,\sigma} \|g_i\|_{n,s,\delta})^{(n-|\beta|)/(n-1)} (\|f_i\|_{n,r,\sigma} \|g_i\|_{1,s,\delta})^{(|\beta|-1)/(n-1)}.$$

By Lemma 3.4, it follows that $\|g_i\|_{1,s,\delta} < C$, and dropping this term, the first part follows using inequality (19).

For the second part write, for $z \in \bar{B}_\delta \times \bar{B}_s$,

$$\psi_i(\varphi_i(z)) - \psi_i(z) = f_i(z + g_i(z)) - f_i(z) + g_i(z).$$

We compute $\partial^{|\alpha|}/\partial z^\alpha$ of the right-hand side, for a multi-index α with $|\alpha|=n$,

$$\frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(\varphi_i(z)) - \frac{\partial^{|\alpha|} f_i}{\partial z^\alpha}(z) + \frac{\partial^{|\alpha|} g_i}{\partial z^\alpha}(z) + \sum_{\beta, \gamma_1, \dots, \gamma_p} \frac{\partial^{|\beta|} f_i}{\partial z^\beta}(\varphi_i(z)) \frac{\partial^{|\gamma_1|} g_i^{j_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_i^{j_p}}{\partial z^{\gamma_p}}(z),$$

where the multi-indices in the sum satisfy (22). We bound the last term as before, and the third by $d(\varphi)_{n,s,\delta}$. Writing the first two terms as

$$\sum_{j=1}^{d+D} \int_0^1 \frac{\partial^{|\alpha|+1} f_i}{\partial z_j \partial z^\alpha}(z + t g_i(z)) g_i^j(z) dt,$$

they are less than $C d(\psi)_{n+1,r,\sigma} d(\varphi)_{0,s,\delta}$. Adding up, the result follows. \square

We now give conditions for infinite compositions of maps to converge.

LEMMA 3.8. *There exists $\theta > 0$, such that for all sequences*

$$\{\varphi_k \in \mathcal{U}_{r_k}\}_{k \geq 1}, \quad \varphi_k: E_{r_k} \longrightarrow E_{r_{k-1}},$$

where $0 < r < r_k < r_{k-1} \leq r_0 < 1$, which satisfy

$$\sigma_0 := \sum_{k \geq 1} d(\varphi_k)_{0,r_k} < \theta \quad \text{and} \quad \sigma_n := \sum_{k \geq 1} d(\varphi_k)_{n,r_k} < \infty \quad \text{for all } n \geq 1,$$

the sequence of maps

$$\psi_k := \varphi_1 \circ \dots \circ \varphi_k: E_{r_k} \longrightarrow E_{r_0}$$

converges in all C^n -norms on E_r to a map $\psi: E_r \rightarrow E_{r_0}$, with $\psi \in \mathcal{U}_r$. Moreover, there are $C_n > 0$ such that, if $d(\varphi_k)_{1,r_k} < 1$ for all $k \geq 1$, then

$$d(\psi)_{n,r} \leq e^{C_n r^{-n} \sigma_n} C_n r^{-n} \sigma_n.$$

Proof. Consider the sequences of numbers

$$\varepsilon_k := \frac{d(\varphi_k)_{0,r_k}}{\sum_{l \geq 1} d(\varphi_l)_{0,r_l}} \quad \text{and} \quad \delta_k := 2 - \sum_{l=1}^k \varepsilon_l.$$

We have that $d(\varphi_k)_{0,r_k} \leq \varepsilon_k \theta$. So, by Lemma 3.5, we may assume that

$$\varphi_k(E_{r_k} |_{\bar{O}_i^2}) \subset E_{r_{k-1}} |_{O_i} \quad \text{and} \quad \varphi_k(E_{r_k} |_{\bar{O}_i^{\delta_k}}) \subset E_{r_{k-1}} |_{O_i^{\delta_{k-1}}},$$

and this implies that

$$\psi_{k-1}(E_{r_{k-1}} |_{\bar{O}_i^{\delta_{k-1}}}) \subset E_{r_0} |_{O_i}.$$

So we can apply Lemma 3.7 to the pair ψ_{k-1} and φ_k for all $k > k_0$. The first part of Lemma 3.7 and Lemma 3.4 imply an inequality of the form

$$1 + d(\psi_k)_{n,r_k,\delta_k} \leq (1 + d(\psi_{k-1})_{n,r_{k-1},\delta_{k-1}}) (1 + C_n r^{-n} d(\varphi_k)_{n,r_k}).$$

Iterating this inequality, we obtain that

$$\begin{aligned} 1 + d(\psi_k)_{n,r_k,\delta_k} &\leq (1 + d(\psi_{k_0})_{n,r_{k_0},\delta_{k_0}}) \prod_{l=k_0+1}^k (1 + C_n r^{-n} d(\varphi_l)_{n,r_l}) \\ &\leq (1 + d(\psi_{k_0})_{n,r_{k_0},\delta_{k_0}}) e^{C_n r^{-n} \sum_{l > k_0} d(\varphi_l)_{n,r_l}} \\ &\leq (1 + d(\psi_{k_0})_{n,r_{k_0},\delta_{k_0}}) e^{C_n r^{-n} \sigma_n}. \end{aligned}$$

The second part of Lemma 3.7 and Lemma 3.4 imply that

$$\begin{aligned} d(\psi_k, \psi_{k-1})_{n,r} &\leq (1 + d(\psi_{k-1})_{n+1, r_{k-1}, \delta_{k-1}}) C_n r^{-n} d(\varphi_k)_{n, r_k, \delta_k} \\ &\leq (1 + d(\psi_{k_0})_{n+1, r_{k_0}, \delta_{k_0}}) e^{C_{n+1} r^{-1-n} \sigma_{n+1}} C_n r^{-n} d(\varphi_k)_{n, r_k}. \end{aligned}$$

This shows that the sum $\sum_{k \geq 1} d(\psi_k, \psi_{k-1})_{n,r}$ converges for all n , and hence the sequence $\{\psi_k|_{E_r}\}_{k \geq 1}$ converges in all C^n -norms to a smooth function $\psi: E_r \rightarrow E_{r_0}$.

If $d(\varphi_k)_{1, r_k} < 1$ for all $k \geq 1$, then we can take $k_0 = 0$. So, we obtain

$$1 + d(\psi_k)_{n, r_k, \delta_k} \leq \prod_{l=1}^k (1 + C_n r^{-n} d(\varphi_l)_{n, r_l}) \leq e^{C_n r^{-n} \sum_{l=1}^k d(\varphi_l)_{n, r_l}} \leq e^{C_n r^{-n} \sigma_n}.$$

Using the trivial inequality $e^x - 1 \leq x e^x$, for $x \geq 0$, the result follows. \square

Tameness of the flow

The C^0 -norm of a vector field controls the size of the domain of its flow.

LEMMA 3.9. *There exists $\theta > 0$ such that for all $0 < s < r \leq 1$ and all $X \in \mathfrak{X}^1(E_r)$ with $\|X\|_{0,r} < (r-s)\theta$, we have that φ_X^t , the flow of X , is defined for all $t \in [0, 1]$ on E_s and belongs to \mathcal{U}_s ,*

$$\varphi_X^t: E_{r-\varepsilon} \longrightarrow E_r.$$

Proof. We denote the restriction of X to a chart by $X_i \in \mathfrak{X}^1(\mathbb{R}^d \times \overline{B}_r)$. Consider $p \in \overline{B}_1 \times \overline{B}_s$. Let $t \in (0, 1]$ be such that the flow of X_i is defined up to time t at p and such that for all $\tau \in [0, t]$ it satisfies $\varphi_{X_i}^\tau(p) \in B_2 \times B_r$. Then we have that

$$|\varphi_{X_i}^t(p) - p| = \left| \int_0^t d(\varphi_{X_i}^\tau(p)) \right| \leq \int_0^t |X_i(\varphi_{X_i}^\tau(p))| d\tau \leq \|X_i\|_{0, r, 2} \leq C \|X\|_{0, r},$$

where for the last step we used Lemma 3.4. Hence, if $\|X\|_{0,r} < (r-s)/C$, we have that $\varphi_{X_i}^t(p) \in B_2 \times B_r$, and this implies the result. \square

We now prove that the map which with a vector field associates its flow is tame (this proof was inspired by the proof of [18, Lemma B.3]).

LEMMA 3.10. *There exists $\theta > 0$ such that for all $0 < s < r \leq 1$, and all $X \in \mathfrak{X}^1(E_r)$ with*

$$\|X\|_{0,r} < (r-s)\theta \quad \text{and} \quad \|X\|_{1,r} < \theta,$$

we have that $\varphi_X := \varphi_X^1$ belongs to \mathcal{U}_s and satisfies

$$d(\varphi_X)_{0,s} \leq C_0 \|X\|_{0,r} \quad \text{and} \quad d(\varphi_X)_{n,s} \leq r^{1-n} C_n \|X\|_{n,r} \quad \text{for all } n \geq 1.$$

Proof. By Lemma 3.9, for $t \in [0, 1]$, we have that $\varphi_X^t \in \mathcal{U}_s$, and by its proof that the local representatives take values in $B_2 \times B_r$,

$$\varphi_{X_i}^t := \text{Id} + g_{i,t}: \bar{B}_1 \times \bar{B}_s \longrightarrow B_2 \times B_r.$$

We will prove, by induction on n , that $g_{i,t}$ satisfies inequalities of the form

$$\|g_{i,t}\|_{n,s} \leq C_n P_n(X), \quad (23)$$

where $P_n(X)$ denotes the following polynomials in the norms of X :

$$P_0(X) = \|X\|_{0,r}, \quad P_1(X) = \|X\|_{1,r} \quad \text{and} \quad P_n(X) = \sum_{\substack{j_1 + \dots + j_p = n-1 \\ 1 \leq j_k \leq n-1}} \|X\|_{j_1+1,r} \dots \|X\|_{j_p+1,r}.$$

Observe that (23) implies the conclusion, since by the interpolation inequalities and the fact that $\|X\|_{1,r} < \theta \leq 1$ we have that

$$\|X\|_{j_k+1,r} \leq C_n r^{-j_k} \|X\|_{1,r}^{1-j_k/(n-1)} \|X\|_{n,r}^{j_k/(n-1)} \leq C_n r^{-j_k} \|X\|_{n,r}^{j_k/(n-1)},$$

and hence

$$P_n(X) \leq C_n r^{1-n} \|X\|_{n,r}.$$

The map $g_{i,t}$ satisfies the ordinary differential equation

$$\frac{dg_{i,t}}{dt}(z) = \frac{d\varphi_{X_i}^t}{dt}(z) = X_i(\varphi_{X_i}^t(z)) = X_i(g_{i,t}(z) + z).$$

Since $g_{i,0} = 0$, it follows that

$$g_{i,t}(z) = \int_0^t X_i(z + g_{i,\tau}(z)) d\tau. \quad (24)$$

Using also Lemma 3.4, we obtain the result for $n=0$:

$$\|g_{i,t}\|_{0,s} \leq \|X\|_{0,r,2} \leq C_0 \|X\|_{0,r}.$$

We will use the following version of the Grönwall inequality: if $u: [0, 1] \rightarrow \mathbb{R}$ is a continuous map and there are positive constants A and B such that

$$u(t) \leq A + B \int_0^t u(\tau) d\tau,$$

then u satisfies $u(t) \leq A e^{Bt}$.

Computing the partial derivative $\partial/\partial z_j$ of equation (24), we obtain

$$\frac{\partial g_{i,t}}{\partial z_j}(z) = \int_0^t \left(\frac{\partial X_i}{\partial z_j}(z + g_{i,\tau}(z)) + \sum_{k=1}^{D+d} \frac{\partial X_i}{\partial z_k}(z + g_{i,\tau}(z)) \frac{\partial g_{i,\tau}^k}{\partial z_j}(z) \right) d\tau.$$

Therefore, using again Lemma 3.4, the function

$$\left| \frac{\partial g_{i,t}}{\partial z_j}(z) \right|$$

satisfies

$$\left| \frac{\partial g_{i,t}}{\partial z_j}(z) \right| \leq C \|X\|_{1,r} + (D+d) \|X\|_{1,r} \int_0^t \left| \frac{\partial g_{i,\tau}}{\partial z_j}(z) \right| d\tau.$$

The case $n=1$ now follows by Grönwall's inequality:

$$\left\| \frac{\partial g_{i,t}}{\partial z_j} \right\|_{0,s} \leq C \|X\|_{1,r} e^{(D+d)\|X\|_{1,r}} \leq C \|X\|_{1,r}.$$

For a multi-index α , with $|\alpha|=n \geq 2$, applying $\partial^{|\alpha|}/\partial z^\alpha$ to (24), we obtain

$$\begin{aligned} \frac{\partial^{|\alpha|} g_{i,t}}{\partial z^\alpha}(z) &= \int_0^t \sum_{2 \leq |\beta| \leq |\alpha|} \frac{\partial^{|\beta|} X_i}{\partial z^\beta}(z + g_{i,\tau}(z)) \frac{\partial^{|\gamma_1|} g_{i,\tau}^{i_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_{i,\tau}^{i_p}}{\partial z^{\gamma_p}}(z) d\tau \\ &\quad + \int_0^t \sum_{j=1}^{D+d} \frac{\partial X_i}{\partial z_j}(z + g_{i,\tau}(z)) \frac{\partial^{|\alpha|} g_{i,\tau}^j}{\partial z^\alpha}(z) d\tau, \end{aligned} \quad (25)$$

where the multi-indices satisfy

$$1 \leq |\gamma_k| \leq n-1 \quad \text{and} \quad (|\gamma_1|-1) + \dots + (|\gamma_p|-1) + |\beta| = n.$$

Since $|\gamma_k| \leq n-1$, we can apply induction to conclude that

$$\left\| \frac{\partial^{|\gamma_k|} g_{i,\tau}^{i_k}}{\partial z^{\gamma_k}} \right\|_{0,s} \leq P_{|\gamma_k|}(X).$$

So, the first part of the sum can be bounded by

$$C_n \sum_{\substack{j_0 + \dots + j_p = n-1 \\ 1 \leq j_k \leq n-1}} \|X\|_{j_0+1,1} P_{j_1+1}(X) \dots P_{j_p+1}(X). \quad (26)$$

It is easy to see that the polynomials $P_k(X)$ satisfy

$$P_{u+1}(X) P_{v+1}(X) \leq C_{u,v} P_{u+v+1}(X). \quad (27)$$

Therefore (26) can be bounded by $C_n P_n(X)$. Using this in (25), we obtain

$$\left| \frac{\partial^{|\alpha|} g_{i,t}}{\partial z^\alpha}(z) \right| \leq C_n P_n(X) + (D+d) \|X\|_{1,r} \int_0^t \left| \frac{\partial^{|\alpha|} g_{i,\tau}}{\partial z^\alpha}(z) \right| d\tau.$$

Applying Grönwall's inequality again, we obtain the conclusion. \square

We now show how to approximate pull-backs by flows of vector fields.

LEMMA 3.11. *There exists $\theta > 0$ such that, for all $0 < s < r \leq 1$ and all $X \in \mathfrak{X}^1(E_r)$ with $\|X\|_{0,r} < (r-s)\theta$ and $\|X\|_{1,r} < \theta$, we have that*

$$\begin{aligned} \|\varphi_X^*(W)\|_{n,s} &\leq C_n r^{-n} (\|W\|_{n,r} + \|W\|_{0,r} \|X\|_{n+1,r}), \\ \|\varphi_X^*(W) - W\|_s|_{n,s} &\leq C_n r^{-2n-1} (\|X\|_{n+1,r} \|W\|_{1,r} + \|X\|_{1,r} \|W\|_{n+1,r}), \\ \|\varphi_X^*(W) - W\|_s - \varphi_X^*([X, W])\|_{n,s} &\leq C_n r^{-3(n+2)} \|X\|_{0,r} \\ &\quad \times (\|X\|_{n+2,r} \|W\|_{2,r} + \|X\|_{2,r} \|W\|_{n+2,r}) \end{aligned}$$

for all $W \in \mathfrak{X}^\bullet(E_r)$, where $C_n > 0$ is a constant depending only on n .

Proof. As in the proof above, the local expression of φ_X is defined as

$$\varphi_{X_i} = \text{Id} + g_i: \bar{B}_1 \times \bar{B}_s \longrightarrow B_2 \times B_r.$$

Let $W \in \mathfrak{X}^\bullet(E_r)$, and denote by W_i its local expression on $E_r|_{\bar{O}_2^i}$,

$$W_i := \sum_{J=\{j_1 < \dots < j_k\}} W_i^J(z) \frac{\partial}{\partial z_{j_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{j_k}} \in \mathfrak{X}^\bullet(\bar{B}_2 \times \bar{B}_r).$$

The local representative of $\varphi_X^*(W)$ is given, for $z \in \bar{B}_1 \times \bar{B}_s$, by

$$(\varphi_X^* W)_i = \sum_J W_i^J(z + g_i(z)) (\text{Id} + d_z g_i)^{-1} \frac{\partial}{\partial z_{j_1}} \wedge \dots \wedge (\text{Id} + d_z g_i)^{-1} \frac{\partial}{\partial z_{j_k}}.$$

By the Cramer rule, the matrix $(\text{Id} + d_z g_i)^{-1}$ has entries of the form

$$\Psi \left(\frac{\partial g_i^l}{\partial z_j}(z) \right) \det(\text{Id} + d_z g_i)^{-1},$$

where Ψ is a polynomial in the variables Y_j^l , which we substitute by

$$\frac{\partial g_i^l}{\partial z_j}(z).$$

Therefore, any coefficient of the local expression of $\varphi_X^*(W)_i$ will be a sum of elements of the form

$$W_i^J(z + g_i(z)) \Psi \left(\frac{\partial g_i^l}{\partial z_j}(z) \right) \det(\text{Id} + d_z g_i)^{-k}.$$

When computing $\partial^{|\alpha|} / \partial z^\alpha$ of such an expression, with $|\alpha| = n$, using an inductive argument, one proves that the outcome is a sum of terms of the form

$$\frac{\partial^{|\beta|} W_i^J}{\partial z^\beta}(z + g_i(z)) \frac{\partial^{|\gamma_1|} g_i^{v_1}}{\partial z^{\gamma_1}}(z) \dots \frac{\partial^{|\gamma_p|} g_i^{v_p}}{\partial z^{\gamma_p}}(z) \det(\text{Id} + d_z g_i)^{-M}, \quad (28)$$

with coefficients depending only on α and on the multi-indices, which satisfy

$$p \geq 0, \quad M \geq 0, \quad |\gamma_j| \geq 1 \quad \text{and} \quad |\beta| + (|\gamma_1| - 1) + \dots + (|\gamma_p| - 1) = n.$$

By Lemma 3.10, $\|g_i\|_{1,s} < C\theta$, so, if we shrink θ , we find that

$$\det(\text{Id} + d_z g_i)^{-1} < 2 \quad \text{for all } z \in \bar{B}_1 \times \bar{B}_s.$$

Using this, Lemma 3.4 for W and

$$\left| \frac{\partial g_i^l}{\partial z_j}(z) \right| \leq C,$$

we bound (28) by

$$C_n \sum_{j, j_1, \dots, j_p} \|W\|_{j,r} \|g_i\|_{j_1+1,s} \dots \|g_i\|_{j_p+1,s},$$

where the indices satisfy

$$j \geq 0, \quad j_k \geq 0 \quad \text{and} \quad j + j_1 + \dots + j_p = n.$$

The term with $p=0$ can simply be bounded by $C_n \|W\|_{n,r}$. For the other terms, we will use the bound $\|g_i\|_{j_k+1,s} \leq P_{j_k+1}(X)$ from the proof of Lemma 3.10. The multiplicative property (27) of the polynomials $P_l(X)$ implies that

$$\|\varphi_X^*(W)\|_{n,s} \leq C_n \sum_{j=0}^n \|W\|_{j,r} P_{n-j+1}(X).$$

Applying interpolation to $W_{j,r}$ and to a term of $P_{n-j+1}(X)$, we obtain

$$\begin{aligned} \|W\|_{j,r} &\leq C_n r^{-j} \|W\|_{0,r}^{1-j/n} \|W\|_{n,r}^{j/n}, \\ \|X\|_{j_k+1,r} &\leq C_n r^{-j_k} \|X\|_{1,r}^{1-j_k/n} \|X\|_{n+1,r}^{j_k/n} \leq C_n r^{-j_k} \|X\|_{n+1,r}^{j_k/n}. \end{aligned}$$

Multiplying all these terms and using (19), we conclude the first part of the proof:

$$\begin{aligned} \|W\|_{j,r} \|X\|_{j_1+1,r} \dots \|X\|_{j_p+1,r} &\leq C_n r^{-n} (\|W\|_{0,r} \|X\|_{n+1,r})^{1-j/n} \|W\|_{n,r}^{j/n} \\ &\leq C_n r^{-n} (\|W\|_{n,r} + \|W\|_{0,r} \|X\|_{n+1,r}). \end{aligned}$$

For the second inequality, set

$$W_t := \varphi_X^{t*}(W) - W|_s \in \mathfrak{X}^*(E_s).$$

Then $W_0=0$, $W_1=\varphi_X^*(W)-W|_s$ and

$$\frac{d}{dt}W_t = \varphi_X^{t*}([X, W]),$$

and therefore

$$\varphi_X^*(W)-W|_s = \int_0^1 \varphi_X^{t*}([X, W]) dt.$$

By the first part, we obtain

$$\|\varphi_X^*(W)-W|_s\|_{n,s} \leq C_n r^{-n} (\|[X, W]\|_{n,r} + \|[X, W]\|_{0,r} \|X\|_{n+1,r}).$$

Using now Lemma 3.3 and that $\|X\|_{1,r} \leq \theta$, we obtain the second part:

$$\|\varphi_X^*(W)-W|_s\|_{n,s} \leq C_n r^{-2n-1} (\|X\|_{n+1,r} \|W\|_{1,r} + \|W\|_{1,r} \|X\|_{n+1,r}).$$

For the last inequality, set

$$W_t := \varphi_X^{t*}(W) - W|_s - t\varphi_X^{t*}([X, W]).$$

Then we have that $W_0=0$, $W_1=\varphi_X^*(W)-W|_s-\varphi_X^*([X, W])$ and

$$\frac{d}{dt}W_t = -t\varphi_X^{t*}([X, [X, W]]),$$

and therefore

$$W_1 = - \int_0^1 t\varphi_X^{t*}([X, [X, W]]) dt.$$

Using again the first part, it follows that

$$\|W_1\|_{n,s} \leq C_n r^{-n} (\|[X, [X, W]]\|_{n,r} + \|[X, [X, W]]\|_{0,r} \|X\|_{n+1,r}). \quad (29)$$

Applying Lemma 3.3 twice, for all $k \leq n$, we obtain that

$$\begin{aligned} \|[X, [X, W]]\|_{k,r} &\leq C_n (r^{-(k+3)} \|X\|_{k+1,r} (\|X\|_{0,r} \|W\|_{1,r} + \|X\|_{1,r} \|W\|_{0,r}) \\ &\quad + r^{-(2k+3)} \|X\|_{0,r} (\|X\|_{0,r} \|W\|_{k+2,r} + \|X\|_{k+2,r} \|W\|_{0,r})) \\ &\leq C_n r^{-(2k+5)} \|X\|_{0,r} (\|W\|_{k+2,r} \|X\|_{0,r} + \|W\|_{2,r} \|X\|_{k+2,r}), \end{aligned}$$

where we have used the interpolation inequality

$$\|X\|_{1,r} \|X\|_{k+1,r} \leq C_n r^{-(k+2)} \|X\|_{0,r} \|X\|_{k+2,r}.$$

The first term in (29) can be bounded using this inequality for $k=n$. For $k=0$, using also that $\|X\|_{1,r} \leq \theta$ and the interpolation inequality

$$\|X\|_{2,r} \|X\|_{n+1,r} \leq C_n r^{-(n+1)} \|X\|_{1,r} \|X\|_{n+2,r},$$

we can bound the second term in (29), and this concludes the proof:

$$\|[X, [X, W]]\|_{0,r} \|X\|_{n+1,r} \leq C_n r^{-(n+6)} \|W\|_{2,r} \|X\|_{0,r} \|X\|_{n+2,r}. \quad \square$$

3.3. An invariant tubular neighborhood and tame homotopy operators

We now start the proof of Theorem 2. We will use two results presented in the appendix: existence of invariant tubular neighborhoods (Lemma A.1) and the tame vanishing lemma (Lemma C.1).

Let (M, π) and $N \subset M$ be as in the statement. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid integrating T^*M . By restricting to the connected components of the identities in the s -fibers of \mathcal{G} [19], we may assume that \mathcal{G} has connected s -fibers.

By Lemma A.1, N has an invariant tubular neighborhood $E \cong \nu_N$ endowed with a metric, such that the closed tubes $E_r := \{v \in E : |v| \leq r\}$, for $r > 0$, are also \mathcal{G} -invariant. We endow E with all the structure from §3.1.

Since E is invariant, the cotangent Lie algebroid of (E, π) is integrable by $\mathcal{G}|_E$, which has compact s -fibers with vanishing H^2 . Therefore, by the tame vanishing lemma and Corollaries C.2 and C.3 from the appendix, there are linear homotopy operators

$$\mathfrak{X}^1(E) \xleftarrow{h_1} \mathfrak{X}^2(E) \xleftarrow{h_2} \mathfrak{X}^3(E),$$

with

$$[\pi, h_1(V)] + h_2([\pi, V]) = V \quad \text{for all } V \in \mathfrak{X}^2(E),$$

which satisfy the following properties:

- they induce linear homotopy operators h_1^r and h_2^r on $(E_r, \pi|_r)$;
- there are constants $C_n > 0$ such that, for all $r \in (0, 1]$,

$$\|h_1^r(X)\|_{n,r} \leq C_n \|X\|_{n+s,r} \quad \text{and} \quad \|h_2^r(Y)\|_{n,r} \leq C_n \|Y\|_{n+s,r}$$

for all $X \in \mathfrak{X}^2(E_r)$ and all $Y \in \mathfrak{X}^3(E_r)$, where $s = \lfloor \frac{1}{2} \dim(M) \rfloor + 1$;

- they induce homotopy operators on the subcomplex of vector fields vanishing along N .

3.4. The Nash–Moser method

We fix radii $0 < r < R < 1$. Let s be as in the previous subsection, and let

$$\alpha := 2(s+5) \quad \text{and} \quad p := 7(s+4).$$

Then p is the integer from the statement of Theorem 2. Consider a second Poisson structure $\tilde{\pi}$ defined on E_R . With $\tilde{\pi}$ we associate the following inductive procedure.

Procedure P_0 . Consider

- the number

$$t(\tilde{\pi}) := \|\pi - \tilde{\pi}\|_{p,R}^{-1/\alpha},$$

- the sequences of numbers

$$\begin{aligned} \varepsilon_0 &:= \frac{1}{4}(R-r), & r_0 &:= R, & t_0 &:= t(\tilde{\pi}), \\ \varepsilon_{k+1} &:= \varepsilon_k^{3/2}, & r_{k+1} &:= r_k - \varepsilon_k, & t_{k+1} &:= t_k^{3/2}, \end{aligned}$$

- the sequences of Poisson bivectors and vector fields

$$\{\pi_k \in \mathfrak{X}^2(E_{r_k})\}_{k \geq 0} \quad \text{and} \quad \{X_k \in \mathfrak{X}^1(E_{r_k})\}_{k \geq 0},$$

defined inductively by

$$\pi_0 := \tilde{\pi}, \quad \pi_{k+1} := \varphi_{X_k}^*(\pi_k) \quad \text{and} \quad X_k := S_{t_k}^{r_k}(h_1^{r_k}(\pi_k - \pi|_{r_k})), \quad (30)$$

- the sequence of maps

$$\psi_k := \varphi_{X_0} \circ \dots \circ \varphi_{X_k} : E_{r_{k+1}} \longrightarrow E_R.$$

By our choice of ε_0 , observe that $r < r_k < R$ for all $k \geq 1$:

$$\sum_{k=0}^{\infty} \varepsilon_k = \sum_{k=0}^{\infty} \varepsilon_0^{(3/2)^k} < \sum_{k=0}^{\infty} \varepsilon_0^{1+k/2} = \frac{\varepsilon_0}{1-\sqrt{\varepsilon_0}} \leq R-r,$$

For procedure P_0 to be well defined, we need that (C_k) the time-one flow of X_k is defined as a map

$$\varphi_{X_k} : E_{r_{k+1}} \longrightarrow E_{r_k}.$$

For part (b) of Theorem 2, we consider also the procedure P_1 , associated with $\tilde{\pi}$ such that $j^1\tilde{\pi}|_N = j^1\pi|_N$. We define *procedure* P_1 the same as procedure P_0 , except that in (30) we use the smoothing operators $S_{t_k}^{r_k, 1}$.

To show that procedure P_1 is well defined, in addition to (C_k) , we need that

$$h_1^{r_k}(\pi_k - \pi|_{r_k}) \in \mathfrak{X}^1(E_{r_k})^{(1)}.$$

Since the operators $h_1^{r_k}$ preserve the space of tensors vanishing up to first order, it suffices to show that $j^1(\pi_k - \pi|_{r_k})|_N = 0$. This is proven inductively: by hypothesis,

$$j^1(\pi_0 - \pi|_R)|_N = 0.$$

Assume that $j^1(\pi_k - \pi|_{r_k})|_N = 0$, for some $k \geq 0$. Then, as before, also $X_k \in \mathfrak{X}^1(E_{r_k})^{(1)}$. Hence the first-order jet of φ_{X_k} along N is that of the identity, and so

$$j^1(\pi_{k+1})|_N = j^1(\pi_k)|_N = j^1(\pi)|_N.$$

Therefore $j^1(\pi_{k+1} - \pi|_{r_{k+1}})|_N = 0$.

Procedure P_0 produces the map ψ from Theorem 2.

PROPOSITION 3.12. *There exists $\delta > 0$ and an integer $d \geq 0$ for which procedure P_0 is well defined for every Poisson bivector $\tilde{\pi}$ satisfying*

$$\|\tilde{\pi} - \pi\|_{p,R} < \delta(r(R-r))^d. \quad (31)$$

If, in addition, $j^1\pi|_N = j^1\tilde{\pi}|_N$, then P_1 is also well defined for $\tilde{\pi}$. In both cases, the resulting sequence $\psi_k|_r$ converges uniformly on E_r with all its derivatives to a local diffeomorphism ψ , which is a Poisson map

$$\psi: (E_r, \pi|_r) \longrightarrow (E_R, \tilde{\pi}),$$

which satisfies

$$d(\psi)_{1,r} \leq \|\pi - \tilde{\pi}\|_{p,R}^{1/\alpha}. \quad (32)$$

In the case of P_1 , the map ψ is the identity along N up to first order.

Proof. We will prove the statement for the two procedures simultaneously. We denote the used smoothing operators by S_k , that is, in P_0 we let $S_k := S_{t_k}^{r_k}$ and in P_1 we let $S_k := S_{t_k}^{r_k,1}$. In both cases, these satisfy the inequalities

$$\begin{aligned} \|S_k(X)\|_{m,r_k} &\leq C_m r^{-c_m} t_k^{l+1} \|X\|_{m-l,r_k}, \\ \|S_k(X) - X\|_{m-l,r_k} &\leq C_m r^{-c_m} t_k^{-l} \|X\|_{m+1,r_k}. \end{aligned}$$

For the procedures to be well defined and to converge, we need that $t_0 = t(\tilde{\pi})$ is big enough, more precisely it will have to satisfy a finite number of inequalities of the form

$$t_0 = t(\tilde{\pi}) > C(r(R-r))^{-c}. \quad (33)$$

Taking $\tilde{\pi}$ such that it satisfies (31), it suffices to ask that δ is small enough and d is big enough, such that a finite number of inequalities of the form

$$\delta((R-r)r)^d < \frac{1}{C}(r(R-r))^c$$

hold, and then t_0 will satisfy (33).

Also, since $t_0 > 4(R-r)^{-1} = \varepsilon_0^{-1}$, it follows that

$$t_k > \varepsilon_k^{-1} \quad \text{for all } k \geq 0.$$

We will prove inductively that the bivectors

$$Z_k := \pi_k - \pi|_{r_k} \in \mathfrak{X}^2(E_{r_k})$$

satisfy the inequalities

$$\|Z_k\|_{s,r_k} \leq t_k^{-\alpha}, \tag{a_k}$$

$$\|Z_k\|_{p,r_k} \leq t_k^\alpha. \tag{b_k}$$

Since $t_0^{-\alpha} = \|Z_0\|_{p,R}$, (a_0) and (b_0) hold. Assuming that (a_k) and (b_k) hold for some $k \geq 0$, we will show that condition (C_k) holds (i.e. the procedure is well defined up to step k) and also that (a_{k+1}) and (b_{k+1}) hold.

First we give a bound for the norms of X_k in terms of the norms of Z_k ,

$$\begin{aligned} \|X_k\|_{m,r_k} = \|S_k(h_1^{r_k}(Z_k))\|_{m,r_k} &\leq C_m r^{-c_m} t_k^{1+l} \|h_1^{r_k}(Z_k)\|_{m-l,r_k} \\ &\leq C_m r^{-c_m} t_k^{1+l} \|Z_k\|_{m+s-l,r_k} \quad \text{for all } 0 \leq l \leq m. \end{aligned} \tag{34}$$

In particular, for $m=l$, we obtain

$$\|X_k\|_{m,r_k} \leq C_m r^{-c_m} t_k^{1+m-\alpha}. \tag{35}$$

As $\alpha > 4$ and $t_k > \varepsilon_k^{-1}$, this inequality implies that

$$\|X_k\|_{1,r_k} \leq C r^{-c} t_k^{2-\alpha} \leq C r^{-c} t_0^{-1} t_k^{-1} < C r^{-c} t_0^{-1} \varepsilon_k. \tag{36}$$

Since $t_0 > C r^{-c} / \theta$, we have that $\|X_k\|_{1,r_k} \leq \theta \varepsilon_k$, and so by Lemma 3.9 (C_k) holds. Moreover, X_k satisfies the inequalities from Lemmas 3.10 and 3.11.

We now deduce an inequality for all norms $\|Z_{k+1}\|_{n,r_{k+1}}$, with $n \geq s$,

$$\begin{aligned} \|Z_{k+1}\|_{n,r_{k+1}} &= \|\varphi_{X_k}^*(Z_k) + \varphi_{X_k}^*(\pi) - \pi\|_{n,r_{k+1}} \\ &\leq C_n r^{-c_n} (\|Z_k\|_{n,r_k} + \|X_k\|_{n+1,r_k} \|Z_k\|_{0,r_k} + \|X_k\|_{n+1,r_k} \|\pi\|_{n+1,r_k}) \\ &\leq C_n r^{-c_n} (\|Z_k\|_{n,r_k} + \|X_k\|_{n+1,r_k}) \\ &\leq C_n r^{-c_n} t_k^{s+2} \|Z_k\|_{n,r_k}, \end{aligned} \tag{37}$$

where we used Lemma 3.11, the inductive hypothesis and inequality (34) with $m=n+1$ and $l=s+1$. For $n=p$, using also that $s+2+\alpha \leq \frac{3}{2}\alpha-1$, this gives (b_{k+1}) :

$$\|Z_{k+1}\|_{p,r_{k+1}} \leq C r^{-c} t_k^{s+2+\alpha} \leq C r^{-c} t_k^{3\alpha/2-1} \leq C r^{-c} t_0^{-1} t_{k+1}^\alpha \leq t_{k+1}^\alpha.$$

To prove (a_{k+1}) , we write $Z_{k+1} = V_k + \varphi_{X_k}^*(U_k)$, where

$$V_k := \varphi_{X_k}^*(\pi) - \pi - \varphi_{X_k}^*([X_k, \pi]) \quad \text{and} \quad U_k := Z_k - [\pi, X_k].$$

Using Lemma 3.11 and inequality (35), we bound the two terms by

$$\|V_k\|_{s,r_{k+1}} \leq Cr^{-c} \|\pi\|_{s+2,r_k} \|X_k\|_{0,r_k} \|X_k\|_{s+2,r_k} \leq Cr^{-c} t_k^{s+4-2\alpha}, \quad (38)$$

$$\begin{aligned} \|\varphi_{X_k}^*(U_k)\|_{s,r_{k+1}} &\leq Cr^{-c} (\|U_k\|_{s,r_k} + \|U_k\|_{0,r_k} \|X_k\|_{s+1,r_k}) \\ &\leq Cr^{-c} (\|U_k\|_{s,r_k} + t_k^{s+2-\alpha} \|U_k\|_{0,r_k}). \end{aligned} \quad (39)$$

To compute the C^s -norm for U_k , we rewrite it as

$$\begin{aligned} U_k &= Z_k - [\pi, X_k] = [\pi, h_1^{r_k}(Z_k)] + h_2^{r_k}([\pi, Z_k]) - [\pi, X_k] \\ &= [\pi, (I - S_k)h_1^{r_k}(Z_k)] - \frac{1}{2}h_2^{r_k}([Z_k, Z_k]). \end{aligned}$$

By tameness of the Lie bracket, the first term can be bounded by

$$\begin{aligned} \|[\pi, (I - S_k)h_1^{r_k}(Z_k)]\|_{s,r_k} &\leq Cr^{-c} \|(I - S_k)h_1^{r_k}(Z_k)\|_{s+1,r_k} \\ &\leq Cr^{-c} t_k^{2-p+2s} \|h_1^{r_k}(Z_k)\|_{p-s,r_k} \\ &\leq Cr^{-c} t_k^{2-p+2s} \|Z_k\|_{p,r_k} \\ &\leq Cr^{-c} t_k^{2-p+2s+\alpha} \\ &= Cr^{-c} t_k^{-3\alpha/2-1}, \end{aligned}$$

and using also the interpolation inequalities, for the second term we obtain

$$\begin{aligned} \|\frac{1}{2}h_2^{r_k}([Z_k, Z_k])\|_{s,r_k} &\leq C\|[Z_k, Z_k]\|_{2s,r_k} \\ &\leq Cr^{-c} \|Z_k\|_{0,r_k} \|Z_k\|_{2s+1,r_k} \\ &\leq Cr^{-c} t_k^{-\alpha} \|Z_k\|_{s,r_k}^{(p-(2s+1))/(p-s)} \|Z_k\|_{p,r_k}^{(s+1)/(p-s)} \\ &\leq Cr^{-c} t_k^{-\alpha(1+(p-(3s+2))/(p-s))}. \end{aligned}$$

Since $-\alpha(1+(p-(3s+2))/(p-s)) \leq -\frac{3}{2}\alpha - 1$, these two inequalities imply that

$$\|U_k\|_{s,r_k} \leq Cr^{-c} t_k^{-3\alpha/2-1}. \quad (40)$$

Using (35), we bound the C^0 -norm of U_k by

$$\|U_k\|_{0,r_k} \leq \|Z_k\|_{0,r_k} + \|[\pi, X_k]\|_{0,r_k} \leq t_k^{-\alpha} + Cr^{-c} \|X_k\|_{1,r_k} \leq Cr^{-c} t_k^{2-\alpha}. \quad (41)$$

By (38)–(41) and $s+4-2\alpha = -\frac{3}{2}\alpha - 1$, (a_{k+1}) follows:

$$\|Z_{k+1}\|_{s,r_{k+1}} \leq Cr^{-c} (t_k^{s+4-2\alpha} + t_k^{-3\alpha/2-1}) \leq Cr^{-c} t_k^{-3\alpha/2-1} \leq \frac{Cr^{-c} t_k^{-3\alpha/2}}{t_0} \leq t_{k+1}^{-\alpha}.$$

This finishes the induction.

Using (37), for every $n \geq 1$, we find $k_n \geq 0$ such that

$$\|Z_{k+1}\|_{n,r_{k+1}} \leq t_k^{s+3} \|Z_k\|_{n,r_k} \quad \text{for all } k \geq k_n.$$

Iterating this, we obtain

$$t_k^{s+3} \|Z_k\|_{n,r_k} \leq (t_k t_{k-1} \dots t_{k_n})^{s+3} \|Z_{k_n}\|_{n,r_{k_n}}.$$

On the other hand, we have that

$$t_k t_{k-1} \dots t_{k_n} = t_{k_n}^{1+3/2+\dots+(3/2)^{k-k_n}} \leq t_{k_n}^{2(3/2)^{k+1-k_n}} = t_k^3.$$

Therefore, we obtain a bound valid for all $k > k_n$,

$$\|Z_k\|_{n,r_k} \leq t_k^{2(s+3)} \|Z_{k_n}\|_{n,r_{k_n}}.$$

Consider now $m > s$ and set $n := 4m - 3s$. Applying the interpolation inequalities, for $k > k_n$, we obtain

$$\begin{aligned} \|Z_k\|_{m,r_k} &\leq C_m r^{-c_m} \|Z_k\|_{s,r_k}^{(n-m)/(n-s)} \|Z_k\|_{n,r_k}^{(m-s)/(n-s)} = C_m r^{-c_m} \|Z_k\|_{s,r_k}^{3/4} \|Z_k\|_{n,r_k}^{1/4} \\ &\leq C_m r^{-c_m} t_k^{-3\alpha/4+2(s+3)/4} \|Z_{k_n}\|_{n,r_{k_n}}^{1/4} = C_m r^{-c_m} t_k^{-(s+6)} \|Z_{k_n}\|_{n,r_{k_n}}^{1/4}. \end{aligned}$$

Using also inequality (34), for $l = s$, we obtain

$$\|X_k\|_{m,r_k} \leq C_m r^{-c_m} t_k^{s+1} \|Z_k\|_{m,r_k} \leq t_k^{-5} (C_m r^{-c_m} \|Z_{k_n}\|_{n,r_{k_n}}^{1/4}).$$

This shows that the series $\sum_{k \geq 0} \|X_k\|_{m,r_k}$ converges for all m . By Lemma 3.10, also $\sum_{k \geq 0} d(\varphi_{X_k})_{m,r_{k+1}}$ converges for all m and, moreover, by (36), we have that

$$\sigma_1 := \sum_{k \geq 1} d(\varphi_{X_k})_{1,r_{k+1}} \leq C r^{-c} \sum_{k \geq 1} \|X_k\|_{1,r_k} \leq C r^{-c} t_0^{-4} \sum_{k \geq 1} \varepsilon_k \leq t_0^{-3}.$$

So, we may assume that $\sigma_1 \leq \theta$ and $d(\varphi_{X_k})_{1,r_{k+1}} < 1$. Then, by applying Lemma 3.8, we conclude that the sequence $\psi_k|_r$ converges uniformly in all C^m -norms to a map $\psi: E_r \rightarrow E_R$ in \mathcal{U}_r which satisfies

$$d(\psi)_{1,r} \leq e^{C r^{-c} \sigma_1} C r^{-c} \sigma_1 \leq e^{t_0^{-2}} t_0^{-2} \leq C t_0^{-2} \leq t_0^{-1}.$$

So (32) holds, and we can also assume that $d(\psi)_{1,r} < \theta$, which, by Lemma 3.6, implies that ψ is a local diffeomorphism. Since $\psi_k|_r$ converges in the C^1 -topology to ψ and $\psi_k^*(\tilde{\pi}) = (d\psi_k)^{-1}(\tilde{\pi}|_{\psi_k})$, it follows that $\psi_k^*(\tilde{\pi})|_r$ converges in the C^0 -topology to $\psi^*(\tilde{\pi})$. On the other hand, $Z_k|_r = \psi_k^*(\tilde{\pi})|_r - \pi|_r$ converges to 0 in the C^0 -norm, and hence $\psi^*(\tilde{\pi}) = \pi|_r$. So ψ is a Poisson map and a local diffeomorphism

$$\psi: (E_r, \pi|_r) \longrightarrow (E_R, \tilde{\pi}).$$

For procedure P_1 , as noted before the proposition, the first jet of φ_{X_k} is that of the identity along N . This clearly holds also for ψ_k , and since $\psi_k|_r$ converges to ψ in the C^1 -topology, also ψ is the identity along N up to first order. \square

We are now ready to finish the proof of Theorem 2.

3.5. Proof of part (a) of Theorem 2

We have to check the properties from the definition of C^p - C^1 -rigidity. Consider $U := \text{int}(E_\varrho)$, for some $\varrho \in (0, 1)$, and let $O \subset U$ be an open set such that $N \subset O \subset \bar{O} \subset U$. Let $r < R$ be such that $O \subset E_r \subset E_R \subset U$. With d and δ from Proposition 3.12, we let

$$\mathcal{V}_O := \{W \in \mathfrak{X}^2(U) : \|W|_R - \pi|_R\|_{p,R} < \delta(r(R-r))^d\}.$$

For $\tilde{\pi} \in \mathcal{V}_O$, define $\psi_{\tilde{\pi}}$ to be the restriction to \bar{O} of the map ψ , obtained by applying procedure P_0 to $\tilde{\pi}|_R$. Then ψ is a Poisson diffeomorphism $(O, \pi|_O) \rightarrow (U, \tilde{\pi})$, and by (32), the assignment $\tilde{\pi} \mapsto \psi$ has the required continuity property.

3.6. Proof of part (b) of Theorem 2

Consider a Poisson structure $\tilde{\pi}$ on some neighborhood of N with $j^1\tilde{\pi}|_N = j^1\pi|_N$. First we show that π and $\tilde{\pi}$ are formally Poisson diffeomorphic around N . By [15], this property is controlled by the groups $H^2(A_N, \mathcal{S}^k(\nu_N^*))$. The Lie groupoid $\mathcal{G}|_N \rightrightarrows N$ integrates A_N and is s -connected. Since $\nu_N^* \subset A_N$ is an ideal, by Lemma B.1 below, the action of A_N on ν_N^* (and hence also on $\mathcal{S}^k(\nu_N^*)$) also integrates to $\mathcal{G}|_N$. Since $\mathcal{G}|_N$ has compact s -fibers with vanishing H^2 , the tame vanishing lemma implies that $H^2(A_N, \mathcal{S}^k(\nu_N^*)) = 0$. So we can apply [15, Theorem 1.1] to conclude that there exists a diffeomorphism φ between open neighborhoods of N , which is the identity on N up to first order, and such that $j^\infty\varphi^*(\tilde{\pi})|_N = j^\infty\pi|_N$.

Let $R \in (0, 1)$ be such that $\varphi^*(\tilde{\pi})$ is defined on E_R . Using the Taylor expansion up to order $2d+1$ around N for the bivector $\pi - \varphi^*(\tilde{\pi})$ and its partial derivatives up to order p , we find a constant $M > 0$ such that

$$\|\varphi^*(\tilde{\pi})|_r - \pi|_r\|_{p,r} \leq Mr^{2d+1} \quad \text{for all } 0 < r < R.$$

If we take $r < 2^{-d}\delta/M$, we obtain that $\|\varphi^*(\tilde{\pi})|_r - \pi|_r\|_{p,r} < \delta(r(r - \frac{1}{2}r))^d$. So, we can apply Proposition 3.12, and procedure P_1 produces a Poisson diffeomorphism

$$\tau: (E_{r/2}, \pi|_{r/2}) \longrightarrow (E_r, \varphi^*(\tilde{\pi})|_r),$$

which is the identity up to first order along N . We obtain (b) with $\psi = \varphi \circ \tau$.

Remark 2. As mentioned already in the introduction, Conn's proof has been formalized in [18] and [20] into an abstract Nash–Moser normal form theorem, and it is likely that one could use [18, Theorem 6.8] to partially prove our rigidity result. Nevertheless, the continuity assertion, which is important in applications (see [17]), is not a

consequence of this result. There are also several technical reasons why we cannot apply [18]: we need the size of the C^p -open set to depend polynomially on r^{-1} and $(R-r)^{-1}$, because we use a formal linearization argument (this dependence is not given in [18]); to obtain diffeomorphisms which fix N , we work with vector fields which vanish along N up to first order, and it is unlikely that this Fréchet space admits smoothing operators of degree 0 (in [18] this is the overall assumption); for the inequalities in Lemma 3.7 we need special norms for the embeddings (indexed also by “ δ ”), which are not considered in [18].

Appendix A. Invariant tubular neighborhoods

In the proof of Theorem 2, we have used the following result.

LEMMA A.1. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid with connected s -fibers and let $N \subset M$ be a compact invariant submanifold. There exists a tubular neighborhood $E \subset M$ (where $E \cong T_N M / TN$) and a metric on E such that, for all $r > 0$, the closed tube $E_r := \{v \in E : |v| \leq r\}$ is \mathcal{G} -invariant.*

This lemma follows from results in [22]; in particular we will use the following lemma.

LEMMA A.2. ([22, Propositions 3.14 and 6.4]) *On the base of a proper Lie groupoid there exist Riemannian metrics such that every geodesic which emanates orthogonally from an orbit stays orthogonal to any orbit it passes through. Such metrics are called adapted.*

Proof of Lemma A.1. Let g be an adapted metric on M and let $E := TN^\perp \subset T_N M$ be the normal bundle. By rescaling g , we may assume that

- (1) the exponential is defined on E_2 and on $\text{int}(E_2)$ it is an open embedding;
- (2) for all $r \in (0, 1]$ we have that

$$\exp(E_r) = \{p \in M : d(p, N) \leq r\},$$

where d denotes the distance induced by the Riemannian structure.

Let $v \in E_1$ with base point x , and denote by $r := |v|$. We claim that the geodesic $\gamma(t) := \exp(tv)$ at $t=1$ is normal to $\exp(\partial E_r)$ at $\gamma(1)$:

$$T_{\gamma(1)} \exp(\partial E_r) = \dot{\gamma}(1)^\perp.$$

Let $S_r(x)$ be the sphere of radius r centered at x . By the Gauss lemma,

$$\dot{\gamma}(1)^\perp = T_{\gamma(1)} S_r(x),$$

and by (2), $\bar{B}_r(x) \subset \exp(E_r)$, where $\bar{B}_r(x)$ is the closed ball of radius r around x . Since $\bar{B}_r(x)$ and $\exp(E_r)$ intersect at $\gamma(1)$, their boundaries must be tangent at this point, and this proves the claim.

By assumption, N is a union of orbits. Therefore the geodesics $\gamma(t) := \exp(tv)$, for $v \in E$, start normal to the orbits of \mathcal{G} , and thus, by the property of the metric, they continue to be orthogonal to the orbits. Hence, by our claim, the orbits which intersect $\exp(\partial E_r)$ are tangent to $\exp(\partial E_r)$. By connectivity of the orbits, $\exp(\partial E_r)$ is invariant, for all $r \in (0, 1)$. Define the embedding $E \hookrightarrow M$ by

$$v \longmapsto \exp\left(\frac{\lambda(|v|)}{|v|}v\right),$$

where $\lambda: [0, \infty) \rightarrow [0, 1)$ is a diffeomorphism which is the identity on $[0, \frac{1}{2})$. \square

Appendix B. Integrating ideals

Representations of a Lie groupoid \mathcal{G} can be differentiated to representations of its Lie algebroid A but, in general, a representation of A does only integrate to a representation of the s -fiber 1-connected groupoid of A , and not necessarily to one of \mathcal{G} . In this subsection, we prove that representations of A on ideals can be integrated to representations of any s -connected integration. This result was used in the proof of part (b) of Theorem 2.

Let $(A, [\cdot, \cdot], \varrho)$ be a Lie algebroid. We call a subbundle $I \subset A$ an *ideal* of A , if $\varrho(I) = 0$ and $\Gamma(I)$ is an ideal of the Lie algebra $\Gamma(A)$. Using the Leibniz rule, one easily sees that, if $I \neq A$, then the second condition implies the first. An ideal I is naturally a representation of A , with A -connection given by the Lie bracket

$$\nabla_X(Y) := [X, Y], \quad X \in \Gamma(A) \text{ and } Y \in \Gamma(I).$$

LEMMA B.1. *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff Lie groupoid with Lie algebroid A and let $I \subset A$ be an ideal. If the s -fibers of \mathcal{G} are connected, then the representation of A on I given by the Lie bracket integrates to \mathcal{G} .*

Proof. First observe that \mathcal{G} acts on the possibly singular bundle of isotropy Lie algebras $\ker(\varrho) \rightarrow M$ via the formula

$$g \cdot Y = \left. \frac{d}{d\varepsilon} (g \exp(\varepsilon Y) g^{-1}) \right|_{\varepsilon=0} \quad \text{for } Y \in \ker(\varrho)_{s(g)}. \quad (42)$$

Let $N(I) \subset \mathcal{G}$ be the subgroupoid consisting of elements g which satisfy $g \cdot I_{s(g)} \subset I_{t(g)}$. We will prove that $N(I) = \mathcal{G}$ and that the induced action of \mathcal{G} on I differentiates to the Lie bracket.

Recall that a derivation on a vector bundle $E \rightarrow M$ (see [13, §3.4]) is a pair (D, V) , where D is a linear operator on $\Gamma(E)$ and V is a vector field on M , satisfying

$$D(f\alpha) = fD(\alpha) + V(f)\alpha \quad \text{for all } \alpha \in \Gamma(E) \text{ and all } f \in C^\infty(M).$$

The flow of a derivation (D, V) , denoted by φ_D^t , is a vector bundle map covering the flow φ_V^t of V , $\varphi_D^t: E_x \rightarrow E_{\varphi_V^t(x)}$ (whenever $\varphi_V^t(x)$ is defined), which is the solution to the differential equation

$$\frac{d}{dt}(\varphi_D^t)^*(\alpha) = (\varphi_D^t)^*(D\alpha), \quad \varphi_D^0 = \text{Id}_E,$$

where $(\varphi_D^t)^*(\alpha)_x = \varphi_D^{-t}(\alpha_{\varphi_V^t(x)})$.

For $X \in \Gamma(A)$, denote by $\Psi^t(X, g)$ the flow of the corresponding right-invariant vector field on \mathcal{G} , and by $\varphi^t(X, x)$ the flow of $\varrho(X)$ on M . Conjugation by $\Psi^t(X)$ is an automorphism of \mathcal{G} covering $\varphi^t(X)$, which we denote by

$$\begin{aligned} C(\Psi^t(X)): \mathcal{G} &\longrightarrow \mathcal{G}, \\ g &\longmapsto \Psi^t(X, t(g))g\Psi^t(X, s(g))^{-1}. \end{aligned}$$

Since $C(\Psi^t(X))$ sends the s -fiber over x to the s -fiber over $\varphi^t(X, x)$, its differential at the identity 1_x gives an isomorphism

$$\text{Ad}(\Psi^t(X)): A_x \longrightarrow A_{\varphi^t(X, x)}, \quad \text{Ad}(\Psi^t(X))_x := dC(\Psi^t(X))|_{A_x}.$$

On $\ker(\varrho)_x$, we recover the action (42) of $g = \Psi^t(X, x)$. We have that

$$\begin{aligned} &\frac{d}{dt}(\text{Ad}(\Psi^t(X))^*Y)_x \\ &= \frac{d}{dt} \text{Ad}(\Psi^{-t}(X, \varphi^t(X, x)))Y_{\varphi^t(X, x)} \\ &= -\frac{d}{ds}(\text{Ad}(\Psi^{-t}(X, \varphi^t(X, x))) \text{Ad}(\Psi^s(X, \varphi^{t-s}(X, x))))Y_{\varphi^{t-s}(X, x)} \Big|_{s=0} \tag{43} \\ &= \text{Ad}(\Psi^{-t}(X, \varphi^t(X, x)))[X, Y]_{\varphi^t(X, x)} \\ &= \text{Ad}(\Psi^t(X))^*([X, Y])_x \end{aligned}$$

for $Y \in \Gamma(A)$, where we have used the adjoint formulas from [13, Proposition 3.7.1]. This shows that $\text{Ad}(\Psi^t(X))$ is the flow of the derivation $([X, \cdot], \varrho(X))$ on A . Since I is an ideal, the derivation $[X, \cdot]$ restricts to a derivation on I , and therefore I is invariant under $\text{Ad}(\Psi^t(X))$. This proves that, for all $Y \in I_x$,

$$\text{Ad}(\Psi^t(X, x))Y = \Psi^t(X, x) \cdot Y \in I.$$

So $N(I)$ contains all the elements in \mathcal{G} of the form $\Psi^t(X, x)$. The set of such elements contains an open neighborhood O of the unit section in \mathcal{G} . Since the s -fibers of \mathcal{G} are connected, O generates \mathcal{G} (see [13, Proposition 1.5.8]). Therefore $N(I)=\mathcal{G}$ and so (42) defines an action of \mathcal{G} on I .

Using that $\Psi^{-t}(X, \varphi^t(X, x))=\Psi^t(X, x)^{-1}$, equation (43) gives

$$\frac{d}{dt}(\Psi^t(X, x)^{-1} \cdot Y_{\varphi^t(X, x)}) \Big|_{t=0} = [X, Y]_x \quad \text{for all } X \in \Gamma(A) \text{ and all } Y \in \Gamma(I).$$

Thus, the action differentiates to the Lie bracket (see [13, Definition 3.6.8]). □

Appendix C. The tame vanishing lemma

In this subsection we prove the tame vanishing lemma, an existence result for tame homotopy operators on the complex computing Lie algebroid cohomology with coefficients. In the proof of Theorem 2, this lemma was applied to the Poisson complex. In combination with the Nash–Moser techniques, the tame vanishing lemma is very useful when applied to various geometric problems (see the appendix in [16]).

C.1. The weak C^∞ -topology

The compact-open C^k -topology on the space of sections of a vector bundle can be generated by a family of seminorms, and we recall here a construction of such seminorms, generalizing the construction from §3. These seminorms will be used to express the tameness property of the homotopy operators.

Let $W \rightarrow M$ be a vector bundle. Consider a locally finite open cover $\mathcal{U}:=\{U_i\}_{i \in I}$ of M by relatively compact domains of coordinate charts $\{\chi_i:U_i \xrightarrow{\sim} \mathbb{R}^m\}_{i \in I}$ and choose trivializations for $W|_{U_i}$. Let $\mathcal{O}:=\{O_i\}_{i \in I}$ be a second open cover, with \bar{O}_i being compact and $\bar{O}_i \subset U_i$. A section $\sigma \in \Gamma(W)$ can be represented in these charts by a family of smooth functions $\{\sigma_i: \mathbb{R}^m \rightarrow \mathbb{R}^k\}_{i \in I}$, where k is the rank of W . For $U \subset M$, an open set with compact closure, we have that \bar{U} intersects only a finite number of the coordinate charts U_i . Denote the set of such indices by $I_U \subset I$. Define the n th norm of σ on U by

$$\|\sigma\|_{n, \bar{U}} := \sup \left\{ \left| \frac{\partial^{|\alpha|} \sigma_i}{\partial x^\alpha}(x) \right| : |\alpha| \leq n, x \in \chi_i(U \cap O_i) \text{ and } i \in I_U \right\}.$$

For a fixed n , the family of seminorms $\|\cdot\|_{n, \bar{U}}$, with U being a relatively compact open set in M , generate the compact-open C^n -topology on $\Gamma(W)$. The union of all these topologies, for $n \geq 0$, is the weak C^∞ -topology on $\Gamma(W)$. Observe that the seminorms $\{\|\cdot\|_{n, \bar{U}}\}_{n \geq 0}$ induce norms on $\Gamma(W|_{\bar{U}})$.

C.2. The statement of the tame vanishing lemma

LEMMA C.1. (The tame vanishing lemma) *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff Lie groupoid with Lie algebroid A and let V be a representation of \mathcal{G} . If the s -fibers of \mathcal{G} are compact and their de Rham cohomology vanishes in degree p , then*

$$H^p(A, V) = 0.$$

Moreover, there exist linear homotopy operators

$$\Omega^{p-1}(A, V) \xleftarrow{h_1} \Omega^p(A, V) \xleftarrow{h_2} \Omega^{p+1}(A, V),$$

with

$$d_{\nabla} h_1 + h_2 d_{\nabla} = \text{Id},$$

which satisfy

- (1) (invariant locality) *for every orbit O of A , they induce linear maps*

$$\Omega^{p-1}(A|_O, V|_O) \xleftarrow{h_{1,O}} \Omega^p(A|_O, V|_O) \xleftarrow{h_{2,O}} \Omega^{p+1}(A|_O, V|_O),$$

such that, for all $\omega \in \Omega^p(A, V)$ and all $\eta \in \Omega^{p+1}(A, V)$, we have that

$$h_{1,O}(\omega|_O) = (h_1 \omega)|_O \quad \text{and} \quad h_{2,O}(\eta|_O) = (h_2 \eta)|_O,$$

- (2) (tameness) *for every invariant open $U \subset M$, with \bar{U} compact, there are constants $C_{n,U} > 0$ such that*

$$\|h_1(\omega)\|_{n,\bar{U}} \leq C_{n,U} \|\omega\|_{n+s,\bar{U}} \quad \text{and} \quad \|h_2(\eta)\|_{n,\bar{U}} \leq C_{n,U} \|\eta\|_{n+s,\bar{U}}$$

for all $\omega \in \Omega^p(A|_{\bar{U}}, V|_{\bar{U}})$ and all $\eta \in \Omega^{p+1}(A|_{\bar{U}}, V|_{\bar{U}})$, where

$$s = \lfloor \frac{1}{2} \text{rank}(A) \rfloor + 1.$$

We also note the following consequences of the proof.

COROLLARY C.2. *The constants $C_{n,U}$ can be chosen so that they are uniform over all invariant open subsets of U . More precisely, if $V \subset U$ is a second invariant open set, then one can choose $C_{n,V} := C_{n,U}$, assuming that the norms on \bar{U} and \bar{V} are computed using the same charts and trivializations.*

COROLLARY C.3. *The homotopy operators preserve the order of vanishing around orbits. More precisely, if O is an orbit of A , and $\omega \in \Omega^p(A, V)$ is a form such that $j^k \omega|_O = 0$, then $j^k h_1(\omega)|_O = 0$; and similarly for h_2 .*

C.3. The de Rham complex of a fiber bundle

To prove the tame vanishing lemma, we first construct tame homotopy operators for the foliated de Rham complex of a fiber bundle. For this, we use a result on the family of inverses of elliptic operators (Proposition C.7), which we prove at the end of the section.

Let $\pi: \mathcal{B} \rightarrow M$ be a locally trivial fiber bundle whose fibers $\mathcal{B}_x := \pi^{-1}(x)$ are diffeomorphic to a compact, connected manifold F and let $V \rightarrow M$ be a vector bundle. The space of vertical vectors on \mathcal{B} will be denoted by $T^\pi \mathcal{B}$ and the space of foliated forms with values in $\pi^*(V)$ by $\Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V))$. An element $\omega \in \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V))$ is a smooth family of forms on the fibers of π with values in V ,

$$\omega = \{\omega_x\}_{x \in M}, \quad \omega_x \in \Omega^\bullet(\mathcal{B}_x, V_x).$$

The fiberwise exterior derivative induces the differential

$$d \otimes I_V: \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)) \longrightarrow \Omega^{\bullet+1}(T^\pi \mathcal{B}, \pi^*(V)),$$

defined by

$$d \otimes I_V(\omega)_x := (d \otimes I_{V_x})(\omega_x), \quad x \in M.$$

We construct the homotopy operators using Hodge theory. Let m be a metric on $T^\pi \mathcal{B}$, or equivalently a smooth family of Riemannian metrics $\{m_x\}_{x \in M}$ on the fibers of π . Integration against the volume density gives an inner product on $\Omega^\bullet(\mathcal{B}_x)$,

$$(\eta, \theta) := \int_{\mathcal{B}_x} m_x(\eta, \theta) |d \text{Vol}(m_x)|, \quad \eta, \theta \in \Omega^q(\mathcal{B}_x).$$

Let δ_x denote the formal adjoint of d with respect to this inner product

$$\delta_x: \Omega^{\bullet+1}(\mathcal{B}_x) \longrightarrow \Omega^\bullet(\mathcal{B}_x),$$

i.e. δ_x is the unique linear first order differential operator satisfying

$$(d\eta, \theta) = (\eta, \delta_x \theta) \quad \text{for all } \eta \in \Omega^\bullet(\mathcal{B}_x) \text{ and all } \theta \in \Omega^{\bullet+1}(\mathcal{B}_x).$$

The Laplace–Beltrami operator associated to m_x will be denoted by

$$\Delta_x := d\delta_x + \delta_x d: \Omega^\bullet(\mathcal{B}_x) \longrightarrow \Omega^\bullet(\mathcal{B}_x).$$

Both these operators induce linear differential operators on $\Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V))$,

$$\begin{aligned} \delta \otimes I_V: \Omega^{\bullet+1}(T^\pi \mathcal{B}, \pi^*(V)) &\longrightarrow \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)), & \delta \otimes I_V(\omega)_x &:= (\delta_x \otimes I_{V_x})(\omega_x), \\ \Delta \otimes I_V: \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)) &\longrightarrow \Omega^\bullet(T^\pi \mathcal{B}, \pi^*(V)), & \Delta \otimes I_V(\omega)_x &:= (\Delta_x \otimes I_{V_x})(\omega_x). \end{aligned}$$

By the Hodge theorem, if the fiber F of \mathcal{B} has vanishing de Rham cohomology in degree p , then Δ_x is invertible in degree p .

LEMMA C.4. *If $H^p(F)=0$ then the following hold:*

(a) $\Delta \otimes I_V$ is invertible in degree p and its inverse is given by

$$\begin{aligned} G \otimes I_V: \Omega^p(T^\pi \mathcal{B}, \pi^*(V)) &\longrightarrow \Omega^p(T^\pi \mathcal{B}, \pi^*(V)), \\ (G \otimes I_V)(\omega)_x &:= (\Delta_x^{-1} \otimes I_{V_x})(\omega_x), \quad x \in M; \end{aligned}$$

(b) the maps $H_1 := (\delta \otimes I_V) \circ (G \otimes I_V)$ and $H_2 := (G \otimes I_V) \circ (\delta \otimes I_V)$,

$$\Omega^{p-1}(T^\pi \mathcal{B}, \pi^*(V)) \xleftarrow{H_1} \Omega^p(T^\pi \mathcal{B}, \pi^*(V)) \xleftarrow{H_2} \Omega^{p+1}(T^\pi \mathcal{B}, \pi^*(V))$$

are linear homotopy operators in degree p ;

(c) H_1 and H_2 satisfy the following local-tameness property: for every relatively compact open $U \subset M$ there are constants $C_{n,U} > 0$ such that

$$\begin{aligned} \|H_1(\eta)\|_{n, \mathcal{B}|_{\bar{U}}} &\leq C_{n,U} \|\eta\|_{n+s, \mathcal{B}|_{\bar{U}}} \quad \text{for all } \eta \in \Omega^p(T^\pi \mathcal{B}|_{\bar{U}}, \pi^*(V|_{\bar{U}})), \\ \|H_2(\omega)\|_{n, \mathcal{B}|_{\bar{U}}} &\leq C_{n,U} \|\omega\|_{n+s, \mathcal{B}|_{\bar{U}}} \quad \text{for all } \omega \in \Omega^{p+1}(T^\pi \mathcal{B}|_{\bar{U}}, \pi^*(V|_{\bar{U}})), \end{aligned}$$

where $s = \lfloor \frac{1}{2} \dim(F) \rfloor + 1$.

Moreover, if $U' \subset U$, then one can take $C_{n,U'} := C_{n,U}$.

Proof. In a trivialization chart the operator $\Delta \otimes I_V$ is given by a smooth family of Laplace–Beltrami operators

$$\Delta_x: \Omega^p(F)^k \longrightarrow \Omega^p(F)^k,$$

where k is the rank of V . These operators are elliptic and invertible, and therefore, by Proposition C.7, $\Delta_x^{-1}(\omega_x)$ is smooth in x , for every smooth family $\omega_x \in \Omega^p(F)^k$. This shows that $G \otimes I_V$ maps smooth forms to smooth forms. Clearly $G \otimes I_V$ is the inverse of $\Delta \otimes I_V$, so we have proven (a).

For part (c), let $U \subset M$ be a relatively compact open set. Applying part (2) of Proposition C.7 to a family of coordinate charts which cover \bar{U} , we find constants $D_{n,U}$ such that

$$\|G \otimes I_V(\eta)\|_{n, \mathcal{B}|_{\bar{U}}} \leq D_{n,U} \|\eta\|_{n+s-1, \mathcal{B}|_{\bar{U}}} \quad \text{for all } \eta \in \Omega^p(T^\pi \mathcal{B}|_{\bar{U}}, \pi^*(V|_{\bar{U}})).$$

Moreover, the constants can be chosen so that they are decreasing in U . Since H_1 and H_2 are defined as the composition of $G \otimes I_V$ with a linear differential operator of degree 1, it follows that we can also find constants $C_{n,U}$ such that the inequalities from (c) are satisfied, and which are also decreasing in U .

For part (b), using that $\delta_x^2=0$, we obtain that Δ_x commutes with $d\delta_x$:

$$\begin{aligned}\Delta_x d\delta_x &= (d\delta_x + \delta_x d)d\delta_x = d\delta_x d\delta_x + \delta_x d^2\delta_x = d\delta_x d\delta_x, \\ d\delta_x \Delta_x &= d\delta_x (d\delta_x + \delta_x d) = d\delta_x d\delta_x + d\delta_x^2 d = d\delta_x d\delta_x.\end{aligned}$$

This implies that $\Delta \otimes I_V$ commutes with $(d \otimes I_V)(\delta \otimes I_V)$, and thus $G \otimes I_V$ commutes with $(d \otimes I_V)(\delta \otimes I_V)$. Using this, we obtain that H_1 and H_2 are homotopy operators:

$$\begin{aligned}I &= (G \otimes I_V)(\Delta \otimes I_V) \\ &= (G \otimes I_V)((d \otimes I_V)(\delta \otimes I_V) + (\delta \otimes I_V)(d \otimes I_V)) \\ &= (d \otimes I_V)(\delta \otimes I_V)(G \otimes I_V) + (G \otimes I_V)(\delta \otimes I_V)(d \otimes I_V) \\ &= (d \otimes I_V)H_1 + H_2(d \otimes I_V).\end{aligned}\quad \square$$

C.4. Proof of the tame vanishing lemma

Let $\mathcal{G} \rightrightarrows M$ be as in the statement. By passing to the connected components of the identities in the s -fibers [19], we may assume that \mathcal{G} is s -connected. Then $s: \mathcal{G} \rightarrow M$ is a locally trivial fiber bundle with compact fibers whose cohomology vanishes in degree p . We will apply Lemma C.4 to the complex of s -foliated forms with coefficients in $s^*(V)$,

$$(\Omega^\bullet(T^s\mathcal{G}, s^*(V)), d \otimes I_V).$$

Recall that the *right translation* by an arrow $g \in \mathcal{G}$ is the diffeomorphism between the s -fibers above $y=t(g)$ and above $x=s(g)$, given by

$$\begin{aligned}r_g: \mathcal{G}_y &\xrightarrow{\sim} \mathcal{G}_x, \\ h &\longmapsto hg.\end{aligned}$$

A form $\omega \in \Omega^\bullet(T^s\mathcal{G}, s^*(V))$ is *invariant* if it satisfies

$$(r_g^* \otimes g)(\omega_{hg}) = \omega_h \quad \text{for all } h, g \in \mathcal{G} \text{ with } s(h) = t(g),$$

where $r_g^* \otimes g$ is the linear isomorphism $\eta \mapsto g \cdot \eta \circ dr_g$. Denote the space of invariant V -valued forms on \mathcal{G} by $\Omega^\bullet(T^s\mathcal{G}, s^*(V))^{\mathcal{G}}$.

It is well known that forms on A with values in V are in one-to-one correspondence with invariant V -valued forms on \mathcal{G} ; this correspondence is given by

$$J: \Omega^\bullet(A, V) \longrightarrow \Omega^\bullet(T^s\mathcal{G}, s^*(V)), \quad J(\eta)_g := (r_{g^{-1}}^* \otimes g^{-1})(\eta_{t(g)}).$$

The map J is also a chain map, and thus it induces an isomorphism of complexes (see [26, Theorem 1.2] and also [16, §2.3.2] for coefficients)

$$J: (\Omega^\bullet(A, V), d_\nabla) \xrightarrow{\sim} (\Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}, d \otimes I_V). \quad (44)$$

A left inverse for J (i.e. a map P such that $P \circ J = \text{Id}$) is given by

$$P: \Omega^\bullet(T^s\mathcal{G}, s^*(V)) \longrightarrow \Omega^\bullet(A, V), \quad P(\omega)_x := \omega_{u(x)}.$$

Let $\langle \cdot, \cdot \rangle$ be an inner product on A . Using right translations, we extend $\langle \cdot, \cdot \rangle$ to an invariant metric m on $T^s\mathcal{G}$,

$$m(X, Y)_g := \langle dr_{g^{-1}}X, dr_{g^{-1}}Y \rangle_{t(g)} \quad \text{for all } X, Y \in T_g^s\mathcal{G}.$$

Invariance of m implies that the right translation by an arrow $g: x \rightarrow y$ is an isometry between the s -fibers

$$r_g: (\mathcal{G}_y, m_y) \xrightarrow{\sim} (\mathcal{G}_x, m_x).$$

The corresponding operators from §C.3 are also invariant.

LEMMA C.5. *The operators $\delta \otimes I_V$, $\Delta \otimes I_V$, H_1 and H_2 , corresponding to m , send invariant forms to invariant forms.*

Proof. Since right translations are isometries and the operators δ_z are invariant under isometries we have that $r_g^* \circ \delta_x = \delta_y \circ r_g^*$, for all arrows $g: x \rightarrow y$.

For $\eta \in \Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}$ we have that

$$\begin{aligned} (r_g^* \otimes g)(\delta \otimes I_V(\eta))|_{\mathcal{G}_x} &= (r_g^* \circ \delta_x \otimes g)(\eta|_{\mathcal{G}_x}) = (\delta_y \circ r_g^* \otimes g)(\eta|_{\mathcal{G}_x}) \\ &= (\delta_y \otimes I_{V_y})(r_g^* \otimes g)(\eta|_{\mathcal{G}_x}) = (\delta_y \otimes I_{V_y})(\eta|_{\mathcal{G}_y}) = (\delta \otimes I_V)(\eta)|_{\mathcal{G}_y}. \end{aligned}$$

This shows that $\delta \otimes I_V(\eta) \in \Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}$. The other operators are constructed in terms of $\delta \otimes I_V$ and $d \otimes I_V$, and thus they also preserve $\Omega^\bullet(T^s\mathcal{G}, s^*(V))^\mathcal{G}$. \square

This lemma and the isomorphism (44) imply that the maps

$$\Omega^{p-1}(A, V) \xleftarrow{h_1} \Omega^p(A, V) \xleftarrow{h_2} \Omega^{p+1}(A, V),$$

defined by

$$h_1 := P \circ H_1 \circ J \quad \text{and} \quad h_2 := P \circ H_2 \circ J,$$

are linear homotopy operators for the Lie algebroid complex in degree p .

For part (1) of the tame vanishing lemma, let $\omega \in \Omega^p(A, V)$ and $O \subset M$ be an orbit of A . Since \mathcal{G} is s -connected we have that $s^{-1}(O) = t^{-1}(O) = \mathcal{G}|_O$. Clearly $J(\omega)|_{s^{-1}(O)}$ depends only on $\omega|_O$. By the construction of H_1 , for all $x \in O$ we have that

$$h_1(\omega)_x = H_1(J(\omega))|_{1_x} = (\delta_x \circ \Delta_x^{-1} \otimes I_{V_x})(J(\omega)|_{s^{-1}(x)})|_{1_x}.$$

Thus $h_1(\omega)|_O$ depends only on $\omega|_O$. The same argument applies also to h_2 .

Before checking part (2), we give a simple lemma.

LEMMA C.6. Consider a vector bundle map $A: F_1 \rightarrow F_2$ between two vector bundles $F_1 \rightarrow M_1$ and $F_2 \rightarrow M_2$, covering a map $f: M_1 \rightarrow M_2$. If A is fiberwise invertible and f is proper, then the pull-back map

$$A^*: \Gamma(F_2) \longrightarrow \Gamma(F_1), \quad A(\sigma)_x := A_x^{-1}(\sigma_{f(x)}),$$

satisfies the following tameness inequalities: for every open $U \subset M_2$, with \bar{U} compact, there are constants $C_{n,U} > 0$ such that

$$\|A^*(\sigma)\|_{n, \overline{f^{-1}(U)}} \leq C_{n,U} \|\sigma\|_{n,U} \quad \text{for all } \sigma \in \Gamma(F_2|_{\bar{U}}).$$

Moreover,

- (a) if $U' \subset U$ is open, and one uses the same charts when computing the norms, then one can choose $C_{n,U'} := C_{n,U}$;
- (b) if $N \subset M_2$ is a submanifold and $\sigma \in \Gamma(F_2)$ satisfies $j^k(\sigma)|_N = 0$, then its pull-back satisfies $j^k(A^*(\sigma))|_{f^{-1}(N)} = 0$.

Proof. Since A is fiberwise invertible, we can assume that $F_1 = f^*(F_2)$ and $A^* = f^*$. By choosing a vector bundle F' such that $F_2 \oplus F'$ is trivial, we reduce the problem to the case when F_2 is the trivial line bundle. So, we have to check that $f^*: C^\infty(M_2) \rightarrow C^\infty(M_1)$ has the desired properties. But this is straightforward: we just cover both $\overline{f^{-1}(U)}$ and \bar{U} by charts, and apply the chain rule. The constants $C_{n,U}$ are the C^n -norms of f over $\overline{f^{-1}(U)}$, and therefore are getting smaller if U gets smaller. This implies (a). For part (b), just observe that $j_{f(x)}^k(\sigma) = 0$ implies $j_x^k(\sigma \circ f) = 0$. \square

Part (2) of the tame vanishing lemma follows by Lemma C.4(c) and by applying Lemma C.6 to J and P . Corollary C.2 follows from Lemma C.6 (a) and Lemma C.4 (c). To prove Corollary C.3, consider a form ω with $j^k \omega|_O = 0$, for an orbit O . Then, by Lemma C.6 (b), it follows that $J(\omega)$ vanishes up to order k along $t^{-1}(O) = \mathcal{G}|_O$. By construction, we have that H_1 is $C^\infty(M)$ linear, and therefore also $H_1(J(\omega))$ vanishes up to order k along $\mathcal{G}|_O$; and again, by Lemma C.6 (b), $h_1(\omega) = u^*(H_1(J(\omega)))$ vanishes along $O = u^{-1}(\mathcal{G}|_O)$ up to order k .

C.5. The inverse of a family of elliptic operators

This subsection is devoted to proving the following result.

PROPOSITION C.7. Consider a smooth family of linear differential operators

$$P_x: \Gamma(V) \longrightarrow \Gamma(W), \quad x \in \mathbb{R}^m,$$

between sections of vector bundles V and W over a compact base F . If P_x is elliptic of degree $d \geq 1$ and invertible for all $x \in \mathbb{R}^m$, then

(1) the family of inverses $\{Q_x := P_x^{-1}\}_{x \in \mathbb{R}^m}$ induces a linear operator

$$Q: \Gamma(p^*(W)) \longrightarrow \Gamma(p^*(V)),$$

$$\{\omega_x\}_{x \in \mathbb{R}^m} \longmapsto \{Q_x \omega_x\}_{x \in \mathbb{R}^m},$$

where $p^*(V) := V \times \mathbb{R}^m \rightarrow F \times \mathbb{R}^m$ and $p^*(W) := W \times \mathbb{R}^m \rightarrow F \times \mathbb{R}^m$;

(2) Q is locally tame, in the sense that for all bounded open sets $U \subset \mathbb{R}^m$, there exist constants $C_{n,U} > 0$ such that the inequalities

$$\|Q(\omega)\|_{n, F \times \bar{U}} \leq C_{n,U} \|\omega\|_{n+s-1, F \times \bar{U}} \quad \text{for all } \omega \in \Gamma(p^*(W)|_{F \times \bar{U}}),$$

hold, with $s = \lfloor \frac{1}{2} \dim(F) \rfloor + 1$. If $U' \subset U$, then one can take $C_{n,U'} := C_{n,U}$.

Fixing C^n -norms $\|\cdot\|_n$ on $\Gamma(V)$, we induce seminorms on $\Gamma(p^*(V))$,

$$\|\omega\|_{n, F \times \bar{U}} := \sup_{0 \leq k+|\alpha| \leq n} \sup_{x \in U} \left\| \frac{\partial^{|\alpha|} \omega_x}{\partial x^\alpha} \right\|_k,$$

where $\omega \in \Gamma(p^*(V))$ is regarded as a smooth family $\omega = \{\omega_x \in \Gamma(V)\}_{x \in \mathbb{R}^m}$. Similarly, fixing norms on $\Gamma(W)$, we define also norms on $\Gamma(p^*(W))$.

Endow $\Gamma(V)$ and $\Gamma(W)$ also with Sobolev norms, denoted by $\{|\cdot|_n\}_{n \geq 0}$. Loosely speaking, $|\omega|_n$ measures the L^2 -norm of ω and its partial derivatives up to order n (for a precise definition, see e.g. [10]). Denote by $H_n(\Gamma(V))$ and by $H_n(\Gamma(W))$ the completion of $\Gamma(V)$, respectively of $\Gamma(W)$, with respect to the Sobolev norm $|\cdot|_n$.

We will use the standard inequalities between the Sobolev and the C^n -norms, which follow from the Sobolev embedding theorem:

$$\|\omega\|_n \leq C_n |\omega|_{n+s} \quad \text{and} \quad |\omega|_n \leq C_n \|\omega\|_n, \tag{45}$$

for all $\omega \in \Gamma(V)$ (resp. $\Gamma(W)$), where $s = \lfloor \frac{1}{2} \dim(F) \rfloor + 1$ and $C_n > 0$ are constants.

Since P_x is of order d , it induces continuous linear maps between the Sobolev spaces, denoted by

$$[P_x]_n: H_{n+d}(\Gamma(V)) \longrightarrow H_n(\Gamma(W)).$$

These maps are invertible.

LEMMA C.8. *If a degree- d elliptic differential operator*

$$P: \Gamma(V) \longrightarrow \Gamma(W)$$

is invertible, then for every $n \geq 0$ the induced map

$$[P]_n: H_{n+d}(\Gamma(V)) \longrightarrow H_n(\Gamma(W))$$

is also invertible and its inverse is induced by the inverse of P .

Proof. Since P is elliptic, it is invertible modulo smoothing operators (see [10, Lemma 1.3.5]), i.e. there exists a pseudo-differential operator

$$\Psi: \Gamma(W) \longrightarrow \Gamma(V),$$

of degree $-d$ such that $\Psi P - \text{Id} = K_1$ and $P\Psi - \text{Id} = K_2$, where K_1 and K_2 are smoothing operators. Since Ψ is of degree $-d$, it induces continuous maps

$$[\Psi]_n: H_n(\Gamma(W)) \longrightarrow H_{n+d}(\Gamma(V)),$$

and since K_1 and K_2 are smoothing operators, they induce continuous maps

$$[K_1]_n: H_n(\Gamma(V)) \longrightarrow \Gamma(V) \quad \text{and} \quad [K_2]_n: H_n(\Gamma(W)) \longrightarrow \Gamma(W).$$

We now show that $[P]_n$ is a bijection.

Injectivity. For $\eta \in H_{n+d}(\Gamma(V))$, with $[P]_n \eta = 0$, we have that

$$\eta = (\text{Id} - [\Psi]_n [P]_n) \eta = -[K_1]_n \eta \in \Gamma(V),$$

and hence $[P]_n \eta = P\eta$. By the injectivity of P , we have that $\eta = 0$.

Surjectivity. For $\theta \in H_n(\Gamma(W))$, we have that

$$([P]_n [\Psi]_n - \text{Id}) \theta = [K_2]_n \theta \in \Gamma(W),$$

and, since P is onto, $[K_2]_n \theta = P\eta$ for some $\eta \in \Gamma(V)$. So θ is in the range of $[P]_n$,

$$\theta = [P]_n ([\Psi]_n \theta - \eta).$$

The inverse of a bounded operator between Banach spaces is bounded, and therefore $[P]_n^{-1}$ is continuous. Since on smooth sections $[P]_n^{-1}$ coincides with P^{-1} , and since the space of smooth sections is dense in all Sobolev spaces, it follows that P^{-1} induces a continuous map $H_n(\Gamma(W)) \rightarrow H_{n+d}(\Gamma(V))$, and that this map is $[P]_n^{-1}$. \square

For two Banach spaces B_1 and B_2 , denote by $\text{Lin}(B_1, B_2)$ the Banach space of bounded linear maps between them, and by $\text{Iso}(B_1, B_2)$ the open subset consisting of invertible maps. The following proves that the family $[P_x]_n$ is smooth.

LEMMA C.9. *Let $\{P_x\}_{x \in \mathbb{R}^m}$ be a smooth family of linear differential operators of order d between the sections of vector bundles V and W , both over a compact manifold F . Then the map induced by P from \mathbb{R}^m to the space of bounded linear operators between the Sobolev spaces*

$$\mathbb{R}^m \ni x \longmapsto [P_x]_n \in \text{Lin}(H_{n+d}(\Gamma(V)), H_n(\Gamma(W)))$$

is smooth and its derivatives are induced by the derivatives of P_x .

Proof. Linear differential operators of degree d from V to W are sections of the vector bundle $\text{Hom}(J^d(V); W) = J^d(V)^* \otimes W$, where $J^d(V) \rightarrow F$ is the d th jet bundle of V . Therefore, P can be viewed as a smooth section of the pull-back bundle

$$p^*(\text{Hom}(J^d(V); W)) := \text{Hom}(J^d(V); W) \times \mathbb{R}^m \longrightarrow F \times \mathbb{R}^m.$$

Since F is compact, by choosing a partition of unity on F with supports inside some open sets on which V and W trivialize, one can write any section of $p^*(\text{Hom}(J^d(V); W))$ as a linear combination of sections of $\text{Hom}(J^d(V); W)$ with coefficients in $C^\infty(\mathbb{R}^m \times F)$. Hence, there are constant differential operators P_i and functions $f_i \in C^\infty(\mathbb{R}^m \times F)$, for $i=1, 2, \dots, N$, such that

$$P_x = \sum_{i=1}^N f_i(x) P_i.$$

So it suffices to prove that for $f \in C^\infty(\mathbb{R}^m \times F)$, multiplication with $f(x)$ induces a smooth map

$$\mathbb{R}^m \ni x \longmapsto [f(x) \text{Id}]_n \in \text{Lin}(H_n(\Gamma(W)), H_n(\Gamma(W))).$$

First, it is easy to see that for any smooth function $g \in C^\infty(\mathbb{R}^m \times F)$ and every compact $K \subset \mathbb{R}^m$, there are constants $C_n(g, K)$ such that $|g(x)\sigma|_n \leq C_n(g, K)|\sigma|_n$ for all $x \in K$ and $\sigma \in H_n(\Gamma(W))$; or equivalently that the operator norm satisfies $\|[g(x) \text{Id}]_n\|_{\text{op}} \leq C_n(g, K)$ for $x \in K$.

Consider $f \in C^\infty(\mathbb{R}^m \times F)$, let $\bar{x} \in \mathbb{R}^m$ and K be a closed ball centered at \bar{x} . Using the Taylor expansion of f at \bar{x} , write

$$f(x) - f(\bar{x}) = \sum_{i=1}^m (x_i - \bar{x}_i) T_{\bar{x}}^i(x),$$

and

$$f(x) - f(\bar{x}) - \sum_{i=1}^m (x_i - \bar{x}_i) \frac{\partial f}{\partial x_i}(\bar{x}) = \sum_{1 \leq i < j \leq m} (x_i - \bar{x}_i)(x_j - \bar{x}_j) T_{\bar{x}}^{i,j}(x),$$

where $T_{\bar{x}}^i, T_{\bar{x}}^{i,j} \in C^\infty(\mathbb{R}^m \times F)$. Thus, for all $x \in K$, we have that

$$\|[f(x) \text{Id}]_n - [f(\bar{x}) \text{Id}]_n\|_{\text{op}} \leq |x - \bar{x}| \sum_{i=1}^m C_n(T_{\bar{x}}^i, K),$$

$$\left| [f(x) \text{Id}]_n - [f(\bar{x}) \text{Id}]_n - \sum_{i=1}^m (x_i - \bar{x}_i) \left[\frac{\partial f}{\partial x_i}(\bar{x}) \text{Id} \right]_n \right|_{\text{op}} \leq |x - \bar{x}|^2 \sum_{1 \leq i < j \leq m} C_n(T_{\bar{x}}^{i,j}, K).$$

The first inequality implies that the map $x \mapsto [f(x) \text{Id}]_n$ is C^0 and the second that it is C^1 , with partial derivatives given by

$$\frac{\partial}{\partial x_i} [f \text{Id}]_n = \left[\frac{\partial f}{\partial x_i} \text{Id} \right]_n.$$

The statement now follows inductively. □

Proof of Proposition C.7

By Lemma C.8, $Q_x = P_x^{-1}$ induces continuous operators

$$[Q_x]_n: H_n(\Gamma(W)) \longrightarrow H_{n+d}(\Gamma(V)).$$

We claim that the following map is smooth

$$\mathbb{R}^m \ni x \longmapsto [Q_x]_n \in \text{Lin}(H_n(\Gamma(W)), H_{n+d}(\Gamma(V))).$$

This follows by Lemmas C.8 and C.9, since we can write

$$[Q_x]_n = [P_x^{-1}]_n = [P_x]_n^{-1} = \iota([P_x]_n),$$

where ι is the (smooth) inversion map

$$\iota: \text{Iso}(H_{n+d}(\Gamma(V)), H_n(\Gamma(W))) \longrightarrow \text{Iso}(H_n(\Gamma(W)), H_{n+d}(\Gamma(V))).$$

Let $\omega = \{\omega_x\}_{x \in \mathbb{R}^m} \in \Gamma(p^*(W))$. By our claim and Lemma C.9, it follows that

$$x \longmapsto [Q_x]_n[\omega_x]_n = [Q_x\omega_x]_{n+d} \in H_{n+d}(\Gamma(V))$$

is a smooth map. On the other hand, the Sobolev inequalities (45) show that the inclusion $\Gamma(V) \rightarrow \Gamma^n(V)$, where $\Gamma^n(V)$ is the space of sections of V of class C^n (endowed with the norm $\|\cdot\|_n$), extends to a continuous map

$$H_{n+s}(\Gamma(V)) \longrightarrow \Gamma^n(V).$$

Since also evaluation $\text{ev}_p: \Gamma^n(V) \rightarrow V_p$ at $p \in F$ is continuous, it follows that the map

$$x \longmapsto Q_x\omega_x(p) \in V_p$$

is smooth. This is enough to conclude the smoothness of the family $\{Q_x\omega_x\}_{x \in \mathbb{R}^m}$, so $Q(\omega) \in \Gamma(p^*(V))$. This finishes the proof of the first part.

For the second part, let $U \subset \mathbb{R}^m$ be an open set with \bar{U} compact. Since the map $x \mapsto [Q_x]_n$ is smooth, it follows that

$$D_{n,m,U} := \sup_{x \in U} \sup_{|\alpha| \leq m} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} [Q_x]_n \right|_{\text{op}} < \infty, \quad (46)$$

where $|\cdot|_{\text{op}}$ denotes the operator norm. Let $\omega = \{\omega_x\}_{x \in \bar{U}}$ be an element of $\Gamma(p^*(W)|_{F \times \bar{U}})$. By Lemma C.9, also the map $x \mapsto [\omega_x]_n \in H_n(\Gamma(W))$ is smooth and for all multi-indices γ ,

$$\frac{\partial^{|\gamma|}}{\partial x^\gamma} [\omega_x]_n = \left[\frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right]_n.$$

Let k and α be such that $|\alpha|+k \leq n$. Using (45) and (46), we obtain

$$\begin{aligned}
\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} (Q_x \omega_x) \right\|_k &\leq \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} (Q_x \omega_x) \right\|_{k+d-1} \\
&\leq C_{k+d-1} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} (Q_x \omega_x) \right|_{k+s+d-1} \\
&\leq C_{k+d-1} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta \ \gamma} \left| \frac{\partial^{|\beta|}}{\partial x^\beta} Q_x \frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right|_{k+s+d-1} \\
&\leq C_{k+d-1} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta \ \gamma} D_{k+s-1, |\beta|, U} \left| \frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right|_{k+s-1} \\
&\leq C_{k+d-1} C_{k+s-1} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta \ \gamma} D_{k+s-1, |\beta|, U} \left\| \frac{\partial^{|\gamma|}}{\partial x^\gamma} \omega_x \right\|_{k+s-1} \\
&\leq C_{n, U} \|\omega\|_{n+s-1, F \times \bar{U}}.
\end{aligned}$$

This proves the second part

$$\|Q(\omega)\|_{n, F \times \bar{U}} \leq C_{n, U} \|\omega\|_{n+s-1, F \times \bar{U}}.$$

The constants $D_{n, m, U}$ are clearly decreasing in U , and hence for $U' \subset U$ we also have that $C_{n, U'} \leq C_{n, U}$. This finishes the proof of Proposition C.7.

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